SPLIT DOMINATION NUMBER IN EDGE SEMI-MIDDLE GRAPH

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ABSTRACT. Let \( G = (p, q) \) be a connected graph and \( M_e(G) \) be its corresponding edge semi-middle graph. A dominating set \( D \subseteq V[M_e(G)] \) is split dominating set \( (V[M_e(G)] - D) \) is disconnected. The minimum size of \( D \) is called the split domination number of \( M_e(G) \) and is denoted by \( \gamma_s[M_e(G)] \). In this paper we obtain several results on split domination number.

1. INTRODUCTION

Domination is an area in graph theory with an extensive research activity. We consider simple, finite, undirected, non-trivial and connected graphs for our study. In literature, the concept of graph theory terminology not presented here can be found in \[1\]. In a graph \( G \), a set \( D \subseteq V \) is dominating set of \( G \) if every vertex in \( V - D \) is adjacent to some vertex in \( D \). The domination number of a graph \( G \) is the minimum size of \( D \). Some studies on domination in graphs were found in the papers \[2–4, 6–15, 18–30\]. The edge semi-middle graph \( M_e(G) \) of a graph \( G \) was studied \[17\] and is defined as follows. Let \( V(G), E(G) \) and \( R(G) \) be the vertex set, edge set and regions set respectively. The edge semi-middle graph of a graph \( G \), denoted by \( M_e(G) \) is a graph whose vertex set is \( V(G) \cup E(G) \cup R(G) \). The vertices of \( M_e(G) \) are adjacent if and only if they correspond to two adjacent edges of \( G \) or one corresponds to a vertex and other to an edge incident with it or one corresponds to edge and other to a region in which edge lie on the region.

Let \( R' = \{r'_1, r'_2, \ldots, r'_m\} \subseteq V[M_e(G)] \) for the region set \( \{r_1, r_2, \ldots, r_m\} \) of \( G \). Let \( V' = \{v'_1, v'_2, \ldots, v'_p\} \subseteq V[M_e(G)] \) for the vertex set \( \{v_1, v_2, \ldots, v_p\} \) of \( G \). Let \( E' = \{e'_1, e'_2, \ldots, e'_q\} \subseteq V[M_e(G)] \) for the edge set \( \{e_1, e_2, \ldots, e_q\} \) of \( G \). The study of some domination parameters on jump graph \[16\] motivated us to introduce split domination number in edge semi-middle graph.

2. PRELIMINARIES

**Theorem 2.1.** \[5\] For any graph \( G \), \( \gamma(G) \geq \lceil \frac{p}{1+\Delta(G)} \rceil \).

**Theorem 2.2.** \[31\] For the path \( P_n \), \( \gamma[M_e(P_n)] = \lceil \frac{n}{2} \rceil \).

**Theorem 2.3.** \[31\] For the cycle \( C_n \), \( \gamma[M_e(C_n)] = \lceil \frac{n}{2} \rceil \).

**Theorem 2.4.** \[31\] For any graph \( G(p, q) \), \( \gamma[M_e(G)] \geq \lceil \frac{p}{1+\Delta(G)} \rceil \).
3. **Split Domination Number in Edge Semi-Middle Graph**

A dominating set \( D \) of \( M_e(G) \) is a split dominating set if \( (V[M_e(G)] - D) \) is disconnected. The minimum cardinality of \( D \) is called split domination number of \( M_e(G) \) and is denoted by \( \gamma_s[M_e(G)] \). A minimum split dominating set is denoted by \( \gamma_s - set \).

In the Figure 1, the split dominating set of \( M_e(G) \) is \( D_4 = \{e_6', e_5',e_3',e_1'\} \), \( \gamma_s[M_e(G)] = 4; \)

![Diagram of graph G and its M_e(G) with a split dominating set](image)

**Figure 1.** The graph \( G \) and its \( M_e(G) \)

We begin with the following observations.

**Observation 3.1.** For every star \( K_{1,n} \), \( \gamma_s[M_e(K_{1,n})] = \gamma_s[M_e(K_{1,n})] = n. \)

**Observation 3.2.** Let \( G \) be a tree, \( \gamma_s[M_e(T)] = \gamma[M_e(T)]. \)
4. Results

Theorem 4.1. For the path $P_n$, $\gamma_s[M_{\ell}(P_n)] = \lceil \frac{n}{2} \rceil$.

Proof. Consider $G = P_n$. Let $D$ be the dominating set of $M_{\ell}(G)$ and is defined as follows.

$$D = \begin{cases} e'_1, e'_3, ... e'_{n-1} & \text{if } n = 2k, \\
e'_1, e'_3, ... e'_{n-2} & \text{if } n = 2k+1. \end{cases}$$

Clearly, $V[M_{\ell}(P_n)] - D$ is disconnected. Thus $\gamma_s[M_{\ell}(P_n)] = \lceil \frac{n}{2} \rceil$.

Theorem 4.2. For the cycle $C_n$,

$$\gamma_s[M_{\ell}(C_n)] = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n = 2k, k \geq 2, \\
\frac{n}{2} & \text{if } n = 2k+1, k \geq 1. \end{cases}$$

Proof. Consider $G = C_n$, $V(C_n) = \{v_i, 1 \leq i \leq n\}$ and $e_i = v_i v_{i+1}; 1 \leq i \leq n - 1$. Let $D$ be the dominating set of $M_{\ell}(C_n)$ and is defined as follows.

$$D = \begin{cases} e'_1, e'_3, ... e'_{n-1} & \text{if } n = 2k, k \geq 2, \\
e'_1, e'_3, ... e'_n & \text{if } n = 2k+1, k \geq 1. \end{cases}$$

Clearly, $D$ is $\gamma_s$-set for $n = 2k + 1$ but $D$ is not for $n = 2k$. Further, consider $D' = D \cup \{e'_n\}$ is a set for $n = 2k$ such that $V[M_{\ell}(C_n)] - D'$ is disconnected. Thus

$$\gamma_s[M_{\ell}(C_n)] = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n = 2k, k \geq 2, \\
\frac{n}{2} & \text{if } n = 2k+1, k \geq 1. \end{cases}$$

Theorem 4.3. For any graph $G$, $\gamma_s[M_{\ell}(G)] \geq \gamma[M_{\ell}(G)]$.

Proof. From $M_{\ell}(G)$ definition, $V[M_{\ell}(G)] = V' \cup E' \cup R'$. Let the dominating set of $M_{\ell}(G)$ be $D = \{u'_i / u'_i \in V[M_{\ell}(G)]\}$. We shall prove this in the below cases.

Case 1. Let $G = P_n$. By the Theorem 2.2 and Theorem 4.1, which implies $\gamma_s[M_{\ell}(P_n)] \geq \gamma[M_{\ell}(P_n)]$.

Case 2. Assume that $G$ be a tree and $D$ be the dominating set of $M_{\ell}(G)$. By Observation 3.2, $\gamma_s[M_{\ell}(T)] = \gamma[M_{\ell}(T)]$ then $D$ itself is a $\gamma_s$-set. Hence the result follows.
Case 3. Let \( G = C_n \). By the Theorem 2.3 and Theorem 4.2, we can say that \( \gamma_s[M_c(C_n)] \geq \gamma[M_c(C_n)] \).

Case 4. Let \( G \) be any graph. By Theorem 2.2, Theorem 4.1, Observation 3.2, Theorem 2.3 and Theorem 4.2, it follows that \( \gamma_s[M_c(G)] \geq \gamma[M_c(G)] \).

It follows that from the above cases, \( \gamma_s[M_c(G)] \geq \gamma[M_c(G)] \).

**Theorem 4.4.** For any graph \( G(p, q) \), \( \gamma_s[M_c(G)] \geq \left[ \frac{p}{1+\Delta(G)} \right] \).

**Proof.** By Theorem 2.4,

\[
\gamma[M_c(G)] \geq \left[ \frac{p}{1+\Delta(G)} \right]
\]

By Theorem 4.3,

\[
\gamma_s[M_c(G)] \geq \gamma[M_c(G)]
\]

From equation (4.1) and (4.2),

\[
\gamma_s[M_c(G)] \geq \left[ \frac{p}{1+\Delta(G)} \right]
\]

**Theorem 4.5.** \( \gamma_s[M_c(G)] \geq \left[ \frac{\text{diam}(G)+1}{2} \right] \) for every graph \( G(p, q) \).

**Proof.** Let \( V(G) = \{v_1, v_2, \ldots, v_p\} \) such that \( \exists \ u, v \in V(G) \) and \( d(u, v) \) forms a diametral path in \( G \). Clearly, \( d(u, v) = \text{diam}(G) \). Consider the set \( D \) be the dominating set of \( M_c(G) \). If there are at least two components in \( V[M_c(G)] - D \) then \( D \) itself is the \( \gamma_s - \) set of \( M_c(G) \). If not, \( \exists \{e_j\} \in V[M_c(G)] - D \) having maximum degree such that \( V[M_c(G)] - D \cup \{e_j\} \) is disconnected. Clearly, \( D \cup \{e_j\} \) forms a \( \gamma_s - \) set of \( M_c(G) \). Therefore the diametral path contains at most \( \gamma_s[M_c(G)] - 1 \) edges connecting the neighbourhood of the vertices of \( D \cup \{e_j\} \). Hence \( \gamma_s[M_c(G)] + \gamma_s[M_c(G)] - 1 \geq \text{diam}(G) \) which gives \( \gamma_s[M_c(G)] \geq \left[ \frac{\text{diam}(G)+1}{2} \right] \).

**Theorem 4.6.** For any graph \( G(p, q) \), \( \gamma_s[M_c(G)] \geq \alpha_1(G) \).

**Proof.** Suppose the minimum set of edges in \( G \) be \( E_1 = \{e_1, e_2, e_3, \ldots, e_k, 1 \leq k \leq q\} \) such that \( |E_1| = \alpha_1(G) \). In \( M_c(G) \), let \( D \) be the dominating set. If at least two components are found in \( V[M_c(G)] - D \) then \( D \) is the \( \gamma_s - \) set of \( M_c(G) \). If not, \( \exists \{e_j\} \in V[M_c(G)] - D \) with a maximum degree such that \( V[M_c(G)] - D \cup \{e_j\} \) has more than one component. Evidently, \( D \cup \{e_j\} \) forms a \( \gamma_s - \) set of \( M_c(G) \). We can say that, \( \gamma_s[M_c(G)] \geq \alpha_1(G) \).

**Theorem 4.7.** Let \( G(p, q) \) be any graph, \( \gamma_s[M_c(G)] \leq q \).

**Proof.** Consider \( G \) be a any graph such that \( |E| = q \). Let \( D \) be a dominating set in \( M_c(G) \). If \( V[M_c(G)] - D \) is disconnected then \( D \) itself forms a \( \gamma_s - \) set of \( M_c(G) \). Otherwise, \( \exists \ e_j \in V[M_c(G)] - D \) having maximum degree such that \( V[M_c(G)] - (D \cup \{e_j\}) \) has more than one component. Evidently, \( D \cup \{e_j\} \) forms a \( \gamma_s - \) set of \( M_c(G) \). Clearly, \( |D| \leq |E| \). Therefore, \( \gamma_s[M_c(G)] \leq q \).

5. Conclusion

In this paper we established some domination results on edge semi-middle graphs. Many bounds on domination number of edge semi-middle graph are obtained.
REFERENCES