EXACT SOLUTIONS OF SYSTEM OF FOURTH-ORDER DIFFERENCE EQUATIONS

MESSAOUD BERKAL1,* AND RAFAF ABO-ZEID2

ABSTRACT. In this paper, we derive the solutions of system difference equations

\[ x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(a + bx_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(c + dy_{n-1}x_{n-3})}, \quad n \in \mathbb{N}_0, \]

where the parameters \( a, b, c, d \) are real numbers and the initial conditions \( x_{-i} \) and \( y_{-i} \) for \( i = 0, 1, 2, 3 \) are non zero real numbers.

1. INTRODUCTION

Nowadays, difference equations becomes a valuable tool in the modeling of many phenomena in various fields such as biology, epidemiology, physics, probability theory, etc. See for example [7, 27–29]. Accordingly, difference equations have attracted the attention of many researchers, see for example [1, 5, 6, 8, 12–18, 21–24, 26, 30, 31]. The following difference equations

\[ x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(1 + x_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(\pm 1 \pm y_{n-1}x_{n-3})}, \]

has been studied by Almatrafi et al. [4].

In addition, Halim et al. [20] obtained the solution of the following system difference equation

\[ x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_{n}(a + by_{n-1}y_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_{n}(a + bx_{n-1}y_{n-2})}. \]

In this paper, we generalize the solutions of the systems of nonlinear rational difference equations presented in [2], [3] and [4], which were established through a mere application of the induction principle.

2. MAIN RESULTS

In this section, we derive the admissible solutions of the system of difference equations given by

\[ x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(a + bx_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(c + dy_{n-1}x_{n-3})}, \]

where \( n \in \mathbb{N}_0 \) and the initial conditions \( x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1} \) and \( y_0 \) are arbitrary non zero real numbers.

The system (2.1) can be written as

\[ u_{n+1} = \frac{u_{n-1}}{a + bu_{n-1}}, \quad v_{n+1} = \frac{v_{n-1}}{c + dv_{n-1}}, \]

1DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF ALICANTE, SAN VICENTE DEL RASPEIG, 03690, SPAIN
2DEPARTMENT OF BASIC SCIENCE, THE HIGHER INSTITUTE FOR ENGINEERING & TECHNOLOGY, AL-OBOUR, CAIRO, EGYPT
E-mail addresses: mb299@gcloud.ua.es, abuzead73@yahoo.com.
Submitted on January 11, 2024.
2020 Mathematics Subject Classification. Primary 39A05, 39A10; Secondary 39A23.
Key words and phrases. Difference equations, System of difference equations, Solution, Forbidden set.
*Corresponding author.
by using the following change of variables

\((2.3)\)

\[
\begin{align*}
  u_n &= x_n y_{n-2}, \\
  v_n &= y_n x_{n-2}.
\end{align*}
\]

2.1. **Solutions of** \(x_{n+1} = \frac{x_{n-1}}{\alpha + \beta x_{n-1}}\). In this section, we derive the closed form of the solution of the equation

\((2.4)\)

\[
  x_{n+1} = \frac{x_{n-1}}{\alpha + \beta x_{n-1}}.
\]

Let us define

\((2.5)\)

\[
  x_{n}^{(j)} = x_{2n-j}, \quad j = 0, 1, \quad n \in \mathbb{N}_0.
\]

Using this notation, we can write (2.4) as

\((2.6)\)

\[
  x_{n}^{(j)} = \frac{x_{n}^{(j)}}{\alpha + \beta x_{n}^{(j)}}.
\]

Equation (2.6) can be reduced to

\((2.7)\)

\[
  w_{n+1} = \frac{(\alpha + 1)w_n - \alpha}{w_n},
\]

by using the change of variable

\((2.8)\)

\[
  x_{n}^{(j)} = \frac{1}{\beta} (w_n - \alpha), \quad j = 0, 1.
\]

Now, we consider the difference equation (2.7) where the initial value \(w_0\) is a non-zero real number. The equality

\((2.9)\)

\[
  w_n = \frac{z_n}{z_{n-1}}
\]

reduces the equation (2.7) to the following one,

\((2.10)\)

\[
  z_{n+1} - (\alpha + 1)z_n - \alpha z_{n-1} = 0, \quad n \in \mathbb{N}_0.
\]

**Case** \(\alpha \neq 1\).

**Lemma 2.1.** Consider the linear difference equation

\((2.11)\)

\[
  z_{n+1} - (\alpha + 1)z_n + \alpha z_{n-1} = 0, \quad n \in \mathbb{N}_0,
\]

with initial conditions \(z_{-1}, z_0 \in \mathbb{R}\). All solutions of equation (2.11) can be written in the form

\((2.12)\)

\[
  z_n = \frac{1}{1 - \alpha} \left( z_0 \left( 1 - \alpha^{(n+1)} \right) - \alpha z_{-1} \left( 1 - \alpha^n \right) \right).
\]

**Proof.** Equation (2.11) is an homogeneous linear second order difference equation with constant coefficients, where \(z_0\) and \(z_{-1}\) \(\in \mathbb{R}\). These type of equations are usually solved by using the characteristic roots \(\lambda_1 = \alpha\) and \(\lambda_2 = 1\) of the characteristic polynomial \(P(\lambda) = \lambda^2 - (1 + \alpha)\lambda + \alpha\), and the general solution is given by

\[
  z_n = c_1 + c_2 \alpha^{2n},
\]

where \(c_1\) and \(c_2\) are expressed in terms of the initial conditions \(z_0\) and \(z_{-1}\) as

\[
  c_1 = \frac{z_0 - z_{-1}\alpha}{1 - \alpha}, \quad c_2 = \frac{\alpha (z_{-1} - z_0)}{1 - \alpha}.
\]
Then, the general solution of equation (2.11) is
\[ z_n = \frac{1}{1 - \alpha} \left( z_0 \left( 1 - \alpha^{n+1} \right) - \alpha z_{-1} (1 - \alpha^n) \right). \]

\[ \square \]

**Case \( \alpha = 1 \).**

**Lemma 2.2.** Consider the linear difference equation
\[ z_{n+1} - 2z_n + z_{n-1} = 0, \quad n \in \mathbb{N}_0, \]
with initial conditions \( z_{-1}, z_0 \in \mathbb{R} \). All solutions of equation (2.13) can be written in the form
\[ z_n = z_0(n + 1) - z_{-1}n. \]

**Proof.** Equation (2.13), where \( z_0 \) and \( z_{-1} \in \mathbb{R} \), is usually solved by using the characteristic roots \( \lambda_1 = \lambda_2 = 1 \) of the characteristic polynomial \( P(\lambda) = (\lambda - 1)^2 \). Thus, its general solution can be written in the following form
\[ z_n = c_1 + c_2n. \]

Using the initial conditions \( z_0 \) and \( z_{-1} \) and after some calculations we get
\[ c_1 = z_0 \]
\[ c_2 = z_0 - z_{-1}. \]

Then, the general solution of equation (2.13) is
\[ z_n = z_0(n + 1) - z_{-1}n. \]

\[ \square \]

Using the above arguments, we can state the following theorem:

**Theorem 2.3.** Let \( \{w_n\}_{n \geq 0} \) be an admissible solution of (2.7). Then,
\[ w_n = \begin{cases} \frac{\alpha(1 - \alpha^n) - w_0(1 - \alpha^{n+1})}{\alpha(1 - \alpha^{n-1}) - w_0(1 - \alpha^n)}, & \text{if } \alpha \neq 1, \\ \frac{n - w_0(n + 1)}{(n - 1) - w_0n}, & \text{if } \alpha = 1. \end{cases} \]

Let
\[ x^{(j)}_n = \frac{1}{\beta} (w_n - \alpha). \]

If \( \alpha \neq 1 \), then
\[ x^{(j)}_n = \frac{(\alpha - 1)x^{(j)}_0}{\alpha^n(\alpha - 1) - \beta(1 - \alpha^n)x^{(j)}_0}. \]

By using
\[ \frac{\alpha^n - 1}{\alpha - 1} = \sum_{i=0}^{n-1} \alpha^i, \quad w_0 = \alpha + \beta x^{(j)}_0, \]
we get that the solution of the difference equation (2.6) is given by
\[ x^{(j)}_n = \frac{x^{(j)}_0}{\alpha^n - \beta \left( \sum_{i=0}^{n-1} \alpha^i \right) x^{(j)}_0}. \]
If $\alpha = 1$, then the solution of equation (2.6) is given by

$$x^{(j)}_n = \frac{x^{(j)}_0}{1 + \beta nx^{(j)}_0},$$

for each $j \in \{0, 1\}$.

We can state the following theorem by taking into account the results described above as well as (2.5).

**Theorem 2.4.** Let $\{x_n\}_{n \geq -1}$ be an admissible solution of (2.4). Then, for $n \geq 0$ the following statement are true:

- If $\alpha \neq 1$, then

  $$x_{2n} = \frac{x_0}{\alpha^n + \beta \left(\sum_{i=0}^{n-1} \alpha^i\right)x_0},$$

  $$x_{2n-1} = \frac{x_{-1}}{\alpha^n + \beta \left(\sum_{i=0}^{n-1} \alpha^i\right)x_{-1}}.$$

- If $\alpha = 1$, then

  $$x_{2n} = \frac{x_0}{1 + \beta nx_0},$$

  $$x_{2n-1} = \frac{x_{-1}}{1 + \beta nx_{-1}}.$$

Now let $\{u_n, v_n\}_{n \geq -1}$ be an admissible solution of (2.2). Then, for $n \geq 0$, we have the following:

- If $\alpha \neq 1$, then,

  $$u_{2n} = \frac{u_0}{a^n + b \left(\sum_{i=0}^{n-1} a^i\right)u_0},$$

  $$u_{2n-1} = \frac{u_{-1}}{a^n + b \left(\sum_{i=0}^{n-1} a^i\right)u_{-1}}$$

  $$v_{2n} = \frac{v_0}{c^n + d \left(\sum_{i=0}^{n-1} c^i\right)v_0},$$

  $$v_{2n-1} = \frac{v_{-1}}{c^n + d \left(\sum_{i=0}^{n-1} c^i\right)v_{-1}}.$$
If $\alpha = 1$, then

\[
\begin{align*}
    u_{2n} &= \frac{u_0}{1 + bnu_0}, \\
    u_{2n-1} &= \frac{u_{-1}}{1 + bnu_{-1}}, \\
    v_{2n} &= \frac{v_0}{1 + dnv_0}, \\
    v_{2n-1} &= \frac{v_{-1}}{1 + dnv_{-1}}.
\end{align*}
\]

Now let

\[
\begin{align*}
    x_n &= \frac{u_n}{y_{n-2}}, \\
    y_n &= \frac{v_n}{z_{n-2}}.
\end{align*}
\]

Using formula (2.17) and (2.18), and after some calculations, we have

\[
\begin{align*}
    x_{4n} &= \frac{u_{4n}}{v_{4n-2}} x_{4n-4}, \\
    y_{4n} &= \frac{v_{4n}}{u_{4n-2}} y_{4n-4}.
\end{align*}
\]

Also, we can obtain

\[
\begin{align*}
    x_{4n-1} &= \frac{u_{4n-1}}{v_{4n-3}} x_{4n-5}, \\
    y_{4n-1} &= \frac{v_{4n-1}}{u_{4n-3}} y_{4n-5},
\end{align*}
\]

for any $n \in \mathbb{N}$.

By multiplying the equalities (2.19), (2.20), (2.21) and (2.22) from 0 to $n - 1$, respectively, it follows that

\[
\begin{align*}
    x_{4n} &= x_0 \prod_{i=0}^{n-1} \left( \frac{u_{4i}}{v_{4i-2}} \right), \\
    y_{4n} &= y_0 \prod_{i=0}^{n-1} \left( \frac{v_{4i}}{u_{4i-2}} \right), \\
    x_{4n-1} &= x_{-1} \prod_{i=0}^{n-1} \left( \frac{u_{4i-1}}{v_{4i-3}} \right), \\
    y_{4n-1} &= y_{-1} \prod_{i=0}^{n-1} \left( \frac{v_{4i-1}}{u_{4i-3}} \right).
\end{align*}
\]

If we substitute equations (2.23), (2.24), (2.25) and (2.26) into (2.19) and (2.20), we get

\[
\begin{align*}
    x_{4n-2} &= \frac{v_{4n}}{y_{4n}} = \frac{v_{4n}}{y_0} \prod_{i=0}^{n-1} \left( \frac{u_{4i-2}}{v_{4i}} \right), \\
    y_{4n-2} &= \frac{u_{4n}}{x_{4n}} = \frac{u_{4n}}{x_0} \prod_{i=0}^{n-1} \left( \frac{v_{4i-2}}{u_{4i}} \right).
\end{align*}
\]
Similarly,

(2.29) \[ x_{4n-3} = \frac{v_{4n-1}}{y_{4n-1}} = \frac{v_{4n-1}}{y_{4n-1}} \prod_{i=0}^{n-1} \left( \frac{u_{4i-3}}{u_{4i-1}} \right) \]

and

(2.30) \[ y_{4n-3} = \frac{u_{4n-1}}{x_{4n-1}} = \prod_{i=0}^{n-1} \left( \frac{v_{4i-3}}{v_{4i-1}} \right). \]

Now, using the above arguments and taking into account that

\[ u_0 = x_0 y_{-2}, \quad v_0 = y_0 x_{-2}, \quad u_{-1} = x_{-1} y_{-3}, \quad v_{-1} = y_{-1} x_{-3}, \]

we have the following:

**Theorem 2.5.** Let \( \{x_n, y_n\}_{n \geq -1} \) be an admissible solution of system (2.1). Then, for \( n \in \mathbb{N}_0 \):

1. If \( a \neq 1 \) and \( c \neq 1 \), then the solution of system (2.1) is given by

   \[
x_{4n-3} = \frac{x_{n-1} y_{n-3} \prod_{i=0}^{n-1} \left( c^{2i} + dy_{-1} x_{-3} \sum_{r=0}^{2i-1} a^r \right)}{y_{n-1} x_{n-3} \prod_{i=0}^{n-1} \left( a^{2i} + bx_{-1} y_{-3} \sum_{r=0}^{2i-1} a^r \right)},
   \]

   \[
y_{4n-3} = \frac{y_{n-1} x_{n-3} \prod_{i=0}^{n-1} \left( a^{2i} + bx_{-1} y_{-3} \sum_{r=0}^{2i-1} a^r \right)}{x_{n-1} y_{n-3} \prod_{i=0}^{n-1} \left( c^{2i} + dy_{-1} x_{-3} \sum_{r=0}^{2i-1} a^r \right)}. 
   \]

2. If \( a \neq 1 \) and \( c = 1 \), then the solution of system (2.1) is given by

   \[
x_{4n-3} = \frac{x_{n-1} y_{n-3} \prod_{i=0}^{n-1} \left( 1 + 2dy_{-1} x_{-3} \right)}{y_{n-1} x_{n-3} \prod_{i=0}^{n-1} \left( a^{2i-1} + bx_{-1} y_{-3} \sum_{r=0}^{2i-1} a^r \right)},
   \]

   \[
y_{4n-3} = \frac{y_{n-1} x_{n-3} \prod_{i=0}^{n-1} \left( a^{2i-1} + bx_{-1} y_{-3} \sum_{r=0}^{2i-1} a^r \right)}{x_{n-1} y_{n-3} \prod_{i=0}^{n-1} \left( 1 + 2dy_{-1} x_{-3} \right)}. 
   \]
\[ x_{4n-2} = \frac{x_0^n y_0^n \prod_{i=0}^{n-1} (1 + 2diy_0x_{-2})}{y_0^n x_0^{n-1} (1 + 2dn_0y_0x_{-2}) \prod_{i=0}^{n-1} (a^{2i-1} + bx_0y_{-2} \sum_{r=0}^{2i-2} a^r) \prod_{r=0}^{2i-1} a^r}, \]
\[ x_{4n-1} = \frac{x_0^{n+1} y_0^n \prod_{i=0}^{n-1} (1 + d(2i-1)y_{-1}x_{-3})}{y_0^n x_0^n \prod_{i=0}^{n-1} (a^{2i} + bx_{-1}y_{-3} \sum_{r=0}^{2i-1} a^r) \prod_{r=0}^{2i-1} a^r}, \]
\[ x_{4n} = \frac{x_0^n y_0^n \prod_{i=0}^{n-1} (1 + d(2i-1)y_0x_{-2})}{y_0^n x_0^n \prod_{i=0}^{n-1} (a^{2i} + bx_0y_{-2} \sum_{r=0}^{2i-1} a^r) \prod_{r=0}^{2i-1} a^r}, \]

\[ y_{4n-3} = \frac{y_0^{n+1} x_0^n \prod_{i=0}^{n-1} (a^{2i} + bx_{-1}y_{-3} \sum_{r=0}^{2i-1} a^r) \prod_{r=0}^{2i-1} a^r}{x_0^n y_0^n \prod_{i=0}^{n-1} (a^{2n} + bx_{-1}y_{-3} \sum_{r=0}^{2n-1} a^r) \prod_{r=0}^{2n-1} a^r}, \]
\[ y_{4n-2} = \frac{y_0^n x_0^n \prod_{i=0}^{n-1} (a^{2i} + bx_0y_{-2} \sum_{r=0}^{2i-1} a^r) \prod_{r=0}^{2i-1} a^r}{x_0^n y_0^n \prod_{i=0}^{n-1} (a^{2n} + bx_0y_{-2} \sum_{r=0}^{2n-1} a^r) \prod_{r=0}^{2n-1} a^r}, \]
\[ y_{4n-1} = \frac{x_0^n y_0^n \prod_{i=0}^{n-1} (1 + 2diy_{-1}x_{-3})}{y_0^n x_0^n \prod_{i=0}^{n-1} (1 + 2diy_{-1}x_{-3})}, \]
\[ x_{4n} = \frac{x_0^n y_0^n \prod_{i=0}^{n-1} (1 + 2diy_0x_{-2})}{y_0^n x_0^n \prod_{i=0}^{n-1} (1 + 2diy_0x_{-2})} \]

3. If \( a = 1 \) and \( c \neq 1 \), then the solution of system (2.1) is given by
\[ x_{4n-3} = \frac{x_0^n y_0^n \prod_{i=0}^{n-1} (c^{2i} + dy_{-1}x_{-3} \sum_{r=0}^{2i-1} c^r)}{y_0^n x_0^n \prod_{i=0}^{n-1} (c^{2n} + dy_{-1}x_{-3} \sum_{r=0}^{2n-1} c^r) \prod_{r=0}^{2n-1} c^r}, \]
\[ x_{4n-2} = \frac{x_0^n y_0^n \prod_{i=0}^{n-1} (c^{2i} + dy_0x_{-2} \sum_{r=0}^{2i-1} c^r) \prod_{r=0}^{2i-1} c^r}{y_0^n x_0^n \prod_{i=0}^{n-1} (c^{2n} + dy_0x_{-2} \sum_{r=0}^{2n-1} c^r) \prod_{r=0}^{2n-1} c^r}, \]
\[ x_{4n-1} = \frac{x_0^n y_0^n \prod_{i=0}^{n-1} (1 + 2bix_{-1}y_{-3})}{y_0^n x_0^n \prod_{i=0}^{n-1} (1 + 2bix_{-1}y_{-3})}, \]
\[ x_{4n} = \frac{x_0^n y_0^n \prod_{i=0}^{n-1} (1 + 2bix_0y_{-2})}{y_0^n x_0^n \prod_{i=0}^{n-1} (1 + 2bix_0y_{-2})}, \]

\[ x_{4n-3} = \frac{x_0^n y_0^n \prod_{i=0}^{n-1} (1 + 2bix_{-1}y_{-3})}{y_0^n x_0^n \prod_{i=0}^{n-1} (c^{2i-1} + dy_{-1}x_{-3} \sum_{r=0}^{2i-2} c^r) \prod_{r=0}^{2i-2} c^r}, \]
\[ y_{4n-2} = \frac{y_0^n x_0^n \prod_{i=0}^{n-1} (1 + 2bix_0y_{-2})}{x_0^n y_0^n \prod_{i=0}^{n-1} (1 + 2bix_0y_{-2})} \]
In this section, we apply the previous results in order to show how some closed-form formulas for the solutions to the systems in (2.1), which were presented in [2] and [4] are obtained:

4. If \( a = 1 \) and \( c = 1 \), then the solution of system (2.1) is given by

\[
\begin{align*}
\frac{x_{4n-1}}{x_{4n-2}} &= \frac{y_{n-1}^{-1} x_{n-3}^{-1} \prod_{i=0}^{n-1} (1 + b(2i - 1)x_{-1}y_{-3})}{y_0^{-1} x_{-3}^{-1} \prod_{i=0}^{n-1} (1 + b(2i - 1)x_{-1}y_{-3})} ,
\end{align*}
\]

and

\[
\begin{align*}
\frac{x_{4n-1}}{x_{4n-2}} &= \frac{y_{n-1}^{-1} x_{n-3}^{-1} \prod_{i=0}^{n-1} (1 + b(2i - 1)x_{-1}y_{-3})}{y_0^{-1} x_{-3}^{-1} \prod_{i=0}^{n-1} (1 + b(2i - 1)x_{-1}y_{-3})} ,
\end{align*}
\]

3. SOME APPLICATIONS

In this section, we apply the previous results in order to show how some closed-form formulas for the solutions to the systems in (2.1), which were presented in [2] and [4] are obtained:

- When \( a = b = c = d = 1 \) in system (2.1), we have that

\[
\begin{align*}
\frac{x_{4n-3}}{x_{4n-2}} &= \frac{y_{n-1}^{-1} x_{n-3}^{-1} \prod_{i=0}^{n-1} (1 + b(2i - 1)x_{-1}y_{-3})}{y_0^{-1} x_{-3}^{-1} \prod_{i=0}^{n-1} (1 + b(2i - 1)x_{-1}y_{-3})} ,
\end{align*}
\]
When this agree with what was obtained is Theorem 2 in [4].

\[ x_{n+1} - y_{n+1} \sum_{i=0}^{n-1} (1 + (2i - 2)y_{i+1)x_{i+3}) = x_0 - y_0 \sum_{i=0}^{n-1} (1 + (2i - 1)y_{i+2}) \]

\[ y_{n+1} - y_{n+1} \sum_{i=0}^{n-1} (1 + b(2i - 1)x_{i+1}y_{i+3}) = y_0 - y_0 \sum_{i=0}^{n-1} (1 + 2b_{i+1}x_{i+2}) \]

\[ 4n - 3, y_{n+1} = \frac{y_0 - y_0 \sum_{i=0}^{n-1} (1 + 2b_{i+1}x_{i+2})}{y_0 x_{i+2} \sum_{i=0}^{n-1} (1 + 2b_{i+1}x_{i+2})}, \]

\[ n(n - 1) \prod_{i=0}^{n-1} (1 + (2i - 1)x_{i+1}y_{i+3}) = \frac{n(n - 1) \prod_{i=0}^{n-1} (1 + 2b_{i+1}x_{i+2})}{y_0 x_{i+2} \sum_{i=0}^{n-1} (1 + 2b_{i+1}x_{i+2})} \]

this agree with what was obtained is theorem 1 in [2].

- When $a = b = 1$ and $c = d = -1$ in system (2.1), we have that

\[ x_{n+1} = \frac{x_0 - x_0 \sum_{i=0}^{n-1} (1 + 2x_{i+1}y_{i+3})}{x_0 x_{i+2} \sum_{i=0}^{n-1} (1 + 2x_{i+1}y_{i+3})}, \]

\[ y_{n+1} = \frac{y_0 - y_0 \sum_{i=0}^{n-1} (1 + 2y_{i+1}x_{i+2})}{y_0 x_{i+2} \sum_{i=0}^{n-1} (1 + 2y_{i+1}x_{i+2})} \]

\[ 4n - 3, y_{n+1} = \frac{y_0 - y_0 \sum_{i=0}^{n-1} (1 + 2y_{i+1}x_{i+2})}{y_0 x_{i+2} \sum_{i=0}^{n-1} (1 + 2y_{i+1}x_{i+2})}, \]

\[ n(n - 1) \prod_{i=0}^{n-1} (1 + (2i - 1)x_{i+1}y_{i+3}) = \frac{n(n - 1) \prod_{i=0}^{n-1} (1 + 2y_{i+1}x_{i+2})}{y_0 x_{i+2} \sum_{i=0}^{n-1} (1 + 2y_{i+1}x_{i+2})} \]

this agree with what was obtained is Theorem 2 in [4].

- When $a = b = 1$ and $c = d = -1$ in system (2.1), we have that

\[ x_{n+1} = \frac{x_0 - x_0 \sum_{i=0}^{n-1} (1 + 2x_{i+1}y_{i+3})}{x_0 x_{i+2} \sum_{i=0}^{n-1} (1 + 2x_{i+1}y_{i+3})}, \]

\[ y_{n+1} = \frac{y_0 - y_0 \sum_{i=0}^{n-1} (1 + 2y_{i+1}x_{i+2})}{y_0 x_{i+2} \sum_{i=0}^{n-1} (1 + 2y_{i+1}x_{i+2})} \]
and

\[
y_{4n-3} = \frac{y_n^ny_3^{n-1} \prod_{i=0}^{n-1} (1 + 2ix_{1}y_{-3})}{x_n^ny_3^{n-1} (1 + 2nx_{-1}y_{-3})} (1 + y_{-1}x_{-3})^n, \\
y_{4n-2} = \frac{y_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + 2ix_{0}y_{-2})}{x_n^ny_2^{n-1} (1 + 2nx_{0}y_{-2})} (1 + dy_{0}x_{-2})^n, \\
y_{4n-1} = \frac{y_n^ny_3^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)x_{-1}y_{-3})}{x_n^ny_3^{n-1}}, \\
y_{4n} = \frac{y_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)x_{0}y_{-2})}{x_n^ny_2^{n-1}}.
\]

this agree with what was obtained is theorem 3 in [4].

- When \( a = b = c = 1 \) and \( d = -1 \) in system (2.1), we have that

\[
x_{4n-3} = \frac{x_n^ny_3^{n-1} \prod_{i=0}^{n-1} (1 + 2iy_{1}x_{-3})}{y_n^nx_3^{n-1} (1 + 2ny_{1}x_{-3})} (1 + (2i-1)x_{-1}y_{-3}) \\
x_{4n-2} = \frac{x_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + 2iy_{x_{0}y_{-2}})}{y_n^ny_2^{n-1} (1 + 2ny_{0}x_{-2})} (1 + (2i-1)x_{0}y_{-2}) \\
x_{4n-1} = \frac{x_n^ny_3^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)y_{1}x_{-3})}{y_0^nx_3^{n-1}}, \\
x_{4n} = \frac{x_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)y_{x_{0}y_{-2}})}{y_0^ny_2^{n-1}}.
\]

and

\[
y_{4n-3} = \frac{y_n^ny_3^{n-1} \prod_{i=0}^{n-1} (1 + 2ix_{1}y_{-3})}{x_n^ny_3^{n-1} (1 + 2nx_{-1}y_{-3})} (1 + x_{-1}y_{-3})^n, \\
y_{4n-2} = \frac{y_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + 2ix_{0}y_{-2})}{x_n^ny_2^{n-1} (1 + 2nx_{0}y_{-2})} (1 + x_{0}y_{-2})^n, \\
y_{4n-1} = \frac{y_n^ny_3^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)x_{-1}y_{-3})}{y_0^nx_3^{n-1}}, \\
y_{4n} = \frac{y_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)x_{0}y_{-2})}{y_0^ny_2^{n-1}}.
\]

this agree with what was obtained is theorem 4 in [4].

- When \( a = b = -1 \) and \( c = d = 1 \) in system (2.1), we have that

\[
x_{4n-3} = \frac{(-1)^nx_n^ny_3^{n-1} \prod_{i=0}^{n-1} (1 + 2iy_{1}x_{-3})}{y_n^nx_3^{n-1} (1 + 2ny_{1}x_{-3})} (1 + x_{-1}y_{-3})^n, \\
x_{4n-2} = \frac{(-1)^nx_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + 2iy_{x_{0}y_{-2}})}{y_n^ny_2^{n-1} (1 + 2ny_{0}x_{-2})} (1 + x_{0}y_{-2})^n, \\
x_{4n-1} = \frac{x_n^ny_3^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)x_{-1}y_{-3})}{y_0^nx_3^{n-1}}, \\
x_{4n} = \frac{x_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)x_{0}y_{-2})}{y_0^ny_2^{n-1}}.
\]

and

\[
y_{4n-3} = \frac{y_n^ny_3^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)x_{-1}y_{-3})}{x_n^ny_3^{n-1}}, \\
y_{4n-2} = \frac{y_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + (2i-1)x_{0}y_{-2})}{x_n^ny_2^{n-1}}, \\
y_{4n-1} = \frac{(-1)^ny_n^nx_3^{n-1} \prod_{i=0}^{n-1} (1 + x_{-1}y_{-3})^n}{y_0^nx_3^{n-1}}, \\
y_{4n} = \frac{x_n^ny_2^{n-1} \prod_{i=0}^{n-1} (1 + x_{0}y_{-2})^n}{y_0^ny_2^{n-1}}.
\]

this agree with what was obtained is 1 in [2].
• When \( a = b = -1 \) and \( c = d = -1 \) in system (2.1), we have that

\[
x_{4n-3} = \frac{(1)^{n}x_{1}^{n}y_{1}^{n-3}}{y_{1}^{n}x_{1}^{n-3} (1 + x_{1}y_{1}^{n-3})^{n}},
\]

\[
x_{4n-2} = \frac{(1)^{n}y_{1}^{n}x_{1}^{n-2}}{y_{1}^{n}x_{1}^{n-2} (1 + x_{0}y_{1})^{n}},
\]

\[
y_{4n-3} = \frac{(1)^{n}y_{1}^{n}x_{1}^{n-3}}{y_{1}^{n}x_{1}^{n-3} (1 + x_{1}y_{1}^{n-3})^{n}},
\]

\[
y_{4n-2} = \frac{(1)^{n}y_{1}^{n}x_{1}^{n-2}}{y_{1}^{n}x_{1}^{n-2} (1 + x_{0}y_{1})^{n}},
\]

this agree with what was obtained is theorem 2 in [2].

4. Conclusions

In this study, we mainly obtained solutions of the system of rational difference equations system.

\[x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(a + bx_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(c + dy_{n-1}x_{n-2})}, \quad n \in \mathbb{N},\]

where the parameters \( a, b, c, d \) are real numbers and the initial conditions \( x_{i} \) and \( y_{i} \) for \( i = 0, 1, 2, 3 \), are non zero real numbers. Our results generalized the results obtained in [2] and [4].

References


