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# FUNCTIONAL $k$-GENERALIZED $\psi$-HILFER FRACTIONAL DIFFERENTIAL EQUATIONS IN $b$-METRIC SPACES

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ABSTRACT. This paper deals with some existence results for a class of $k$-generalized $\psi$-Hilfer implicit fractional differential equations in $b$-metric spaces. The results are based on the $\alpha - \phi$-Geraghty type contraction and the fixed point theory. We illustrate our results by an example in the last section.

## 1. INTRODUCTION

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences. Considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential and integral equations; see the publications [1,3,4,15,18,21,22,25–31].

The notion of $b$-metric was proposed by Czerwik [11,12]. Following these initial papers, the existence fixed point for the various classes of operators in the setting of $b$-metric spaces have been investigated extensively; see [2,9,10,13,23], and the related references therein.

In [19], the authors considered the following conformable impulsive problem:

$$\begin{cases} \mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta) = \aleph\left(\zeta, \chi_\zeta, \mathcal{T}^{\vartheta}_{\jmath}\chi(\zeta)\right), \quad \zeta \in \Omega_\jmath; \jmath = 0, 1, \ldots, \beta, \\ \Delta\chi|_{\zeta=\zeta_\jmath} = \Upsilon_\jmath(\chi_{\zeta_\jmath^-}), \quad \jmath = 1, 2, \ldots, \beta, \\ \chi(\zeta) = \mu(\zeta), \quad \zeta \in (-\infty, \varkappa], \end{cases}$$

where $0 \leq \varkappa = \zeta_0 < \zeta_1 < \cdots < \zeta_\beta < \zeta_{\beta+1} = \bar{\varkappa} < \infty$, $\mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta)$ is the conformable fractional derivative of order $0 < \vartheta < 1$, $\aleph : \Omega \times \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\Omega := [\varkappa, \bar{\varkappa}]$, $\Omega_0 := [\varkappa, \zeta_1]$,

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$\Omega_\jmath := (\zeta_\jmath, \zeta_{\jmath+1}]; \jmath = 1, 2, \ldots, \beta$, $\mu : (-\infty, \varkappa] \to \mathbb{R}$ and $\Upsilon_\jmath : \mathcal{Q} \to \mathbb{R}$ are given continuous functions, and $\mathcal{Q}$ is called a phase space.

In [20] the authors used the $\alpha - \phi$-Geraghty type contraction and the fixed point theory to investigate the following terminal value problem for implicit Katugampola fractional differential equation in b-metric spaces:

$$\begin{cases} ({}^\rho D^r_{0^+}\vartheta)(\tau) = \varkappa(\tau, \vartheta(\tau), ({}^\rho D^r_{0^+}\vartheta)(\tau)); \ \tau \in I := [0, T], \\ \vartheta(T) = \vartheta_T \in \mathbb{R}, \end{cases}$$

where ${}^\rho D^r_{0^+}$ is the Katugampola fractional derivative of order $r \in (0, 1]$.

In this paper, we discuss the existence of solutions for the following more general class of initial value problems of $k$-generalized $\psi$-Hilfer implicit fractional differential equations

$$\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\wp\right)(t) = f\left(t, \wp(t), \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\wp\right)(t)\right), \ \ t \in (a, b], \tag{1.1}$$

$$\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}\wp\right)(a^+) = \wp_0, \tag{1.2}$$

where ${}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}, \mathcal{J}^{k(1-\xi),k;\psi}_{a+}$ are the $k$-generalize $\psi$-Hilfer fractional derivative of order $\vartheta \in (0, 1)$ and type $r \in [0, 1]$, and $k$-generalize $\psi$-fractional integral of order $k(1 - \xi)$ defined in [24] respectively, where $\xi = \frac{1}{k}(r(k - \vartheta) + \vartheta)$, $x_0 \in \mathbb{R}$, $k > 0$ and $f \in C([a, b] \times \mathbb{R}^2, \mathbb{R})$.

## 2. Preliminaries

Let $0 < a < b < \infty$, $I = [a, b]$, $\vartheta \in (0, 1)$, $r \in [0, 1]$, $k > 0$ and $\xi = \frac{1}{k}(r(k - \vartheta) + \vartheta)$. By $C(I, \mathbb{R})$ we denote the Banach space of all continuous functions from $I$ into $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in I\}.$$

$AC^n(I, \mathbb{R})$, $C^n(I, \mathbb{R})$ are the spaces of continuous functions, $n$-times absolutely continuous and $n$-times continuously differentiable functions on $I$, respectively.
Consider the weighted Banach space

$$C_{\xi,k;\psi}(I) = \left\{\wp : (a, b] \to \mathbb{R} : t \to (\psi(t) - \psi(a))^{1-\xi}\wp(t) \in C(I, \mathbb{R})\right\},$$

with the norm

$$\|\wp\|_{C_{\xi,k;\psi}} = \sup_{t\in I}\left|(\psi(t) - \psi(a))^{1-\xi}\wp(t)\right|,$$

and

$$C^n_{\xi,k;\psi}(I) = \left\{\wp \in C^{n-1}(I) : \wp^{(n)} \in C_{\xi,k;\psi}(I)\right\}, n \in \mathbb{N},$$
$$C^0_{\xi,k;\psi}(I) = C_{\xi,k;\psi}(I),$$

with the norm

$$\|\wp\|_{C^n_{\xi,k;\psi}} = \sum_{i=0}^{n-1}\|\wp^{(i)}\|_\infty + \|\wp^{(n)}\|_{C_{\xi,k;\psi}}.$$

The weighted space $C^{\vartheta,r}_{\xi,k;\psi}(I)$ is defined by

$$C^{\vartheta,r}_{\xi,k;\psi}(I) = \left\{\wp \in C_{\xi,k;\psi}(I), \ {}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\wp \in C_{\xi,k;\psi}(I)\right\}.$$

Consider the space $X_{\psi}^{p}(a,b),\ (c\in\mathbb{R},\ 1\leq p\leq\infty)$ of those real-valued Lebesgue measurable functions $g$ on $[a,b]$ for which $\|g\|_{X_{\psi}^{p}}<\infty$, where the norm is defined by

$$\|g\|_{X_{\psi}^{p}}=\left(\int_{a}^{b}\psi'(t)|g(t)|^{p}dt\right)^{\frac{1}{p}},$$

where $\psi$ is an increasing and positive function on $[a,b]$ such that $\psi'$ is continuous on $[a,b]$ with $\psi(0)=0$. In particular, when $\psi(\wp)=\wp$, the space $X_{\psi}^{p}(a,b)$ coincides with the $L_{p}(a,b)$ space. Recently, in [14], Diaz and Pariguan have defined new functions called $k$-gamma and $k$-beta functions given by

$$\Gamma_{k}(\alpha)=\int_{0}^{\infty}t^{\alpha-1}e^{-\frac{t^{k}}{k}}dt,\alpha>0,$$

and

$$B_{k}(\alpha,\beta)=\frac{1}{k}\int_{0}^{1}t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}dt.$$

When $k\to1$ then $\Gamma(\alpha)=\Gamma_{k}(\alpha)$, we have also some useful following relations

$$\Gamma_{k}(\alpha)=k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right),$$

$$\Gamma_{k}(\alpha+k)=\alpha\Gamma_{k}(\alpha)$$

$$\Gamma_{k}(k)=\Gamma(1)=1.$$

$$B_{k}(\alpha,\beta)=\frac{1}{k}B\left(\frac{\alpha}{k},\frac{\beta}{k}\right)$$

$$B_{k}(\alpha,\beta)=\frac{\Gamma_{k}(\alpha)\Gamma_{k}(\beta)}{\Gamma_{k}(\alpha+\beta)}.$$

The Mittag-Leffler function can also be refined into the $k$-Mittag-Leffler function defined as follows

$$E_{k}^{\alpha,\beta}(x)=\sum_{i=0}^{\infty}\frac{x^{i}}{\Gamma_{k}(\alpha i+\beta)},\alpha,\beta>0.$$

**Definition 2.1.** ( [24]) ($k$-Generalized $\psi$-fractional Integral) Let $g\in X_{\psi}^{p}(a,b)$ and $[a,b]$ be a finite or infinite interval on the real axis $\mathbb{R}=(-\infty,\infty)$, $\psi(t)>0$ be an increasing function on $(a,b]$ and $\psi'(t)>0$ be continuous on $(a,b)$ and $\vartheta>0$. The generalized $k$-fractional integral operators of a function $f$ (left-sided and right-sided) of order $\vartheta$ are defined by

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}g(t)=\frac{1}{k\Gamma_{k}(\vartheta)}\int_{a}^{t}\frac{\psi'(s)g(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}},$$
$$\mathcal{J}_{b-}^{\vartheta,k;\psi}g(t)=\frac{1}{k\Gamma_{k}(\vartheta)}\int_{t}^{b}\frac{\psi'(s)g(s)ds}{(\psi(s)-\psi(t))^{1-\frac{\vartheta}{k}}},$$

with $k>0$.

**Definition 2.2.** ($k$-Generalized $\psi$-Hilfer Derivative) Let $n-1<\vartheta\leq n$ with $n\in\mathbb{N}$, $I=[a,b]$ an interval such that $-\infty\leq a<b\leq\infty$ and $g,\psi\in C^{n}([a,b],\mathbb{R})$ two functions such that $\psi$ is increasing and $\psi'(t)\neq0$, for all $t\in I$. The $k$-generalized $\psi$-Hilfer fractional derivatives (left-sided and right-sided) ${}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}(\cdot)$ and ${}_{k}^{H}\mathcal{D}_{b-}^{\vartheta,r;\psi}(\cdot)$ of a function $g$ of order $\vartheta$ and type $0\leq r\leq1$, with $k>0$ are defined by

$$\begin{aligned}{}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}g\left(t\right)&=\left(\mathcal{J}_{a+}^{r(kn-\vartheta),k;\psi}\left(\frac{1}{\psi'\left(t\right)}\frac{d}{dt}\right)^{n}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}g\right)\right)(t)\\&=\left(\mathcal{J}_{a+}^{r(kn-\vartheta),k;\psi}\delta_{\psi}^{n}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}g\right)\right)(t)\end{aligned}$$

and

$$
\begin{aligned}
{}_{k}^{H}\mathcal{D}_{b-}^{\vartheta,r;\psi}g\left(t\right) &= \left(\mathcal{J}_{b-}^{r(kn-\vartheta),k;\psi}\left(-\frac{1}{\psi'\left(t\right)}\frac{d}{dt}\right)^{n}\left(k^{n}\mathcal{J}_{b-}^{(1-r)(kn-\vartheta),k;\psi}g\right)\right)(t)\\
&= \left(\mathcal{J}_{b-}^{r(kn-\vartheta),k;\psi}(-1)^{n}\delta_{\psi}^{n}\left(k^{n}\mathcal{J}_{b-}^{(1-r)(kn-\vartheta),k;\psi}g\right)\right)(t),
\end{aligned}
$$

where $\delta_{\psi}^{n}=\left(\frac{1}{\psi'\left(t\right)}\frac{d}{dt}\right)^{n}$.

**Lemma 2.3.** *Let $\xi=\dfrac{r(k-\vartheta)+\vartheta}{k}$. By a solution of the problem (1.1)-(1.2) we mean a function $\wp\in C_{\xi,k;\psi}(I)$ that satisfies*

$$
\wp(t)=\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_{k}(k\xi)}\wp_{0}+\frac{1}{k\Gamma_{k}(\vartheta)}\int_{a}^{t}\frac{\psi'(s)g(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}},
$$

*where $0<\vartheta<1$, $0\leq r\leq 1$ and $k>0$ and $g(t)=f(t,\wp(t),g(t))$.*

**Definition 2.4.** [5,6] Let $c\geq 1$ and $M$ be a set. A distance function $d:M\times M\to\mathbb{R}_{+}^{*}$ is called $b$-metric if for all $\mu,\nu,\xi\in M$, the following are fulfilled:

- $d(\mu,\nu)=0$ if and only if $\mu=\nu$;
- $d(\mu,\nu)=d(\nu,\mu)$;
- $d(\mu,\xi)\leq c[d(\mu,\nu)+d(\nu,\xi)]$.

The tripled $(M,d,c)$ is called a $b$-metric space.

**Example 2.5.** [5,6] Let $d:C(I)\times C(I)\to\mathbb{R}_{+}^{*}$ be defined by

$$
d(\wp,\Im)=\|(\wp-\Im)^{2}\|_{\infty}:=\sup_{t\in I}\|\wp(t)-\Im(t)\|^{2};\ for\ all\ \wp,\Im\in C(I).
$$

It is clear that $d$ is a $b$-metric with $c=2$.

**Example 2.6.** [5,6] Let $X=[0,1]$ and $d:X\times X\to\mathbb{R}_{+}^{*}$ be defined by

$$
d(x,y)=|x^{2}-y^{2}|;\ for\ all\ x,y\in X.
$$

It is clear that $d$ is not a metric, but it is easy to see that $d$ is a $b$-metric space with $r\geq 2$.

Let $\Phi$ be the set of all increasing and continuous function $\phi:\mathbb{R}_{+}^{*}\to\mathbb{R}_{+}^{*}$ satisfying the property: $\phi(c\mu)\leq c\phi(\mu)\leq c\mu$, for $c>1$ and $\phi(0)=0$. We denote by $\mathcal{F}$ the family of all nondecreasing functions $\lambda:\mathbb{R}_{+}^{*}\to[0,\frac{1}{c^{2}})$ for some $c\geq 1$.

**Definition 2.7.** [5,6] For a b-metric space $(M,d,c)$, an operator $T:M\to M$ is called a generalized $\alpha-\phi-$Geraghty contraction type mapping whenever there exists $\alpha:M\times M\to\mathbb{R}_{+}^{*}$, and some $L\geq 0$ such that for

$$
D(x,y)=\max\left\{d(x,y),d(x,T(x)),d(y,T(y)),\frac{d(x,T(y))+d(y,T(x))}{2s}\right\},
$$

and

$$
N(x,y)=\min\{d(x,y),d(x,T(x)),d(y,T(y))\},
$$

we have

$$
\alpha(\mu,\nu)\phi(c^{3}d(T(\mu),T(\nu)))\leq\lambda(\phi(D(\mu,\nu))\phi(D(\mu,\nu))+L\psi(N(\mu,\nu))); \tag{2.1}
$$

for all $\mu,\nu\in M$, where $\lambda\in\mathcal{F},\ \phi\,\psi\in\Phi$.

*Remark* 2.8. In the case when $L = 0$ in Definition 2.7, and the fact that

$$d(x, y) \leq D(x, y); \text{ for all } x, y \in M,$$

the inequality (2.1) becomes

$$\alpha(\mu, \nu)\phi(c^3 d(T(\mu), T(\nu)) \leq \lambda(\phi(d(\mu, \nu))\phi(d(\mu, \nu)). \tag{2.2}$$

**Definition 2.9.** [5,6] Let $M$ be a non empty set, $T : M \to M$, and $\alpha : M \times M \to \mathbb{R}_+^*$ be a given mappings. We say that $T$ is $\alpha-$admissible if for all $\mu, \nu \in M$, we have

$$\alpha(\mu, \nu) \geq 1 \Rightarrow \alpha(T(\mu), T(\nu)) \geq 1.$$

**Definition 2.10.** [5,6] Let $(M, d)$ be a $b$-metric space and let $\alpha : M \times M \to \mathbb{R}_+^*$ be a function. $M$ is said to be $\alpha-$regular if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $M$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_n$ with $\alpha(x_{n(k)}, x) \geq 1$ for all $k$.

The following fixed point theorem plays a key role in the proof of our main results.

**Theorem 2.11.** *[5,6] Let $(M, d)$ be a complete b-metric space and $T : M \to M$ be a generalized $\alpha - \phi-$Geraghty contraction type mapping such that*

- *(i) $T$ is $\alpha-$admissible;*
- *(ii) there exists $\mu_0 \in M$ such that $\alpha(\mu_0, T(\mu_0)) \geq 1$;*
- *(iii) either $T$ is continuous or $M$ is $\alpha-$regular.*

*Then $T$ has a fixed point. Moreover, if*

- *(iv) for all fixed points $\mu, \nu$ of $T$, either $\alpha(\mu, \nu) \geq 1$ or $\alpha(\nu, \mu) \geq 1$,*

*then $T$ has a unique fixed point.*

## 3. Main Results

Let $(C_{\xi,k;\psi}(I), d, 2)$ be the complete $b$-metric space with $c = 2$, such that $d : C_{\xi,k;\psi}(I) \times C_{\xi,k;\psi}(I) \to \mathbb{R}_+^*$ is given by:

$$d(\wp, \Im) = \|(\wp - \Im)^2\|_C := \sup_{t\in I} t^{\rho(1-r)}|\wp(t) - \Im(t)|^2.$$

Then $(C_{\xi,k;\psi}(I), d, 2)$ is a $b$-metric space.

In this section, we are concerned with the existence results of the problem (1.1)-(1.2).

**Definition 3.1.** Let $\xi = \dfrac{r(k-\vartheta)+\vartheta}{k}$. By a solution of the problem (1.1)-(1.2) we mean a function $\wp \in C_{\xi,k;\psi}(I)$ that satisfies

$$\wp(t) = \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}\wp_0 + \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\psi'(s)g(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}},$$

where $0 < \vartheta < 1$, $0 \leq r \leq 1$ and $k > 0$ and $g(t) = f(t, \wp(t), g(t))$.

The following hypotheses will be used in the sequel.

$(H_1)$ There exist $\phi \in \Phi$, $p : C(I) \times C(I) \to (0, \infty)$ and $q : I \to (0, 1)$ such that for each $\wp, \Im, \wp_1, \Im_1 \in C_{\xi,k;\psi}(I)$, and $t \in I$

$$|f(t, \wp, \Im) - f(t, \wp_1, \Im_1)| \leq p(\wp, \Im)|\wp - \wp_1| + q(t)|\Im - \Im_1|,$$

with

$$\left\|\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}\right\|_C^2+$$

$$\left\|\frac{1}{k\Gamma_k(\vartheta)}\int_a^t\frac{\psi'(s)p(\wp,\Im)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}(1-q*)}\right\|_C^2\leq\phi(\|(\wp-\Im)^2\|_C),$$

where $g,h\in C^1_{\xi,k;\psi}(I)$

$(H_2)$ There exist $\mu_0\in C_{\xi,k;\psi}(I)$ and a function $\theta:C_{\xi,k;\psi}(I)\times C_{\xi,k;\psi}(I)\to\mathbb{R}$, such that

$$\theta\left(\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}\wp_0+\frac{1}{k\Gamma_k(\vartheta)}\int_a^t\frac{\psi'(s)g(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}}\right)\geq 0,$$

where $g\in C^1_{\xi,k;\psi}(I)$, with $g(t)=f(t,\mu_0(t),g(t))$.

$(H_3)$ For each $t\in I$, and $u,v\in C_{\xi,k;\psi}(I)$, we have:

$$\theta(\wp(t),\Im(t))\geq 0$$

implies

$$\theta\left(\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}\wp_0+\frac{1}{k\Gamma_k(\vartheta)}\int_a^t\frac{\psi'(s)g(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}},\right.$$
$$\left.\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}\wp_0+\frac{1}{k\Gamma_k(\vartheta)}\int_a^t\frac{\psi'(s)h(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}}\right)\geq 0,$$

where $g,h\in C^1_{\xi,k;\psi}(I)$, with $g(t)=f(t,\wp(t),g(t))$ and $h(t)=f(t,\Im(t),h(t))$.

$(H_4)$ If $\wp_{n n\in N}\subset C(I)$ with $\wp_n\to u$ and $\theta(\wp_n,\wp_{n+1})\geq$, then

$$\theta(\wp_n,\wp)\geq 1.$$

**Theorem 3.2.** *Assume that hypotheses $(H_1)-(H_4)$ hold. Then the problem (1.1)-(1.2) has a least one solution defined on $I$.*

**Proof.** Consider the operator $N:C_{\xi,k;\psi}(I)\to C_{\xi,k;\psi}(I)$ defined by

$$(N\wp)(t)=\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}\wp_0+\frac{1}{k\Gamma_k(\vartheta)}\int_a^t\frac{\psi'(s)g(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}},$$

where $g\in C(I)$, with $g(t)=f(t,u(t),g(t))$.

By using Lemma 2.3, it is clear that the fixed points of the operator $N$ are solutions of (1.1)-(1.2).

Let $\alpha:C_{\xi,k;\psi}(I)\times C_{\xi,k;\psi}(I)\to(0,\infty)$ be the function defined by:

$$\begin{cases}\alpha(\wp,\Im)=1; & if\ \theta(\wp(t),\Im(t))\geq 0,\ t\in I,\\ \alpha(\wp,\Im)=0; & elese.\end{cases}$$

First, we prove that $N$ is a generalized $\alpha$-$\phi$-Geraghty operator:

For any $\wp,\Im\in C(I)$ and each $t\in I$, we have

$$\begin{aligned}|t^{\rho(1-r)}(N\wp)(t)-t^{\rho(1-r)}(N\Im)(t)| &\leq \tfrac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}|\wp_0-\Im_0|\\ &+\frac{1}{k\Gamma_k(\vartheta)}\int_a^t\frac{\psi'(s)|g(s)-h(s)|ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}}\end{aligned}$$

where $g,h\in C^1_{\xi,k;\psi}(I)$, with

$$g(t)=f(t,\wp(t),g(t)),$$

and

$$h(t)=f(t,\Im(t),h(t)).$$

From $(H_1)$ we have

$$\begin{aligned}|g(t)-h(t)| &= |f(t,\wp(t),g(t))-f(t,\Im(t),h(t))| \\ &\leq p(\wp,\Im)|\wp(t)-\Im(t)|+q(t)|g(t)-h(t)| \\ &\leq p(\wp,\Im)|\wp(t)-\Im(t)|^2)^{\frac{1}{2}}+q(t)|g(t)-h(t)|.\end{aligned}$$

Thus,

$$|g(t)-h(t)| \leq \frac{p(\wp,\Im)}{1-q*}\|(\wp-\Im)^2\|_C^{\frac{1}{2}},$$

where $q* = \sup_{t\in I}|q(t)|$.
Next, we have

$$\begin{aligned}(N\wp)(t)-(N\Im)(t) &\leq \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}\|(\wp-\Im)^2\|_C^{\frac{1}{2}} \\ &+ \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\psi'(s)p(\wp,\Im)\|(\wp-\Im)^2\|_C^{\frac{1}{2}}ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}(1-q*)}\end{aligned}$$

Thus

$$\begin{aligned}&\alpha(\wp,\Im)|(N\wp)(t)-(N\Im)(t)|^2 \\ &\leq \|(\wp-\Im)^2\|_C\alpha(\wp,\Im)\left\|\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}\right\|_C^2 \\ &\quad+\|(\wp-\Im)^2\|_C\alpha(u,v)\left\|\frac{1}{k\Gamma_k(\vartheta)}\int_a^t\frac{\psi'(s)p(\wp,\Im)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}(1-q*)}\right\|_C^2 \\ &\leq \|(\wp-\Im)^2\|_C\phi(\|(\wp-\Im)^2\|_C).\end{aligned}$$

Hence,

$$\alpha(\wp,\Im)\phi(2^3d(N(\wp),N(\Im))\leq\lambda(\phi(d(\wp,\Im))\phi(d(\wp,\Im)),$$

where $\lambda\in\digamma$, $\phi\in\Phi$, with $\lambda(t)=\frac{1}{8}t$, and $\phi(t)=t$.
So, $N$ is generalized $\alpha$-$\phi$-Geraghty operator.

Let $\wp,\Im\in C_{\xi,k;\psi}(I)$ such that

$$\alpha(\wp,\Im)\geq 1.$$

Thus, for each $t\in I$, we have

$$\theta(\wp(t),\Im(t))\geq 0.$$

This implies from $(H_3)$ that

$$\theta(N\wp(t),N\Im(t))\geq 0,$$

which gives

$$\alpha(N(\wp),N(\Im))\geq 1.$$

Hence, $N$ is a $\alpha$-admissible.

Now, from $(H_2)$, there exists $\mu_0\in C_{\xi,k;\psi}(I)$ such that

$$\alpha(\mu_0,N(\mu_0))\geq 1.$$

Finally, from $(H_4)$, If $\mu_{n_{n\in N}}\subset M$ with $\mu_n\to\mu$ and $\alpha(\mu_n,\mu_{n+1})\geq 1$, then

$$\alpha(\mu_n,\mu)\geq 1.$$

From an application of Theorem 2.11, we deduce that $N$ has a fixed point $u$ which is a solution of problem (1.1)-(1.2).

## 4. An Example

Let $(C_{\xi,k;\psi}([0,1]),d,2)$ be the complete $b$-metric space, such that
$d: C_{\xi,k;\psi}([0,1]) \times C_{\xi,k;\psi}([0,1]) \to \mathbb{R}_+^*$ is given by:

$$d(\wp,\Im) = \|(\wp-\Im)^2\|_C.$$

Consider the following fractional differential problem

$$\begin{cases} \left({}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}\wp\right)(t) = f(t,\wp(t),\left({}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}\wp\right)(t);\ t\in[0,1], \\ \left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}\wp\right)(0)=0, \end{cases} \tag{4.1}$$

where

$$f(t,\wp(t),\Im(t)) = \frac{(1+\sin(|\wp(t)|))}{4(1+|\wp(t)|)} + \frac{e^{-t}}{2(1+|\Im(t))|};\ t\in[0,1].$$

Let $t\in(0,1]$, and $\wp,\Im\in C_{\xi,k;\psi}([0,1])$. If $|\wp(t)|\le|v(t)|$, then

$$\begin{aligned} |f(t,\wp(t),\wp_1(t)) - f(t,\Im(t),\Im_1(t))| &= \left|\frac{1+\sin(|\wp(t)|)}{4(1+|\wp(t)|)} - \frac{1+\sin(|\Im(t)|)}{4(1+|\Im(t)|)}\right| \\ &\quad + \left|\frac{e^{-t}}{2(1+|\wp_1(t))|} - \frac{e^{-t}}{2(1+|\Im_1(t))|}\right| \\ &\le \frac{1}{4}||\wp(t)|-|\Im(t)|| + \frac{1}{4}|\sin(|\wp(t)|)-\sin(|\Im(t)|)| \\ &\quad + \frac{1}{4}\,||\wp(t)|\sin(|\Im(t)|) - |\Im(t)|\sin(|\wp(t)|)| \\ &\quad + \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t))| \\ &\le \frac{1}{4}|\wp(t)-\Im(t)| + \frac{1}{4}|\sin(|\wp(t)|)-\sin(|\Im(t)|)| \\ &\quad + \frac{1}{4}\,||\Im(t)|\sin(|\Im(t)|) - |\Im(t)|\sin(|\wp(t)|)| \\ &\quad + \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t))| \\ &= \frac{1}{4}\wp(t)-\Im(t)| + \frac{1}{4}(1+|\Im(t)|) \\ &\quad \times|\sin(|\wp(t)|)-\sin(|\Im(t)|)| \\ &\quad + \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t))| \\ &\le \frac{1}{4}|\wp(t)-\Im(t)| + \frac{1}{2}(1+|\Im(t)|) \\ &\quad \times\left|\sin\left(\frac{||\wp(t)|-|\Im(t)|}{2}\right)\right|\left|\cos\left(\frac{|\wp(t)|+|\Im(t)|}{2}\right)\right| \\ &\quad + \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t)| \\ &\le \frac{1}{4}(2+|\Im(t)|)|\wp(t)-\Im(t)| + \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t)|. \end{aligned}$$

The case when $|\Im(t)|\le|\wp(t)|$, we get

$$|f(t,\wp(t),\wp_1(t)) - f(t,\Im(t),\Im_1(t))| \le \frac{1}{4}(2+|\wp(t)||\wp(t)-\Im(t)| + \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t)|.$$

Hence

$$|f(t,\wp(t),\wp_1(t)) - f(t,\Im(t),\Im_1(t))|$$

$$\leq \frac{1}{4}\min_{t\in I}\{2+|\wp(t)|, 2+|\Im(t)|\}|\wp(t)-\Im(t)| + \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t)|.$$

Thus, hypothesis $(H_1)$ is satisfied with

$$p(\wp,\Im) = \frac{1}{4}\min_{t\in I}\{2+|\wp(t)|, 2+|\Im(t)|\},$$

and

$$q(t) = \frac{1}{2}e^{-t}.$$

Define the functions $\lambda(t) = \frac{1}{8}t$, $\phi(t) = t$, $\alpha : C_{\xi,k;\psi}([0,1]) \times C_{\xi,k;\psi}([0,1]) \to \mathbb{R}_+^*$ with

$$\begin{cases} \alpha(\wp,\Im) = 1;\ if\ \delta(\wp(t),\Im(t)) \geq 0,\ t\in I, \\ \alpha(\wp,\Im) = 0;\ else, \end{cases}$$

and $\delta : C_{\xi,k;\psi}([0,1]) \times C_{\xi,k;\psi}([0,1]) \to \mathbb{R}$ with $\delta(\wp,\Im) = \|\wp-\Im\|_C$.
Hypothesis $(H_2)$ is satisfied with $\mu_0(t) = \wp_0$. Also, $(H_3)$ holds from the definition of the function $\delta$. Hence by Theorem 3.2, problem (4.1) has at least one solution defined on $[0,1]$.

## DECLARATIONS

**Ethical approval:** This article does not contain any studies with human participants or animals performed by any of the authors.

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