SOME APPROXIMATE DIVISION AND SEMIGROUP IDENTITIES FOR THE MITTAG-LEFFLER FUNCTION

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ABSTRACT. It is known that the Mittag-Leffler (ML) function, $E_\alpha(z)$, a non-local extension of the Euler exponential function e^z, does not enjoy the semigroup property while e^z does. The purpose of this note is to show that $E_\alpha(z)$ does, however, for real t, s with s small enough, enjoy the approximate semigroup property, $E_\alpha(t+s) \approx E_\alpha(t)(1 + \frac{E_{\alpha,\alpha}(t)}{E_\alpha(t)})s$. This follows from an approximation of $\lim_{h\to 0^+} \frac{E_\alpha(\pm(u+h)^\alpha)}{E_\alpha(\pm u^\alpha)}$, which also yields related expressions for $\alpha \to 1^-$, and is obtained from a recently proposed universal difference quotient representation for fractional derivatives. Graphical demonstrations are presented to show that the approximations are 'reasonably accurate' for $0 < h \le 0.1$, with virtually no distinction from identity for $0 < h \le 0.01$.

1. INTRODUCTION

Just as exponential functions are ubiquitous in the theory and application of integer-order derivatives, so are the one- and two-parameter Mittag-Leffler (ML) functions, $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, respectively defined by

$$E_\alpha(z) = E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

and

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

in the theory and application of fractional derivatives (FDs). This is reflected, for instance, by its constant presence in FDE textbooks (e.g., [12]), particularly featured in their solutions [1,2,6,12], and stability analysis (see, eg., [9], [14]). The intimate connection of $E_\alpha(z)$ to fractional differential equations (FDEs) and integral equations of Abel type is best summarized in [7] as follows: "... it seems important as a first step to develop their theory and stable methods for their numerical computation". For a recent survey of the properties and applications of both $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ as well as other related functions, the reader is referred to [13].

While it is a non-local extension of the Euler exponential function e^z, and enjoys some of its properties, $E_\alpha(z)$ does not enjoy the semigroup property, which e^z does; that is, while $e^{(t+s)} = e^t e^s$ holds true, the equality $E_\alpha(t+s) = E_\alpha(t)E_\alpha(s)$ does not similarly follow, that is, $E_\alpha(t+s) \ne E_\alpha(t)E_\alpha(s)$. Consequent to

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the semigroup property for e^z, there exists the following division formula for $e^{\lambda x}$:

$$\text{(1.1)} \qquad \frac{e^{\pm\lambda(x+h)^\alpha}}{e^{\pm\lambda x^\alpha}} = e^{\pm(\lambda(x+h)^\alpha - x^\alpha)}.$$

However, because $E_\alpha(z)$ does not possess the semigroup property, there does not exist a parallel division formula for the ML function, that is, we have

$$\text{(1.2)} \qquad \frac{E_\alpha(\pm\lambda(x+h)^\alpha)}{E_\alpha(\pm\lambda\,x^\alpha)} \neq E_\alpha\left(\pm\lambda\left((x+h)^\alpha - x^\alpha\right)\right).$$

The need for such a division formula for the ML function arises due to the following general definition for non-integer derivatives (NIDs), recently proposed in [4], such as in the case where the solution in Definition 1.1 below is $\mho(t,\alpha) = E_\alpha(-t\alpha)$ (see Table 2.4 of [4]):

Definition 1.1 (Corollary 2.1.7 and Theorem 2.2.1 of [4]). Let $\mho(t,\alpha)$ denote the unique solution of the initial value problem for the fractional relaxation equation (FRE):

$$\text{(1.3)} \qquad D_t^\alpha \Psi_\alpha(t) = -\Psi_\alpha(t), \quad \Psi_\alpha(0^+) = 1, \quad t \geq 0; \alpha \in (0,\ 1]$$

for D_t^α an arbitrary FD. Then D_t^α has the universal difference quotient representation (UDQR), $^{\mho C}_0\Delta_t^\alpha f(t,h)$, and generalized fractional derivative representation (GFDR) $^{GC}_0 D_t^\alpha$, respectively given by:

$$\text{(1.4)} \qquad ^{\mho C}_0\Delta_t^\alpha f(t,h) = \frac{f(t+h) - f(t)}{1 - \frac{\mho(t+h,\alpha)}{\mho(t,\alpha)}}$$

and

$$\text{(1.5)} \qquad ^{GC}_0 D_t^\alpha f(t) = \lim_{h\to 0} {}^{\mho C}_0\Delta_t^\alpha f(t,h).$$

In fact, it is shown in [11], using a Laplace transform argument, that $E_\alpha(\lambda z)$ possesses the semigroup property only if $\alpha = 1$ or $\lambda = 0$. Following a conclusion that this lack of the semigroup property "seems to tell us that any equality relationship involving $E_\alpha(\lambda t^\alpha)$, $E_\alpha(\lambda s^\alpha)$, and $E_\alpha\left((\lambda(s+t)^\alpha\right)$ should be of memory and hence characterized by integrals", the following result is proven in [11]:

Theorem 1.2 (Theorem 1 of [4]). *For every real λ there holds that*

$$\int_0^{t+s} \frac{E_\alpha(\lambda x^\alpha)}{(t+s-x)^\alpha}dx - \int_0^t \frac{E_\alpha(\lambda x^\alpha)}{(t+s-x)^\alpha}dx - \int_0^s \frac{E_\alpha(\lambda x^\alpha)}{(t+s-x)^\alpha}dx$$
$$= \alpha \int_0^t \int_0^s \frac{E_\alpha(\lambda u^\alpha)E_\alpha(\lambda v^\alpha)}{(t+s-u-v)^{1+\alpha}}dudv, \quad t,s \geq 0.$$

However, the identity of Theorem 1.2 does not involve $E_\alpha\left(\lambda(s+t)^\alpha\right)$, and therefore does not resolve the lack of a division formula for $E_\alpha(\lambda z)$ that parallels Eqn. (1.1). The main purpose of this note is to present formal and graphical arguments that the following h-approximate division property holds in lieu of equality in Eqn. (1.2):

$$\text{(1.6)} \qquad \lim_{h\to 0} \frac{E_\alpha(\pm(u+h)^\alpha)}{E_\alpha(\pm u^\alpha)} = \lim_{h\to 0}\left(1 \pm \frac{1}{\alpha}\frac{E_{\alpha,\alpha}(\pm\,u^\alpha)}{E_\alpha(\pm u^\alpha)}\left((u+h)^\alpha - u^\alpha\right)\right)$$

From property (1.6), it follows that the approximate semigroup property (1.7) below holds for $s > 0$ small enough:

$$\text{(1.7)} \qquad E_\alpha\left((t+s)\right) \approx E_\alpha(t)\left(1 + \frac{1}{\alpha}\frac{E_{\alpha,\alpha}(t)}{E_\alpha(t)}s\right),$$

and an α-approximate division property is obtained relating $E_\alpha(\lambda t^\alpha)$, $E_{\alpha,\alpha}(\lambda t^\alpha)$, $E_\alpha\left(\lambda(s+t)^\alpha\right)$, and the exponential function. The remainder of this article is organized as follows. In Section 2, the CFD and

Caputo FD are recalled and properties of the UDQR for NIDs are presented, along with the consequent CFD and Caputo FD relationships with the integer derivative. Section 3 presents formal derivation of the proposed ML function approximate identities. Results of the numerical comparison of the approximate ML identities are presented in Section 4, with some observations and remarks in Section 5 concluding the article.

2. PRELIMINARIES

Together with the universal DQR and GFD for non-integer derivatives given in Definition 1.1, two fractional derivatives (FDs) will be used in deriving the new approximate identities, the conformable FD (CFD) and the Caputo FD. The CFD was introduced in [8] and is defined as

$$(2.1) \qquad T_t^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right)}{\varepsilon}, \quad \alpha \in (0,1],$$

and the solution to the conformable FRE, that is, Eqn. (1.3) with CFD, is

$$(2.2) \qquad T_t^\alpha(f)(t) = t^{1-\alpha} \frac{d}{dt} f(t).$$

The Caputo FD, introduced in [3], is defined as

$$(2.3) \qquad {}_0^C D_t^\alpha(f(t)) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-x)^{-\alpha} \frac{d}{dx} f(x)\, dx,$$

and the solution to the Caputo FRE, that is, Eqn. (1.3) with Caputo FD, is

$$\Psi_\alpha(t) = E_\alpha(-t^\alpha).$$

Among its properties is the following relationship with the CFD,

$$(2.4) \qquad {}_0^C D_t^\alpha(f(t)) = \frac{E_\alpha(-t^\alpha)}{E_{\alpha,\alpha}(-t^\alpha)} T_t^\alpha(f)(t),$$

which is concluded from the following result derived in [10]:

Theorem 2.1 (see Eqns. (1), (7), (13) of [10]). *The initial value problem for the Caputo FRE with constant coefficient, IVP (1.3), is equivalent to that for the first order ODE with varying coefficient,*

$$(2.5) \qquad \frac{d}{dt} \Psi(t) = -r(t)\, \Psi(t), \ \Psi(0^+) = 1, \ t \geq 0,$$

if and only if

$$(2.6) \qquad r(t) = r(t,\alpha) = \frac{t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha)}{E_\alpha(-t^\alpha)}.$$

The identity (2.4) can then be obtained by substituting Eqn. (2.6) into Eqn. (2.5) and comparing the result with Eqn. (1.3). The identity (2.4) can also be derived from Theorem 2 below, which gives the properties of a recently proposed universal quotient difference representation (UQDR) of NIDs, and also relates all Caputo type fractional derivatives to the integer derivative.

Theorem 2.2 (Theorem 2.1.6 of [4]). *Let $\alpha \in (0,1]$ and assume that ${}_0^\mho\Delta_t^\alpha f(t)$, ${}_0^\mho\Delta_t^\alpha g(t)$ exist at a point t in $(0,\infty)$. Then, for all constants A,B,C the generic DQR expressions in Definition 2 has properties (1)-(4) below while the generic GFD expression has properties (5)-(6):*

    (1) ${}_0^\mho\Delta_t^\alpha(Af(t) + Bg(t)) = A\,{}_0^\mho\Delta_t^\alpha f + B\,{}_0^\mho\Delta_t^\alpha g$

    (2) ${}_0^\mho\Delta_t^\alpha(fg) = g\,{}_0^\mho\Delta_t^\alpha f + f\,{}_0^\mho\Delta_t^\alpha g$

    (3) ${}_0^\mho\Delta_t^\alpha\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(g\,{}_0^\mho\Delta_t^\alpha f - f\,{}_0^\mho\Delta_t^\alpha g\right)$

(4) $\, {}^{\mho}_0\Delta^\alpha_t C = 0$

(5) If $\mho(t_0, t, y_0)$ is first order differentiable, then the following holds:

$$ {}^{\mho}_0D^\alpha_t(t^p) = -\mho(t, \alpha)\frac{1}{\frac{d\mho(t,\alpha)}{dt}}pt^{p-1} $$

(6) If $f(t)$ and $\mho(t, \alpha)$ are both first order differentiable, then the following holds:

$$ {}^{\mho}_0D^\alpha_t\big(f(t)\big) = -\mho(t, \alpha)\frac{1}{\frac{d\mho(t,\alpha)}{dt}}\frac{df(t)}{dt} $$

Note 2.3. The Khalil et. al. result, Eqn. (2.2) from [8], relating the CFD and first order derivative is obtained from Theorem 2.2 (6) by setting
$\mho(t, \alpha) = \exp\left(-\frac{1}{\alpha}t^\alpha\right)$.

Note 2.4. The Mainardi result, Eqn. (2.4) from [10], relating the CFD and Caputo derivative is obtained from Theorem 2 (6) by setting $\mho(t_0, t, y_0) = E_\alpha(-t^\alpha)$ and using the identity (see [10])

(2.7)
$$ \frac{dE_\alpha(-t^\alpha)}{dt} = -t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha). $$

In view of the foregoing, Theorem 2.2 (6) is a generalization of the Khalil et. al. identity (2.2) to all conformal (local) fractional derivatives and of the Mainardi identity (2.4) to all non-local fractional derivatives of Caputo type.

A consequence of Definition 1 is the following alternative definition of the CFD, given in [5], which results from direct substitution of the solution of the CFD FRE, $\mho(t, \alpha) = \exp\left(-\frac{1}{\alpha}t^\alpha\right)$ into Eqns. (1.4) and (1.5).

Definition 2.5 (Definition 4.2 of [5]). The conformable fractional derivative has the following alternative definition on $[0, \infty)$, for, $0 < \alpha \le 1$.

$$ {}^C_0T^\alpha_t f(t) \equiv {}^{CFD}_{0}\Delta^\alpha_t y(t) = \lim_{h\to 0}\frac{f(t+h) - f(t)}{1 - e^{-\frac{1}{\alpha}\left((t+\alpha)^\alpha - t^\alpha\right)}} = \alpha\lim_{h\to 0}\frac{y(t+h) - y(t)}{(t+h)^\alpha - t^\alpha}, $$

where ${}^C_0T^\alpha_t f(0)$ is understood to mean ${}^C_0T^\alpha_t f(0) = \lim_{t\to 0^+}{}^C_0T^\alpha_t f(t)$.

3. The approximate ML relationships

Direct substitution of the solution of the Caputo FRE, $\mho(t, \alpha) = E_\alpha(-t^\alpha)$, into Eqns. (1.4) and (1.5), and then using the Mainardi identity (2.4) and Definition 3, yields the following:

$$ {}^C_0D^\alpha_t f(t) = \lim_{h\to 0}\frac{f(t+h) - f(t)}{1 - E_\alpha(-(t+h)^\alpha)/E_\alpha(-t^\alpha)} $$

(3.1)
$$ = \frac{E_\alpha(-t^\alpha)}{E_{\alpha\alpha,}(-t^\alpha)}{}^C_0T^\alpha_t f(t) = \alpha\frac{E_\alpha(-t^\alpha)}{E_{\alpha,\alpha}(-t^\alpha)}\lim_{h\to 0}\frac{f(t+h) - f(t)}{(t+h)^\alpha - t^\alpha}. $$

The proposed result for ML division then follows by equating the two denominators in Eqn. (3.1) to get

$$ \lim_{h\to 0}\left(1 - \frac{E_\alpha\left(-(t+h)^\alpha\right)}{E_\alpha(-t^\alpha)}\right) = \frac{1}{\alpha}\frac{E_{\alpha,\alpha}\left(-t^\alpha\right)}{E_\alpha(-t^\alpha)}\lim_{h\to 0}\left((t+h)^\alpha - t^\alpha\right), $$

from which follows

(3.2)
$$ \lim_{h\to 0}\frac{E_\alpha(-(t+h)^\alpha)}{E_\alpha(-t^\alpha)} = 1 - \frac{1}{\alpha}\frac{E_{\alpha,\alpha}(-t^\alpha)}{E_\alpha(-t^\alpha)}\lim_{h\to 0}\left((t+h)^\alpha - t^\alpha\right), $$

and hence the approximate division formula (1.6), for negative arguments. From (3.2), identifying $(u+h)^\alpha = y$, $u^\alpha = x$, there follows

$$ E_\alpha(-y) \approx E_\alpha(-x)\left(1 - \frac{1}{\alpha}\frac{E_{\alpha,\alpha}(-x)}{E_\alpha(-x)}(y - x)\right). $$

Identifying $-y = t + s$, $-x = t$ the approximate semi-group property (1.7) then follows:

$$E_\alpha(t + s) \approx E_\alpha(t) \left(1 - \frac{1}{\alpha} \frac{E_{\alpha,\alpha}(t)}{E_\alpha(t)} ((-t - s) + t) \right) = E_\alpha(t) \left(1 + \frac{1}{\alpha} \frac{E_{\alpha,\alpha}(t)}{E_\alpha(t)} s \right).$$

The approximate semi-group property (1.7) may also be deduced from consideration of the growth equation,

$$D_t^\alpha \Psi_\alpha(t) = \Psi_\alpha(t), \quad \Psi_\alpha \left(0^+ \right) = 1, \ t \geq 0; \ \alpha \in (0, 1]$$

which yields, denoting the solution as $\overline{\mho}(t, \alpha)$ the following DQR:

(3.3)
$$\overset{\mho C}{_0}\bar{\Delta}_t^\alpha f(t, h) = \frac{f(t + h) - f(t)}{\frac{\mho(t+h,\alpha)}{\mho(t,\alpha)} - 1}.$$

Since both $\overset{\mho C}{_0}\bar{\Delta}_t^\alpha$ and $\overset{\mho C}{_0}\Delta_t^\alpha$ are exact QDR approximations of the FD D_t^α, it follows from the invariance of derivative definition that the following must hold true:

(3.4)
$$\lim_{h \to 0} \overset{\mho C}{_0}\bar{\Delta}_t^\alpha f(t, h) = \overset{\mho C}{_0}\bar{D}_t^\alpha f(t) = \overset{\mho C}{_0}D_t^\alpha f(t) = \lim_{h \to 0} \overset{\mho C}{_0}\Delta_t^\alpha f(t, h)$$

Consideration of the Caputo FD in (3.3) and (3.4) yields therefore

$$\lim_{h \to 0} \left(\frac{E_\alpha((t + h)^\alpha)}{E_\alpha(t^\alpha)} - 1 \right) = \frac{1}{\alpha} \frac{E_{\alpha,\alpha}(t^\alpha)}{E_\alpha(t^\alpha)} \lim_{h \to 0} ((t + h)^\alpha - t^\alpha),$$

hence the division formula (1.6) for positive arguments,

(3.5)
$$\frac{E_\alpha((u + h)^\alpha)}{E_\alpha(u^\alpha)} \approx 1 + \frac{1}{\alpha} \frac{E_{\alpha,\alpha}(u^\alpha)}{E_\alpha(u^\alpha)} ((u + h)^\alpha - u^\alpha),$$

$$E_\alpha((u + h)^\alpha) \approx E_\alpha(u^\alpha) \left(1 + \frac{1}{\alpha} \frac{E_{\alpha,\alpha}(u^\alpha)}{E_\alpha(u^\alpha)} ((u + h)^\alpha - u^\alpha) \right).$$

From the approximation (3.5), identifying $(u + h)^\alpha = y$, $u^\alpha = x$, there follows

$$E_\alpha(y) \approx E_\alpha(x) \left(1 + \frac{1}{\alpha} \frac{E_{\alpha,\alpha}(x)}{E_\alpha(x)} (y - x) \right),$$

and hence, identifying $y = t + s$, $x = t$, the approximate semi-group property (1.7).

Next, note that the formula in Eqn. (1.6) is the result of the following identities obtained from the UDQR representations of the Caputo FD and CFD:

(3.6)
$$\lim_{h \to 0} \left(1 - \frac{E_\alpha(-\lambda(t + h)^\alpha)}{E_\alpha(-\lambda t^\alpha)} \right) = \frac{E_{\alpha,\alpha}(-\lambda t^\alpha)}{E_\alpha(-\lambda t^\alpha)} \lim_{h \to 0} \left(1 - e^{\frac{-\lambda}{\alpha}((t+h)^\alpha - t^\alpha)} \right)$$

and

(3.7)
$$\lim_{h \to 0} \left(\frac{E_\alpha(\lambda(t + h)^\alpha)}{E_\alpha(\lambda t^\alpha)} - 1 \right) = \frac{E_{\alpha,\alpha}(\lambda t^\alpha)}{E_\alpha(\lambda t^\alpha)} \lim_{h \to 0} \left(e^{\frac{\lambda}{\alpha}((t+h)^\alpha - t^\alpha)} - 1 \right).$$

Taken without the limits for $\alpha = 1$, (3.6) and (3.7) respectively yield, as expected,

$$\frac{E_1\left(-\lambda(t + h)\right)}{E_1(-\lambda t)} = 1 - \frac{E_{1,1}(-\lambda t)}{E_1(-\lambda t)} \left(1 - e^{-\lambda((t+h)-t)} \right) = e^{-\lambda((t+h)-t)} = \frac{e^{-\lambda((t+h))}}{e^{-\lambda t}},$$

and

$$\frac{E_1\left(\lambda(t + h)\right)}{E_1(\lambda t)} = 1 + \frac{E_{1,1}(\lambda t)}{E_1(\lambda t)} \left(e^{\lambda((t+h)-t)} - 1 \right) = e^{\lambda((t+h)-t)} = \frac{e^{\lambda((t+h))}}{e^{\lambda t}}.$$

Following the reasoning of Peng and Li in [11], we conclude that "the semigroup property of $E_1(z)$ above is just the limit state of equality in (3.6) and (3.7) as $\alpha \to 1^-$", and the following results are therefore deduced from (3.6) and (3.7):
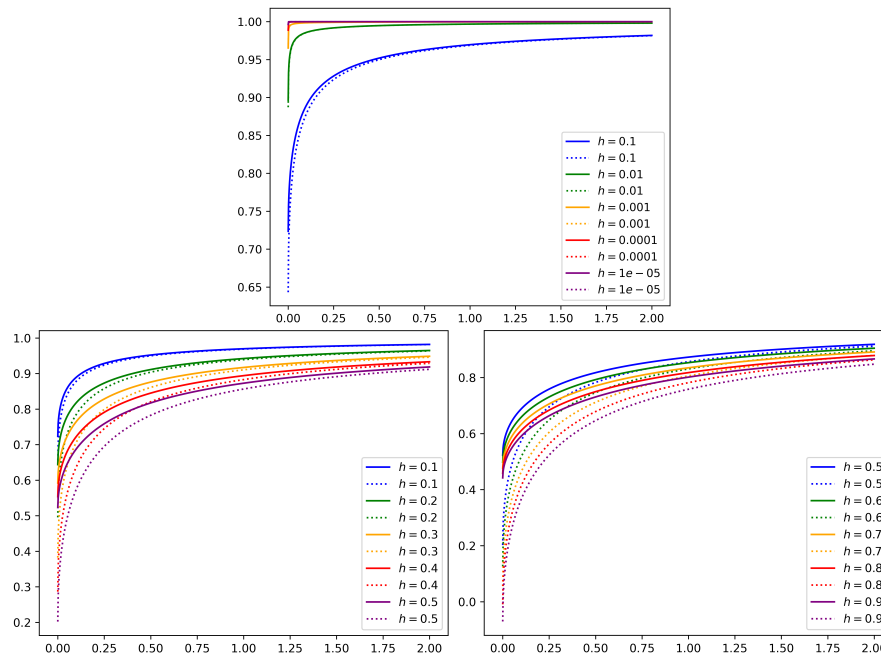
FIGURE 1.1. (a) (top), (b) (bottom, left), (c) (bottom, right). Comparison, for $0 \leq u \leq 2$ (abscissa), of the LHS (solid lines) and the RHS (dotted lines) of (3.2) (ordinate) for $\alpha = 0.5$ and various values of **(a)** $h \in [3.0 \times 10^{-5}, 0.1]$ **(b)** $h \in [0.1, 0.5]$ and **(c)** $h \in [0.5, 0.9]$

Proposition 3.1. *For h sufficiently small and every real λ, there holds that*

$$(3.8) \qquad \lim_{\alpha \to 1^-} \frac{E_\alpha(-\lambda(t+h)^\alpha)}{E_\alpha(-\lambda t^\alpha)} = 1 - \lambda \lim_{\alpha \to 1^-} \frac{E_{\alpha,\alpha}(-\lambda\, t^\alpha)}{E_\alpha(-\lambda t^\alpha)} \left(1 - e^{\frac{-\lambda}{\alpha}((t+h)^\alpha - t^\alpha)}\right)$$

and

$$(3.9) \qquad \lim_{\alpha \to 1^-} \frac{E_\alpha(\lambda(t+h)^\alpha)}{E_\alpha(\lambda t^\alpha)} = 1 + \lambda \lim_{\alpha \to 1^-} \frac{E_{\alpha,\alpha}(\lambda t^\alpha)}{E_\alpha(\lambda t^\alpha)} \left(e^{\frac{\lambda}{\alpha}((t+h)^\alpha - t^\alpha)} - 1\right)$$

4. NUMERICAL EXPERIMENTS

In this section, numerical simulation results are presented for the h-approximate division formulas (3.2) and (3.5) for $0 \leq u \leq 2$ and positive, small parameter h, and the α-approximate division formulas (3.6) and (3.7). The results for identities (3.2), (3.5), (3.8), and (3.9) are presented in, respectively, Figures 1.1, 1.2 (a), (b), (c), Figures 2.1, 2.2 (a), (b), (c), Figures 3.1 (a), (b), 3.2, and Figures 4.1 (a), (b), 4.2.

As can be seen from Figure 1.1, the right-hand side (RHS) of (3.2), represented by solid lines, and the left-hand side (LHS), represented by solid lines, both increase with decreasing h towards LHS = RHS = 1, being closer in Figure 1 (b) than in (c), where the percentage difference at $\alpha = 0.5$ is PD $< 0.7\%$ on $0 < u \leq 2$ for $h \geq 0.5$. The two sides are even closer in (a), with virtually no distinction between them for $h < 0.01$; that is, the dotted and corresponding solid lines both increase towards LHS = RHS = 1, becoming almost indistinguishable after the green lines. Percentage differences for Figure 1.1 data are shown in Figure 1.2.

In Figure 2.1, the right-hand side (RHS) of (3.5), represented by solid lines, and the left-hand side (LHS), represented by solid lines as in Figure 1, are seen to both decrease with decreasing h towards LHS = RHS = 1, being closer in Figure 2 (b) than in (c), where $h \geq 0.5$ and the percentage difference rapidly decreases from PD $\approx 11\%$ at $u = 0.2$ to PD $\approx 0.7\%$ at $u = 2$ for $\alpha = 0.5$. The two sides are even closer in (a), becoming almost indistinguishable after the green lines, that is, for $h < 0.01$. Figure 2.2 shows the percentage differences for Figure 2.1 data.
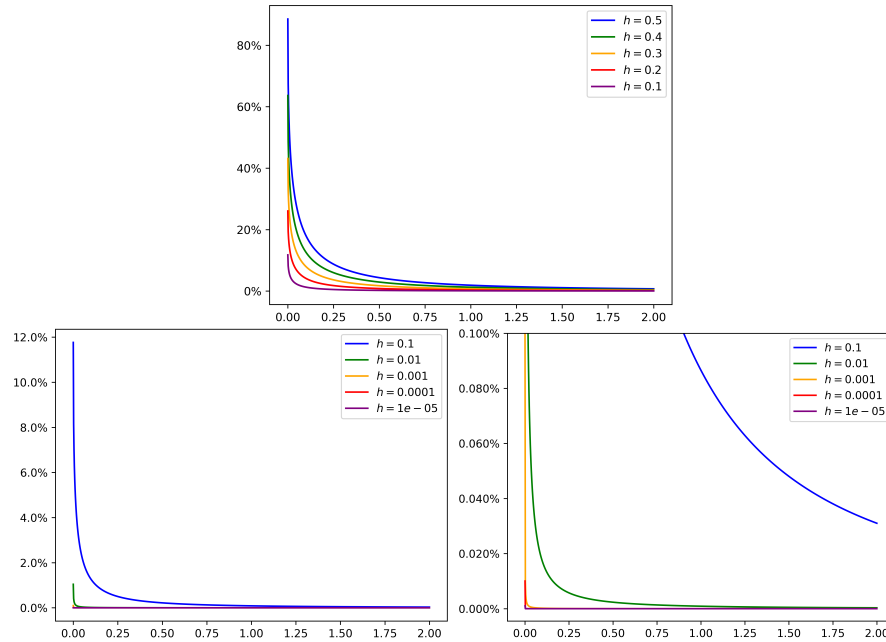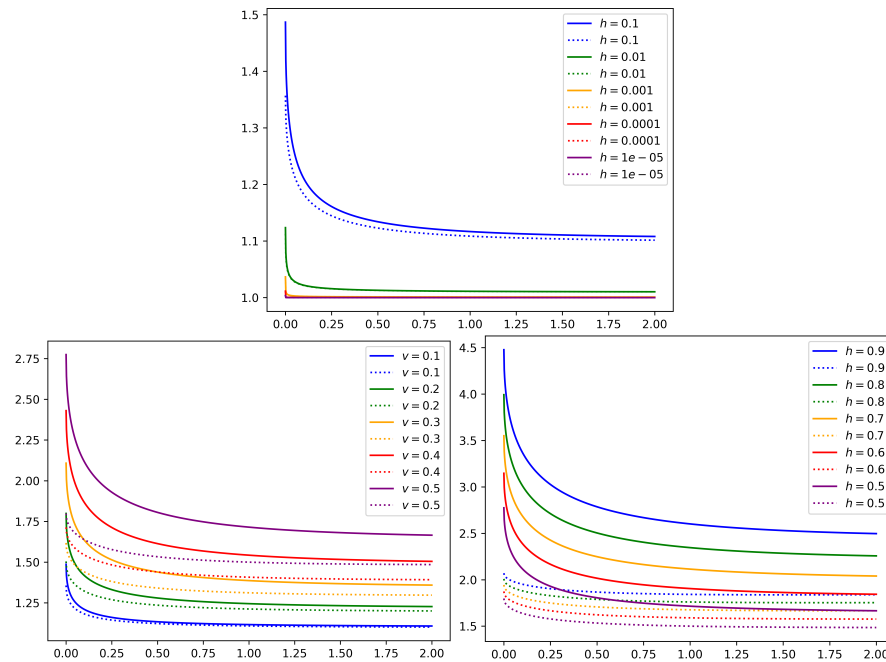
FIGURE 1.2. (a) (top), (b) (bottom, left), (c) (bottom, right). Percentage differences, for $0 \leq u \leq 2$ (abscissa), between the LHS and the RHS of (3.2) (ordinate) for $\alpha = 0.5$ and various values of **(a)** $h \in [0.1, 0.5]$, **(b)** $h \in [1.0 \times 10^{-5}, 0.1]$, and **(c)** $h \in [1.0 \times 10^{-5}, 0.1]$ zoomed in.

FIGURE 2.1. (a) (top), (b) (bottom, left), (c) (bottom, right). Comparison, for $0 \leq u \leq 2$ (abscissa), of the LHS (solid lines) and the RHS (dotted lines) of (3.5) (ordinate) for $\alpha = 0.5$ and various values of **(a)** $h \in [3.0 \times 10^{-5}, 0.1]$, **(b)** $h \in [0.1, 0.5]$ and **(c)** $h \in [0.5, 0.9]$
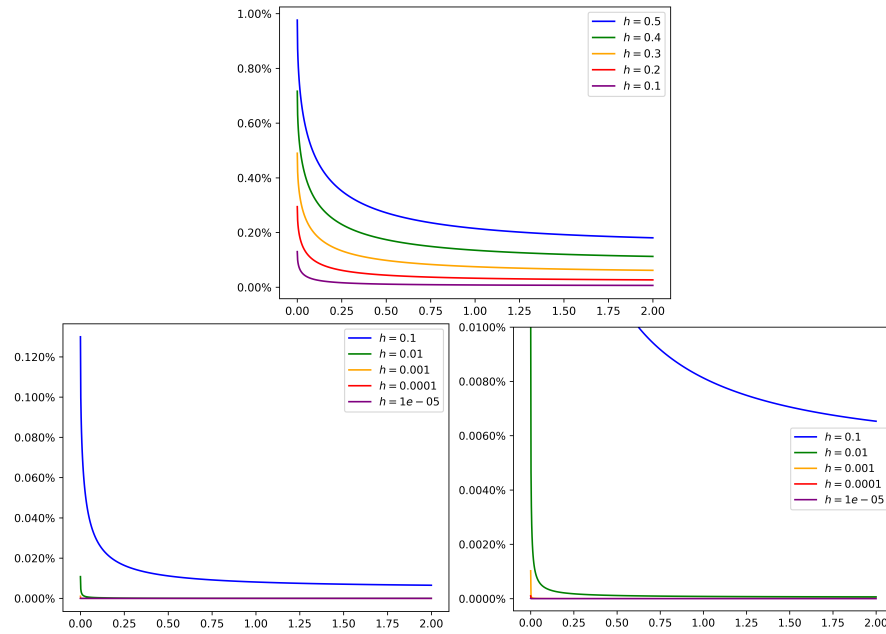
FIGURE 2.2. (a) (top), (b) (bottom, left), (c) (bottom, right). Percentage differences, for $0 \leq u \leq 2$ (abscissa), between the LHS and the RHS of (3.5) (ordinate) for $\alpha = 0.5$ and various values of **(a)** $h \in [0.1, 0.5]$, **(b)** $h \in [1.0 \times 10^{-5}, 0.1]$, and **(c)** $h \in [1.0 \times 10^{-5}, 0.1]$ zoomed in.
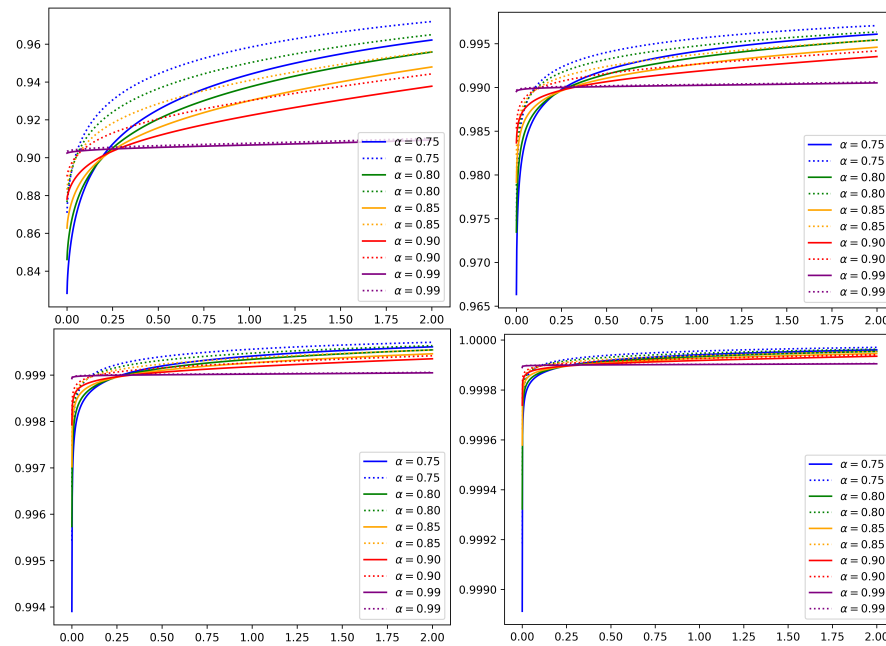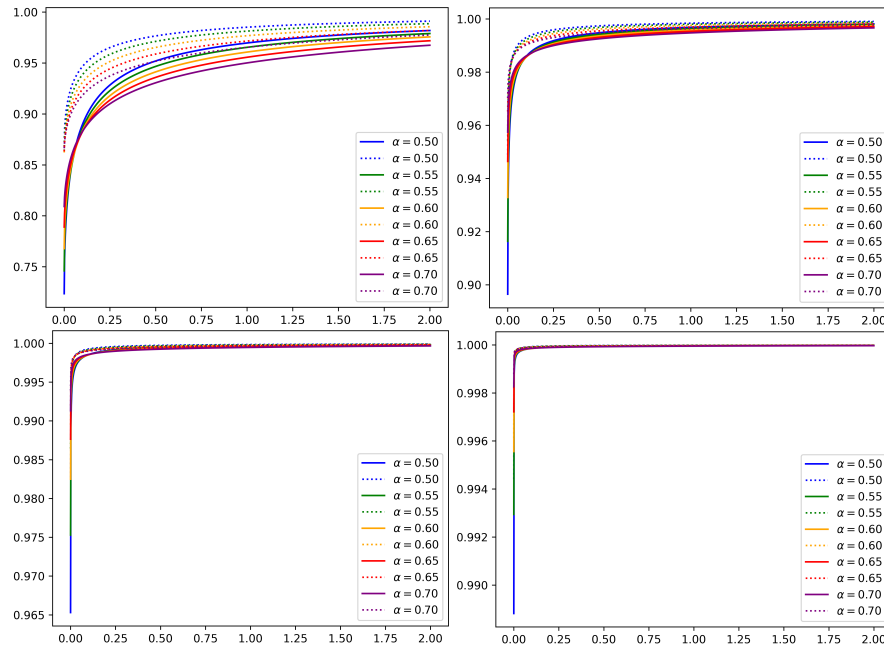

FIGURE 3.1 (A). (i) (top, left), (ii) (top, right), (iii) (bottom, left), (iv) (bottom, right). Comparison, for $0 \leq u \leq 2$ (abscissa), of the LHS (solid lines) and the RHS (dotted lines) of (3.8) (ordinate) for various values of $\alpha \in [0.75, 0.99]$, for (i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$, (iv) $h = 0.0001$.

FIGURE 3.1 (B). (i) (top, left), (ii) (top, right), (iii) (bottom, left), (iv) (bottom, right). Comparison, for $0 \leq u \leq 2$ (abscissa), of the LHS (solid lines) and the RHS (dotted lines) of (3.8) (ordinate) for various values of $\alpha \in [0.5, 0.75]$, for (i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$, (iv) $h = 0.0001$
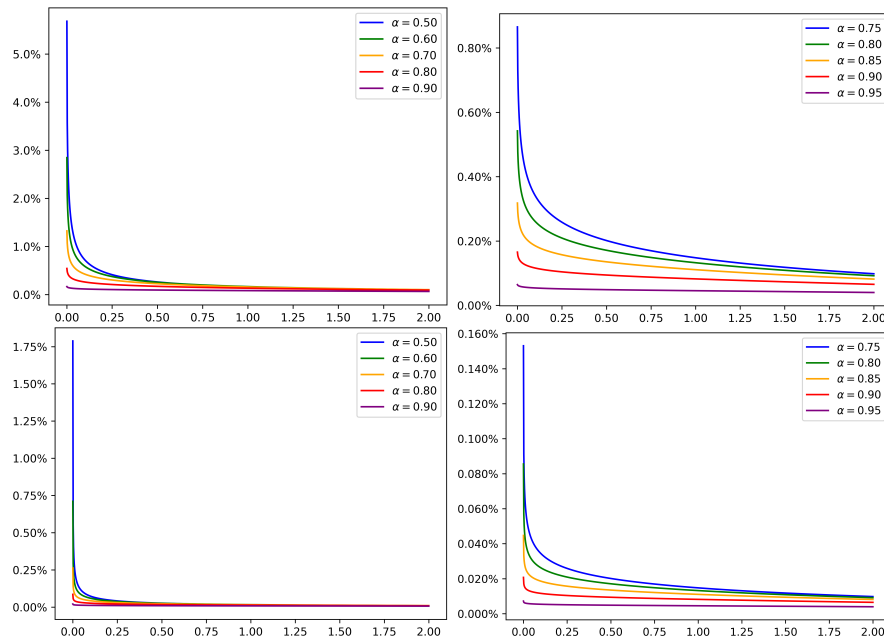

FIGURE 3.2. (a) (top, left), (b) (top, right), (c) (bottom, left), (d) (bottom, right). Percentage differences, for $0 \leq u \leq 2$ (abscissa), between the LHS and the RHS of (3.8) (ordinate) for various values of (a) $\alpha \in [0.5, 0.9]$, $h = 0.1$, (b) $\alpha \in [0.75, 0.95]$, $h = 0.1$, (c) $\alpha \in [0.5, 0.9]$, $h = 0.001$, (d) $\alpha \in [0.75, 0.99]$, $h = 0.001$ zoomed in.
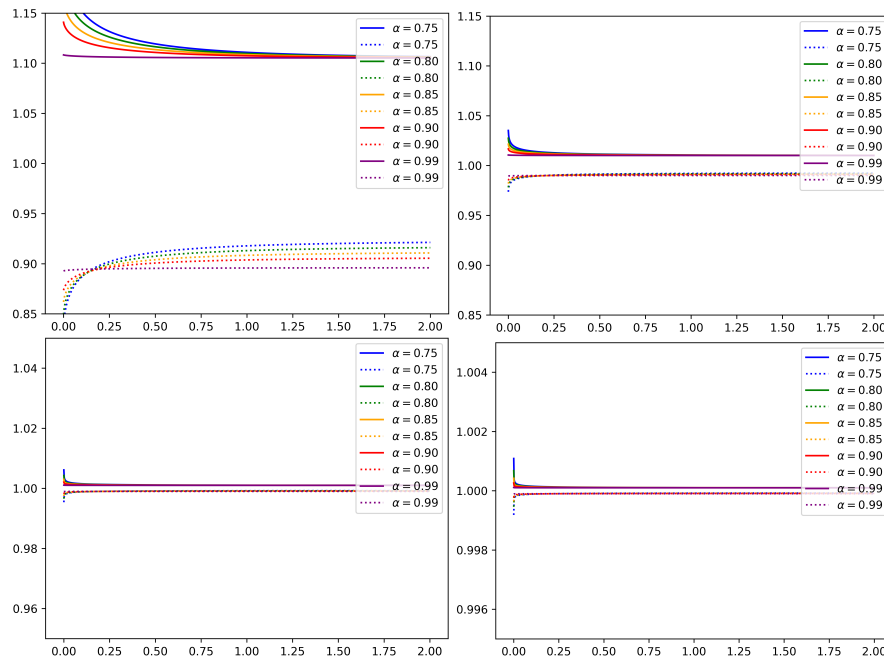
FIGURE 4.1 (A). (i) (i) (top, left), (ii) (top, right), (iii) (bottom, left), (iv) (bottom, right). Comparison, for $0 \leq u \leq 2$ (abscissa), of the LHS (solid lines) and the RHS (dotted lines) of (3.9) (ordinate) for various values of $\alpha \in [0.75, 0.99]$, for (i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$, (iv) $h = 0.0001$.
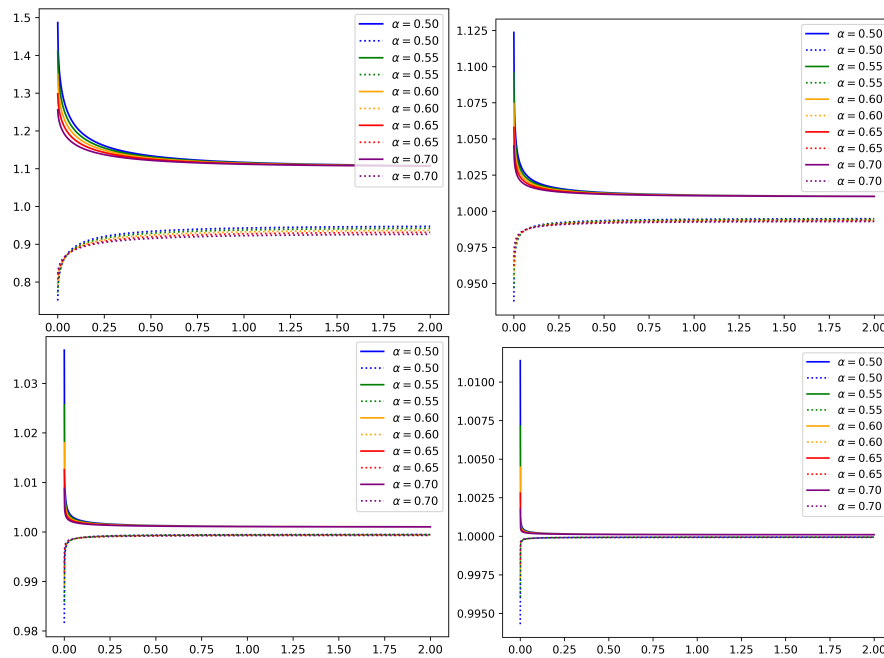

FIGURE 4.1 (B). (i) (top, left), (ii) (top, right), (iii) (bottom, left), (iv) (bottom, right). Comparison, for $0 \leq u \leq 2$ (abscissa), of the LHS (solid lines) and the RHS (dotted lines) of (3.9) (ordinate) for various values of $\alpha \in [0.5, 0.75]$, for (i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$, (iv) $h = 0.0001$.
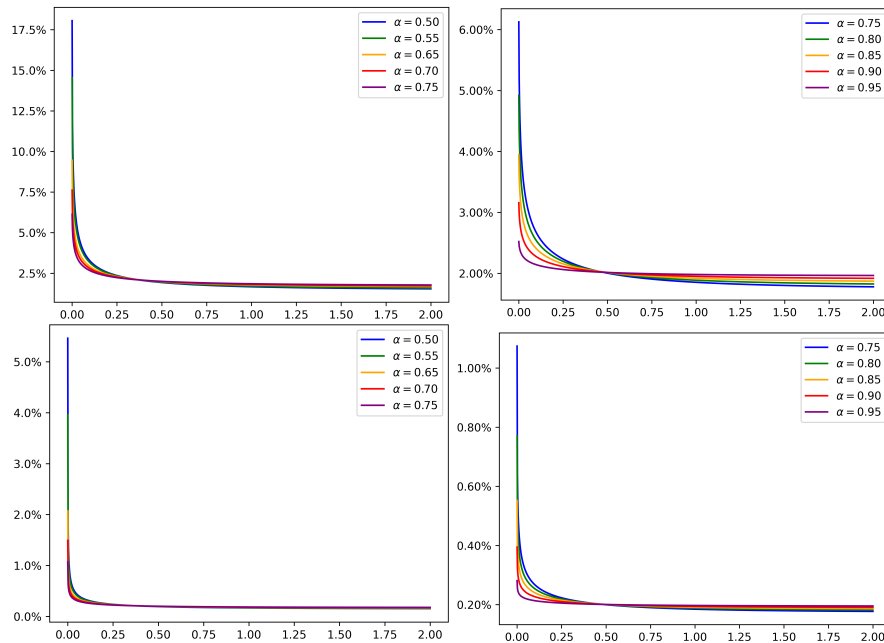
FIGURE 4.2. (a) (top, left), (b) (top, right), (c) (bottom, left), (d) (bottom, right). Percentage differences, for $0 \leq u \leq 2$ (abscissa), between the LHS and the RHS of (3.9) (ordinate) for various values of (a) $\alpha \in [0.5, 0.75]$, $h = 0.01$, (b) $\alpha \in [0.75, 0.95]$, $h = 0.01$, (c) $\alpha \in [0.5, 0.75]$, $h = 0.001$, (d) $\alpha \in [0.75, 0.95]$, $h = 0.001$.

In both Figures 3.1 (a) and 3.1 (b), the RHS and LHS of (3.8) are seen to both increase with decreasing h towards LHS = RHS = 1; in all cases, the two sides are indistinguishable for $\alpha = 0.99$. For argument difference $h = 0.01$ (top right: 3(a-ii), 3 (b-ii)), the difference between each solid line (LHS) and its corresponding dotted line (RHS) is almost negligible, being indistinguishable for $\alpha = 0.99$ and with percentage difference for all $0 < u \leq 2$ of PD $< 0.3\%$ for $\alpha \geq 0.75$ (Fig.3(a-ii)) and PD $< 0.5\%$ for $0.5 \leq \alpha \leq 0.70$ (Fig.3(b-ii)). That difference is seen to decrease significantly for $h < 0.01$ (bottom left: 3(a-iii), 3 (b-iii)) and the two sides become almost indistinguishable for $h < 0.001$ (bottom right: 3(a-iv), 3 (b-iv)) for all $\alpha \geq 0.50$. Figure 3.2 shows the percentage differences for Figures 3.1 (a), (b) data.

In both Figures 4.1 (a) and 4.1 (b), the LHS of (3.9) is seen to decrease while the RHS increases with decreasing h towards LHS = RHS = 1. For argument difference $h = 0.01$ (top right: 4(a-ii), 4(b-ii)), the difference between each solid line (LHS) and its corresponding dotted line (RHS) is small, with percentage difference PD $\leq 2.5\%$ for all $\alpha \geq 0.5$. That difference is seen to decrease significantly and the two sides become almost indistinguishable for $h < 0.01$ (bottom left and right: 4(a-iii, iv), 4(b-iii, iv)), with percentage difference PD $\leq 0.25\%$ for all $\alpha \geq 0.5$. Further comparison for Figures 4 (a), (b) data is shown as percentage differences in Figure 4.2.

5. CONCLUSION

An approximate semigroup property has been described for the Mittag-Leffler (ML) function, $E_\alpha(z)$, a non-local extension of the Euler exponential function e^z. The property, which relates the three ML functions $E_\alpha(\lambda(t+h)^\alpha)$, $E_\alpha(\lambda t^\alpha)$, and $E_{\alpha,\alpha}(\lambda\, t^\alpha)$, is obtained from a recently proposed universal quotient difference representation for fractional derivatives (UQDR). Using the UDQR, whose basic properties are given, the Caputo FD and CFD are represented as difference quotients in terms of, respectively, the ML function and the exponential function. A property of the UDQR is then used to express the Caputo FD in terms of the CFD, which leads to a division approximation that yields the following formula for approximating $E_\alpha(\pm z)$ on a discrete set of points $\{t_n\}$:

$$E_\alpha\left(\pm(t_{n+1})^\alpha\right) \approx E_\alpha(\pm t_n{}^\alpha)\left(1 \pm \frac{1}{\alpha}\frac{E_{\alpha,\alpha}(\pm t_n{}^\alpha)}{E_\alpha(\pm t_n)}\big((t_{n+1})^\alpha - t_n{}^\alpha\big)\right).$$

The derivation also yields an approximate expression for $\alpha \uparrow 1$ that relates the three ML functions referred to above with the exponential function.

Numerical experiments are presented examining the behavior of both the h-approximate and α-approximate formulas as $h \to 0$ and as $\alpha \to 1^-$. It is shown that the division h-approximations are 'reasonably accurate' for $h \leq 0.1$, with virtually no distinction from identity for $h \leq 0.01$. The α-approximations are also shown to be accurate, with virtually no distinction from identity for argument difference $0 < h < 0.01$ and $\alpha \geq 0.5$. Further comparison of the values for Figures 1.1, 2.1, 3.1, 4.1 data is included in Figures 1.2, 2.2, 3.2, 4.2 as respective percentage difference graphs.

Competing interests. The authors declare no competing interests.

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