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THE SOLUTION EXPRESSIONS AND THE PERIODICITY SOLUTIONS OF SOME NONLINEAR DISCRETE SYSTEMS

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ABSTRACT. The principle purpose of this paper is to investigate the long-term behaviors of some nonlinear systems of difference equations and obtain these solution expressions and periodicity character. Furthermore, we use MATLAB programming to simulate the dynamics and confirm our results.

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1. INTRODUCTION

This paper discusses the form of the solutions and the periodic solution character of some fractional difference equations systems

$$z_{n+1} = \frac{t_n t_{n-3}}{z_{n-2}(1 + t_n t_{n-3})}, t_{n+1} = \frac{z_n z_{n-3}}{t_{n-2}(\pm 1 \pm z_n z_{n-3})} \quad (1.1)$$

where the initial values z_{-3}, z_{-2}, z_{-1}, z_0, t_{-3}, t_{-2}, t_{-1} and t_0 are non-zero real numbers.

In the past decades, the theory of discrete dynamical systems consisting of difference equations has been used to explain natural phenomena that change over discrete time. A large number of studies have analyzed many real-life problems that occur in population dynamics, genetics in biology, engineering, queuing problems, electrical networks, physics, economics, etc. Some scientists have recently discussed the long-term behaviors of nonlinear systems of difference equations when the form of these solutions are difficult to obtain. For instance, Gumus et al. [12] investigated the qualitative properties of the behavior, such as local and global stability of the equilibrium points, the existence of unbounded solutions, and periodicity solutions of the following system

$$u_{n+1} = \frac{\alpha u_{n-1}^2}{\beta + \gamma v_{n-2}}, v_{n+1} = \frac{\alpha_1 v_{n-1}^2}{\beta_1 + \gamma_1 u_{n-2}}.$$

Din [9] analyzed and obtained the equilibrium points, local asymptotic stability, and global behavior of the equilibrium points of Lotka-Volterra model which is illustrated by the system

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, y_{n+1} = \frac{\delta y_n - \epsilon x_n y_n}{1 + \eta y_n}$$

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However, obtaining the solution forms of nonlinear systems of difference equations is of great interest to researchers. Here are some recent studies in [5] the obtained theoretical results have been formed and verified numerically by Alayachi et al. for the discrete dynamical systems

$$X_{n+1}=\frac{X_{n-3}Y_{n-4}}{Y_n(1+X_{n-1}Y_{n-2}X_{n-3}Y_{n-4})}\ ,Y_{n+1}=\frac{Y_{n-3}X_{n-4}}{X_n(1+Y_{n-1}X_{n-2}Y_{n-3}X_{n-4})}.$$

Kara et al. [16] illustrated that the following difference equations system can be solved in closed-form

$$x_{n+1}=\frac{x_{n-2}y_{n-3}}{y_{n-1}(a_n+b_nx_{n-2}y_{n-3})}\ ,y_{n+1}=\frac{y_{n-2}x_{n-3}}{x_{n-1}(\alpha_n+\beta_ny_{n-2}x_{n-3})}.$$

In [11] El-Dessoky et al. provided the form of solutions and the periodicity character of rational systems of difference equations

$$T_{n+1}=\frac{T_{n-1}Z_n}{\pm T_n\pm T_{n-1}}\ ,Z_{n+1}=\frac{Z_{n-1}T_n}{\pm Z_n\pm Z_{n-1}}.$$

Touafek et al. [22] investigated the periodic nature and got the solution expressions of the following rational difference equations systems

$$x_{n+1}=\frac{x_{n-3}}{\pm 1\pm x_{n-3}y_{n-1}}\ ,y_{n+1}=\frac{y_{n-3}}{\pm 1\pm y_{n-3}x_{n-1}}.$$

For more related studies on systems of non-linear difference equations, we refer the reader to [1-22] and references cited therein.

2. Main Results

2.1. **The First System.** This subsection discusses the form of solutions of the following system

$$z_{n+1}=\frac{t_nt_{n-3}}{z_{n-2}(1+t_nt_{n-3})}\ ,t_{n+1}=\frac{z_nz_{n-3}}{t_{n-2}(1+z_nz_{n-3})} \tag{2.1}$$

where $n=0,1,2,...$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 2.1. *Assume that $\{z_n,t_n\}$ are solutions of system (2.1). Then for $n\geq 0$, the solutions of system (2.1) can be formed as follows*

$$z_{6n-3}=\frac{A^nD^n}{a^nd^{n-1}}\prod_{i=0}^{n-1}\left(\frac{1+(6i)ad}{1+(6i+3)AD}\right)\ ,\ t_{6n-3}=\frac{a^nd^n}{A^nD^{n-1}}\prod_{i=0}^{n-1}\left(\frac{1+(6i)AD}{1+(6i+3)ad}\right)$$

$$z_{6n-2}=\frac{a^nd^nc}{A^nD^n}\prod_{i=0}^{n-1}\left(\frac{1+(6i+1)AD}{1+(6i+4)ad}\right)\ ,\ t_{6n-2}=\frac{A^nD^nC}{a^nd^n}\prod_{i=0}^{n-1}\left(\frac{1+(6i+1)ad}{1+(6i+4)AD}\right)$$

$$z_{6n-1}=\frac{A^nD^nb}{a^nd^n}\prod_{i=0}^{n-1}\left(\frac{1+(6i+2)ad}{1+(6i+5)AD}\right)\ ,\ t_{6n-1}=\frac{a^nd^nB}{A^nD^n}\prod_{i=0}^{n-1}\left(\frac{1+(6i+2)AD}{1+(6i+5)ad}\right)$$

$$z_{6n}=\frac{a^{n+1}d^n}{A^nD^n}\prod_{i=0}^{n-1}\left(\frac{1+(6i+3)AD}{1+(6i+6)ad}\right)\ ,\ t_{6n}=\frac{A^{n+1}D^n}{a^nd^n}\prod_{i=0}^{n-1}\left(\frac{1+(6i+3)ad}{1+(6i+6)AD}\right)$$

$$z_{6n+1}=\frac{A^{n+1}D^{n+1}}{a^nd^nc(1+AD)}\prod_{i=0}^{n-1}\left(\frac{1+(6i+4)ad}{1+(6i+7)AD}\right)\ ,\ t_{6n+1}=\frac{a^{n+1}d^{n+1}}{A^nD^nC(1+ad)}\prod_{i=0}^{n-1}\left(\frac{1+(6i+4)AD}{1+(6i+7)ad}\right)$$

$$z_{6n+2}=\frac{a^{n+1}d^{n+1}}{A^nD^nb(1+2ad)}\prod_{i=0}^{n-1}\left(\frac{1+(6i+5)AD}{1+(6i+8)ad}\right)\ ,\ t_{6n+2}=\frac{A^{n+1}D^{n+1}}{a^nd^nB(1+2AD)}\prod_{i=0}^{n-1}\left(\frac{1+(6i+5)ad}{1+(6i+8)AD}\right)$$

where $z_{-3}=d$, $z_{-2}=c$, $z_{-1}=b$, $z_0=a$, $t_{-3}=D$, $t_{-2}=C$, $t_{-1}=B$, and $t_0=A$.

Proof. It is clear that for $n = 0$ the results are true. Now for $n > 0$, assume the results hold for $n-1$ and they are given as follows

$$z_{6n-9} = \frac{A^{n-1}D^{n-1}}{a^{n-1}d^{n-2}} \prod_{i=0}^{n-2} \Big(\frac{1+(6i)ad}{1+(6i+3)AD}\Big) \ , \ t_{6n-9} = \frac{a^{n-1}d^{n-1}}{A^{n-1}D^{n-2}} \prod_{i=0}^{n-2} \Big(\frac{1+(6i)AD}{1+(6i+3)ad}\Big)$$

$$z_{6n-8} = \frac{a^{n-1}d^{n-1}c}{A^{n-1}D^{n-1}} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+1)AD}{1+(6i+4)ad}\Big) \ , \ t_{6n-2} = \frac{A^{n-1}D^{n-1}C}{a^{n-1}d^{n-1}} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+1)ad}{1+(6i+4)AD}\Big)$$

$$z_{6n-7} = \frac{A^{n-1}D^{n-1}b}{a^{n-1}d^{n-1}} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+2)ad}{1+(6i+5)AD}\Big) \ , \ t_{6n-1} = \frac{a^{n-1}d^{n-1}B}{A^{n-1}D^{n-1}} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+2)AD}{1+(6i+5)ad}\Big)$$

$$z_{6n-6} = \frac{a^{n}d^{n-1}}{A^{n-1}D^{n-1}} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+3)AD}{1+(6i+6)ad}\Big) \ , \ t_{6n-6} = \frac{A^{n}D^{n-1}}{a^{n-1}d^{n-1}} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+3)ad}{1+(6i+6)AD}\Big)$$

$$z_{6n-5} = \frac{A^{n}D^{n}}{a^{n-1}d^{n-1}c(1+AD)} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+4)ad}{1+(6i+7)AD}\Big) \ ,$$

$$t_{6n-5} = \frac{a^{n}d^{n}}{A^{n-1}D^{n-1}C(1+ad)} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+4)AD}{1+(6i+7)ad}\Big)$$

$$z_{6n-4} = \frac{a^{n}d^{n}}{A^{n-1}D^{n-1}b(1+2ad)} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+5)AD}{1+(6i+8)ad}\Big) \ ,$$

$$t_{6n-4} = \frac{A^{n}D^{n}}{a^{n-1}d^{n-1}B(1+2AD)} \prod_{i=0}^{n-2} \Big(\frac{1+(6i+5)ad}{1+(6i+8)AD}\Big)$$

Now, the first relation will be proven.
After substituting $6n-3$ into system (2.1). We get

$$z_{6n-3} = \frac{t_{6n-4}t_{6n-7}}{z_{6n-6}(1+t_{6n-4}t_{6n-7})} \ ,$$

$$z_{6n-3} = \frac{\frac{A^{n}D^{n}}{a^{n-1}d^{n-1}B(1+2AD)} \prod_{i=0}^{n-2} \left(\frac{1+(6i+5)ad}{1+(6i+8)AD}\right) \frac{a^{n-1}d^{n-1}B}{A^{n-1}D^{n-1}} \prod_{i=0}^{n-2} \left(\frac{1+(6i+2)AD}{1+(6i+5)ad}\right)}{\frac{a^{n}d^{n-1}}{A^{n-1}D^{n-1}} \prod_{i=0}^{n-2} \left(\frac{1+(6i+3)AD}{1+(6i+6)ad}\right)\left(1+\frac{AD}{1+2AD} \prod_{i=0}^{n=2} \left(\frac{1+(6i+2)AD}{1+(6i+8)AD}\right)\right.},$$

so,

$$z_{6n-3} = \frac{\frac{AD}{1+(6n-4)AD}}{\frac{a^{n}d^{n-1}}{A^{n-1}D^{n-1}} \prod_{i=0}^{n-2} \left(\frac{1+(6i+3)AD}{1+(6i+6)ad}\right)\left(\frac{1+(6n-4)AD+AD}{1+(6n-4)AD}\right)}$$

$$z_{6n-3} = \frac{A^{n}D^{n}}{(a^{n}d^{n-1})(1+(6n-3)AD} \prod_{i=0}^{n-2} \frac{1+(6i+6)ad}{1+(6i+3)AD}.$$

Therefore,

$$z_{6n-3} = \frac{A^{n}D^{n}}{a^{n}d^{n-1}} \prod_{i=0}^{n-1} \Big(\frac{1+(6i)ad}{1+(6i+3)AD}\Big).$$

$$t_{6n-3} = \frac{z_{6n-4}z_{6n-7}}{t_{6n-6}(1+z_{6n-4}z_{6n-7})} \ ,$$

$$t_{6n-3} = \frac{\frac{a^{n}d^{n}}{A^{n-1}D^{n-1}b(1+2ad)} \prod_{i=0}^{n-2} \left(\frac{1+(6i+5)AD}{1+(6i+8)ad}\right) \frac{A^{n-1}D^{n-1}b}{a^{n-1}d^{n-1}} \prod_{i=0}^{n-2} \left(\frac{1+(6i+2)ad}{1+(6i+5)AD}\right)}{\frac{A^{n}D^{n-1}}{a^{n-1}d^{n-1}} \prod_{i=0}^{n-2} \left(\frac{1+(6i+3)ad}{1+(6i+6)AD}\right)\left(1+\frac{ad}{1+2ad} \prod_{i=0}^{n=2} \left(\frac{1+(6i+2)ad}{1+(6i+8)ad}\right)\right.},$$

then,

$$t_{6n-3} = \frac{\frac{ad}{1+(6n-4)ad}}{\frac{A^n D^{n-1}}{a^{n-1}d^{n-1}} \prod_{i=0}^{n-2} \left(\frac{1+(6i+3)ad}{1+(6i+6)AD}\right)\left(\frac{1+(6n-4)ad+ad}{1+(6n-4)ad}\right)}$$

$$t_{6n-3} = \frac{a^n d^n}{(A^n D^{n-1})(1+(6n-3)ad} \prod_{i=0}^{n-2} \frac{1+(6i+6)AD}{1+(6i+3)ad}.$$

Consequently,

$$t_{6n-3} = \frac{a^n d^n}{A^n D^{n-1}} \prod_{i=0}^{n-1} \left(\frac{1+(6i)AD}{1+(6i+3)ad}\right)$$

Similarly, we can prove the remaining relations. The proof is complete. □

We simulate the difference equations system (2.1) numerically by using MATLAB programming. Figure 1 shows the behavior of the system under the random values $z_{-3} = 0.4$, $z_{-2} = -1.5$, $z_{-1} = 4$, $z_0 = 1$, $t_{-3} = 2$, $t_{-2} = -1.3$, $t_{-1} = 5$ and $t_0 = 0.5$.

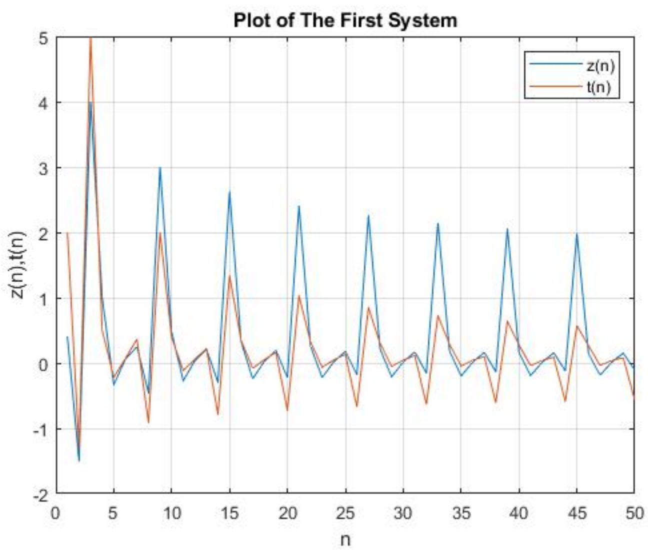

FIGURE 1. The Dynamics of The Solution of The System (2.1)

2.2. **The Second System.** In this subsection, we provide the solution expression and periodic solutions of period six for the following system.

$$z_{n+1} = \frac{t_n t_{n-3}}{z_{n-2}(1+t_n t_{n-3})}, t_{n+1} = \frac{z_n z_{n-3}}{t_{n-2}(1-z_n z_{n-3})} \tag{2.2}$$

where $n = 0, 1, 2, ...$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 2.2. *Let $\{z_n, t_n\}$ be solutions of the system of non-linear difference equations (2.2). Then for $n = 0, 1, 2, ..$, the solutions of system (2.2) can be formed as follows*

$$z_{6n-3} = \frac{A^n D^n}{a^n d^{n-1}(1+AD)^n}, \quad t_{6n-3} = \frac{a^n d^n}{A^n D^{n-1}(1-ad)^n}$$

$$z_{6n-2} = \frac{a^n d^n c(1+AD)^n}{A^n D^n}, \quad t_{6n-3} = \frac{A^n D^n C(1-ad)^n}{a^n d^n}$$

$$z_{6n-1}=\frac{A^nD^nb}{a^nd^n(1+AD)^n}\ ,\ t_{6n-1}=\frac{a^nd^nB}{A^nD^n(1-ad)^n}$$

$$z_{6n}=\frac{a^{n+1}d^n(1+AD)^n}{A^nD^n}\ ,\ t_{6n}=\frac{A^{n+1}D^n(1-ad)^n}{a^nd^n}$$

$$z_{6n+1}=\frac{A^{n+1}D^{n+1}}{a^nd^nc(1+AD)^{n+1}}\ ,\ t_{6n+1}=\frac{a^{n+1}d^{n+1}}{A^nD^nC(1-ad)^{n+1}}$$

$$z_{6n+2}=\frac{a^{n+1}d^{n+1}(1+AD)^n}{A^nD^nb}\ ,\ t_{6n+2}=\frac{A^{n+1}D^{n+1}(1-ad)^n}{a^nd^nB}$$

Proof. The results are true for $n=0$. Now for $n>0$, assume the results hold for $n-1$ and they are given as follows

$$z_{6n-9}=\frac{A^{n-1}D^{n-1}}{a^{n-1}d^{n-2}(1+AD)^{n-1}}\ ,\ t_{6n-9}=\frac{a^{n-1}d^{n-1}}{A^{n-1}D^{n-2}(1-ad)^{n-1}}$$

$$z_{6n-8}=\frac{a^{n-1}d^{n-1}c(1+AD)^{n-1}}{A^{n-1}D^{n-1}}\ ,\ t_{6n-8}=\frac{A^{n-1}D^{n-1}C(1-ad)^{n-1}}{a^{n-1}d^{n-1}}$$

$$z_{6n-7}=\frac{A^{n-1}D^{n-1}b}{a^{n-1}d^{n-1}(1+AD)^{n-1}}\ ,\ t_{6n-7}=\frac{a^{n-1}d^{n-1}B}{A^{n-1}D^{n-1}(1-ad)^{n-1}}$$

$$z_{6n-6}=\frac{a^nd^{n-1}(1+AD)^{n-1}}{A^{n-1}D^{n-1}}\ ,\ t_{6n-6}=\frac{A^nD^{n-1}(1-ad)^{n-1}}{a^{n-1}d^{n-1}}$$

$$z_{6n-5}=\frac{A^nD^n}{a^{n-1}d^{n-1}c(1+AD)^n}\ ,\ t_{6n-5}=\frac{a^nd^n}{A^{n-1}D^{n-1}C(1-ad)^n}$$

$$z_{6n-4}=\frac{a^nd^n(1+AD)^{n-1}}{A^{n-1}D^{n-1}b}\ ,\ t_{6n-4}=\frac{A^nD^n(1-ad)^{n-1}}{a^{n-1}d^{n-1}B}.$$

Now, we prove the first relation.
Substituting $6n-3$ into system (2.2). We get

$$z_{6n-3}=\frac{t_{6n-4}t_{6n-7}}{z_{6n-6}(1+t_{6n-4}t_{6n-7})}\ ,$$

$$z_{6n-3}=\frac{\frac{A^nD^n(1-ad)^{n-1}}{a^{n-1}d^{n-1}B}\frac{a^{n-1}d^{n-1}B}{A^{n-1}D^{n-1}(1-ad)^{n-1}}}{\frac{a^nd^{n-1}(1+AD)^{n-1}}{A^{n-1}D^{n-1}}\left(1+\frac{A^nD^n(1-ad)^{n-1}}{a^{n-1}d^{n-1}B}\frac{a^{n-1}d^{n-1}B}{A^{n-1}D^{n-1}(1-ad)^{n-1}}\right)}\ ,$$

thus,

$$z_{6n-3}=\frac{AD}{\frac{a^nd^{n-1}(1+AD)^{n-1}}{A^{n-1}D^{n-1}}\left(1+AD\right)}\ .$$

Therefore,

$$z_{6n-3}=\frac{A^nD^n}{a^nd^{n-1}(1+AD)^n}\ .$$

$$t_{6n-3}=\frac{z_{6n-4}z_{6n-7}}{t_{6n-6}(1+z_{6n-4}z_{6n-7})}\ ,$$

$$t_{6n-3}=\frac{\frac{a^nd^n(1+AD)^{n-1}}{A^{n-1}D^{n-1}b}\frac{A^{n-1}D^{n-1}b}{a^{n-1}d^{n-1}(1+AD)^{n-1}}}{\frac{A^nD^{n-1}(1-ad)^{n-1}}{a^{n-1}d^{n-1}}\left(1+\frac{a^nd^n(1+AD)^{n-1}}{A^{n-1}D^{n-1}b}\frac{A^{n-1}D^{n-1}b}{a^{n-1}d^{n-1}(1+AD)^{n-1}}\right)}\ ,$$

so,

$$t_{6n-3}=\frac{ad}{\frac{A^nD^{n-1}(1+ad)^{n-1}}{a^{n-1}d^{n-1}}\left(1+ad\right)}\ .$$

Hence,

$$t_{6n-3}=\frac{a^nd^n}{A^nD^{n-1}(1+ad)^n}\ .$$

Following the same approach, we can verify the other forms. The proof is complete. □

Theorem 2.3. *The system of non-linear difference equation (2.2) has a periodic solution of period six iff $AD=-2$, $ad=2$ and it will take the following form*

$$\{z_n\}=\{d,c,b,a,\frac{-AD}{c},\frac{ad}{b},d,c,b,a,\frac{-AD}{c},\frac{ad}{b},...\},$$
$$\{t_n\}=\{D,C,B,A,\frac{-ad}{C},\frac{AD}{B},D,C,B,A,\frac{-ad}{C},\frac{AD}{B},...\}.$$

Proof. Suppose that a prime period six solution exists

$$\{z_n\}=\{d,c,b,a,\frac{-AD}{c},\frac{ad}{b},d,c,b,a,\frac{-AD}{c},\frac{ad}{b},...\},$$
$$\{t_n\}=\{D,C,B,A,\frac{-ad}{C},\frac{AD}{B},D,C,B,A,\frac{-ad}{C},\frac{AD}{B},...\}$$

of system (2.2). Then, we can recognize from the form of the solution of system (2.2) that

$$d=\frac{A^nD^n}{a^nd^{n-1}(1+AD)^n}\ ,\ D=\frac{a^nd^n}{A^nD^{n-1}(1-ad)^n}$$

$$c=\frac{a^nd^nc(1+AD)^n}{A^nD^n}\ ,\ C=\frac{A^nD^nC(1-ad)^n}{a^nd^n}$$

$$b=\frac{A^nD^nb}{a^nd^n(1+AD)^n}\ ,\ B=\frac{a^nd^nB}{A^nD^n(1-ad)^n}$$

$$a=\frac{a^{n+1}d^n(1+AD)^n}{A^nD^n}\ ,\ A=\frac{A^{n+1}D^n(1-ad)^n}{a^nd^n}$$

$$\frac{-AD}{c}=\frac{A^{n+1}D^{n+1}}{a^nd^nc(1+AD)^{n+1}}\ ,\ \frac{-ad}{C}=\frac{a^{n+1}d^{n+1}}{A^nD^nC(1-ad)^{n+1}}$$

$$\frac{ad}{b}=\frac{a^{n+1}d^{n+1}(1+AD)^n}{A^nD^nb}\ ,\ \frac{AD}{B}=\frac{A^{n+1}D^{n+1}(1-ad)^n}{a^nd^nB}.$$

So,

$$AD=-2,\ \ ad=2.$$

Then, assmue that $AD=-2$, $ad=2$. Consequently, we see from the form of solution of system (2.2) that

$$z_{6n-3}=d\ ,\ t_{6n-3}=D$$

$$z_{6n-2}=c,\ t_{6n-3}=C$$

$$z_{6n-1}=b,\ t_{6n-1}=B$$

$$z_{6n}=a,\ t_{6n}=A$$

$$z_{6n+1}=\frac{-AD}{c}\ ,\ t_{6n+1}=\frac{-ad}{C}$$

$$z_{6n+2}=\frac{ad}{b},\ t_{6n+2}=\frac{AD}{B}.$$

Therefore, we have a periodic solution of period six. The proof is complete. □

The following figures are confirmed our theoretical results numerically by simulating the system (2.2). Figure (2) illustrates the behavior of the system when the initial values are $z_{-3} = 1$, $z_{-2} = -0.5$, $z_{-1} = 2$, $z_0 = 2.3$, $t_{-3} = 4$, $t_{-2} = -1.1$, $t_{-1} = 4$ and $t_0 = -0.5$. While figure (3), displays that the solution is periodic of period six when the initial conditions are $z_{-3} = 4$, $z_{-2} = -5$, $z_{-1} = 7$, $z_0 = 0.5$, $t_{-3} = 4$, $t_{-2} = -9$, $t_{-1} = 3$ and $t_0 = -0.5$. And it is clear that the initial values satisfied the condition of **Theorem (2.3)**.

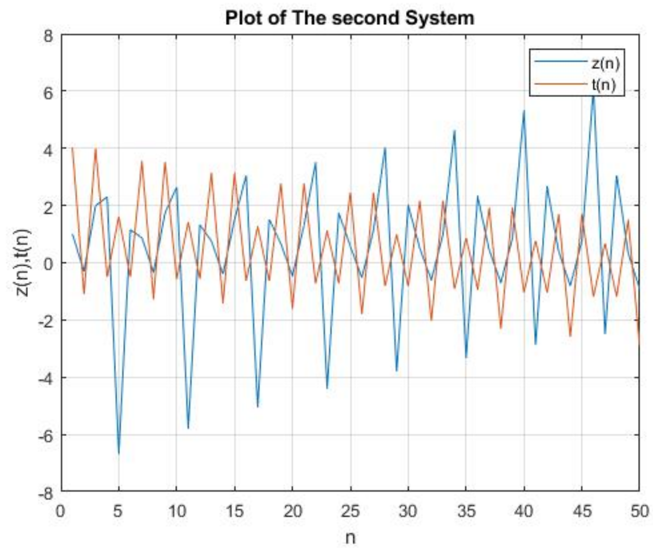

FIGURE 2. The Dynamics of The Solution of The System (2.2)

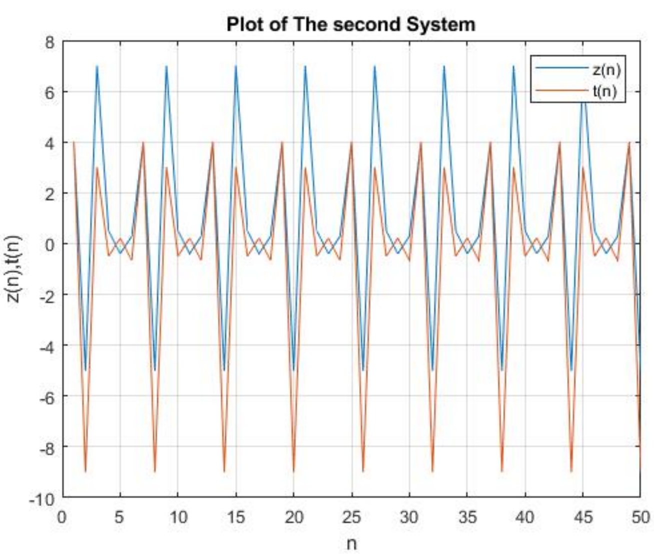

FIGURE 3. The Dynamics of The Solution of The System (2.2)

2.3. **The Third System.** In this part, the solution expressions is explored. Moreover, we illustrate the periodicity character of the following system

$$z_{n+1} = \frac{t_n t_{n-3}}{z_{n-2}(1 + t_n t_{n-3})}\ ,t_{n+1} = \frac{z_n z_{n-3}}{t_{n-2}(-1 - z_n z_{n-3})} \tag{2.3}$$

where $n = 0, 1, 2, ...$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 2.4. *Assume that* $\{z_n, t_n\}$ *are solutions of the system (2.3). Then for* $n = 0, 1, 2, 3, ..$ *the solutions of system (2.3) can be formed as follows*

$$z_{12n-3} = \frac{A^{2n}D^{2n}}{a^{2n}d^{2n-1}(1+AD)^{2n}}\ ,\ t_{12n-3} = \frac{(-1)^n a^{2n}d^{2n}(-1-2AD)^n}{A^{2n}D^{2n-1}(-1+ad)^n(-1-ad)^n}$$

$$z_{12n-2} = \frac{a^{2n}d^{2n}c(1+AD)^{2n}}{A^{2n}D^{2n}}\ ,\ t_{12n-2} = \frac{A^{2n}D^{2n}C(-1-ad)^n(-1+ad)^n}{(-1)^n a^{2n}d^{2n}(-1-2AD)^n}$$

$$z_{12n-1} = \frac{A^{2n}D^{2n}b}{a^{2n}d^{2n}(-1-AD)^{2n}}\ ,\ t_{12n-1} = \frac{(-1)^n a^{2n}d^{2n}B(-1-2AD)^n}{A^{2n}D^{2n}(-1+ad)^n(-1-ad)^n}$$

$$z_{12n} = \frac{a^{2n+1}d^{2n}(1+AD)^{2n}}{A^{2n}D^{2n}}\ ,\ t_{12n} = \frac{A^{2n+1}D^{2n}(-1-ad)^n(-1+ad)^n}{(-1)^n a^{2n}d^{2n}(-1-2AD)^n}$$

$$z_{12n+1} = \frac{A^{2n+1}D^{2n+1}}{a^{2n}d^{2n}c(1+AD)^{2n+1}}\ ,\ t_{12n+1} = \frac{(-1)^n a^{2n+1}d^{2n+1}(-1-2AD)^n}{A^{2n}D^{2n}C(-1+ad)^n(-1-ad)^{n+1}}$$

$$z_{12n+2} = \frac{a^{2n+1}d^{2n+1}(-1-AD)^{2n}}{-A^{2n}D^{2n}b}\ ,\ t_{12n+2} = \frac{A^{2n+1}D^{2n+1}(-1-ad)^n(-1+ad)^n}{(-1)^n a^{2n}d^{2n}B(-1-2AD)^{n+1}}$$

$$z_{12n+3} = \frac{A^{2n+1}D^{2n+1}}{-a^{2n+1}d^{2n}(1+AD)^{2n+1}}\ ,\ t_{12n+3} = \frac{(-1)^{n+1} a^{2n+1}d^{2n+1}(-1-2AD)^n}{A^{2n+1}D^{2n}(-1+ad)^{n+1}(-1-ad)^n}$$

$$z_{12n+4} = \frac{a^{2n+1}d^{2n+1}c(1+AD)^{2n+1}}{A^{2n+1}D^{2n+1}}\ ,\ t_{12n+4} = \frac{A^{2n+1}D^{2n+1}C(-1-ad)^{n+1}(-1+ad)^n}{(-1)^n a^{2n+1}d^{2n+1}(-1-2AD)^n}$$

$$z_{12n+5} = \frac{A^{2n+1}D^{2n+1}b}{a^{2n+1}d^{2n+1}(-1-AD)^{2n+1}}\ ,\ t_{12n+5} = \frac{(-1)^n a^{2n+1}d^{2n+1}B(-1-2AD)^{n+1}}{A^{2n+1}D^{2n+1}(-1+ad)^n(-1-ad)^{n+1}}$$

$$z_{12n+6} = \frac{a^{2n+2}d^{2n+1}(1+AD)^{2n+1}}{A^{2n+1}D^{2n+1}}\ ,\ t_{12n+6} = \frac{A^{2n+2}D^{2n+1}(-1-ad)^n(-1+ad)^{n+1}}{(-1)^{n+1} a^{2n+1}d^{2n+1}(-1-2AD)^{n+1}}$$

$$z_{12n+7} = \frac{A^{2n+2}D^{2n+2}}{-a^{2n+1}d^{2n+1}c(1+AD)^{2n+2}}\ ,\ t_{12n+7} = \frac{(-1)^{n+1} a^{2n+2}d^{2n+2}(-1-2AD)^n}{A^{2n+1}D^{2n+1}C(-1+ad)^{n+1}(-1-ad)^{n+1}}$$

$$z_{12n+8} = \frac{a^{2n+2}d^{2n+2}(-1-AD)^{2n+1}}{A^{2n+1}D^{2n+1}b}\ ,\ t_{12n+8} = \frac{A^{2n+2}D^{2n+2}(-1-ad)^{n+1}(-1+ad)^n}{(-1)^n a^{2n+1}d^{2n+1}B(-1-2AD)^{n+1}}$$

Proof. The results are true for $n = 0$. Now for $n > 0$, assume the results hold for $n - 1$ and they are given as follows

$$z_{12n-15} = \frac{A^{2n-2}D^{2n-2}}{a^{2n-2}d^{2n-3}(1+AD)^{2n-2}}\ ,\ t_{12n-15} = \frac{(-1)^{n-1} a^{2n-2}d^{2n-2}(-1-2AD)^{n-1}}{A^{2n-2}D^{2n-3}(-1+ad)^{n-1}(-1-ad)^{n-1}}$$

$$z_{12n-14} = \frac{a^{2n-2}d^{2n-2}c(1+AD)^{2n-2}}{A^{2n-2}D^{2n-2}}\ ,\ t_{12n-14} = \frac{A^{2n-2}D^{2n-2}C(-1-ad)^{n-1}(-1+ad)^{n-1}}{(-1)^{n-1} a^{2n-2}d^{2n-2}(-1-2AD)^{n-1}}$$

$$z_{12n-13} = \frac{A^{2n-2}D^{2n-2}b}{a^{2n-2}d^{2n-2}(-1-AD)^{2n-2}}\ ,\ t_{12n-13} = \frac{(-1)^{n-1} a^{2n-2}d^{2n-2}B(-1-2AD)^{n-1}}{A^{2n-2}D^{2n-2}(-1+ad)^{n-1}(-1-ad)^{n-1}}$$

$$z_{12n-12} = \frac{a^{2n-1}d^{2n-2}(1+AD)^{2n-2}}{A^{2n-2}D^{2n-2}}\ ,\ t_{12n-12} = \frac{A^{2n-1}D^{2n-2}(-1-ad)^{n-1}(-1+ad)^{n-1}}{(-1)^{n-1} a^{2n-2}d^{2n-2}(-1-2AD)^{n-1}}$$

$$z_{12n-11} = \frac{A^{2n-1}D^{2n-1}}{a^{2n-2}d^{2n-2}c(1+AD)^{2n-1}}\ ,\ t_{12n-11} = \frac{(-1)^{n-1} a^{2n-1}d^{2n-1}(-1-2AD)^{n-1}}{A^{2n-2}D^{2n-2}C(-1+ad)^{n-1}(-1-ad)^n}$$

$$z_{12n-10}=\frac{a^{2n-1}d^{2n-1}(-1-AD)^{2n-2}}{-A^{2n-2}D^{2n-2}b}\ ,\ t_{12n-10}=\frac{A^{2n-1}D^{2n-1}(-1-ad)^{n-1}(-1+ad)^{n-1}}{(-1)^{n-1}a^{2n-2}d^{2n-2}B(-1-2AD)^{n}}$$

$$z_{12n-9}=\frac{A^{2n-1}D^{2n-1}}{-a^{2n-1}d^{2n-2}(1+AD)^{2n-1}}\ ,\ t_{12n-9}=\frac{(-1)^{n}a^{2n-1}d^{2n-1}(-1-2AD)^{n-1}}{A^{2n-1}D^{2n-2}(-1+ad)^{n}(-1-ad)^{n-1}}$$

$$z_{12n-8}=\frac{a^{2n-1}d^{2n-1}c(1+AD)^{2n-1}}{A^{2n-1}D^{2n-1}}\ ,\ t_{12n-8}=\frac{A^{2n-1}D^{2n-1}C(-1-ad)^{n}(-1+ad)^{n-1}}{(-1)^{n-1}a^{2n-1}d^{2n-1}(-1-2AD)^{n-1}}$$

$$z_{12n-7}=\frac{A^{2n-1}D^{2n-1}b}{a^{2n-1}d^{2n-1}(-1-AD)^{2n-1}}\ ,\ t_{12n-7}=\frac{(-1)^{n-1}a^{2n-1}d^{2n-1}B(-1-2AD)^{n}}{A^{2n-1}D^{2n-1}(-1+ad)^{n-1}(-1-ad)^{n}}$$

$$z_{12n-6}=\frac{a^{2n}d^{2n-1}(1+AD)^{2n-1}}{A^{2n-1}D^{2n-1}}\ ,\ t_{12n-6}=\frac{A^{2n}D^{2n-1}(-1-ad)^{n-1}(-1+ad)^{n}}{(-1)^{n}a^{2n-1}d^{2n-1}(-1-2AD)^{n}}$$

$$z_{12n-5}=\frac{A^{2n}D^{2n}}{-a^{2n-1}d^{2n-1}c(1+AD)^{2n}}\ ,\ t_{12n-5}=\frac{(-1)^{n}a^{2n}d^{2n}(-1-2AD)^{n-1}}{A^{2n-1}D^{2n-1}C(-1+ad)^{n}(-1-ad)^{n}}$$

$$z_{12n-4}=\frac{a^{2n}d^{2n}(-1-AD)^{2n-1}}{A^{2n-1}D^{2n-1}b}\ ,\ t_{12n-4}=\frac{A^{2n}D^{2n}(-1-ad)^{n}(-1+ad)^{n-1}}{(-1)^{n-1}a^{2n-1}d^{2n-1}B(-1-2AD)^{n}}.$$

We prove the first form.

Substituting $12n-3$ into the system of difference equations (2.3). We get

$$z_{12n-3}=\frac{t_{12n-4}t_{12n-7}}{z_{12n-6}(1+t_{12n-4}t_{12n-7})}\ ,$$

$$z_{12n-3}=\frac{\frac{A^{2n}D^{2n}(-1-ad)^{n}(-1+ad)^{n-1}}{(-1)^{n-1}a^{2n-1}d^{2n-1}B(-1-2AD)^{n}}\frac{(-1)^{n-1}a^{2n-1}d^{2n-1}B(-1-2AD)^{n}}{A^{2n-1}D^{2n-1}(-1+ad)^{n-1}(-1-ad)^{n}}}{\frac{a^{2n}d^{2n-1}(1+AD)^{2n-1}}{A^{2n-1}D^{2n-1}}(1+AD)}\ ,$$

so,

$$z_{12n-3}=\frac{AD}{\frac{a^{2n}d^{2n-1}(1+AD)^{2n-1}}{A^{2n-1}D^{2n-1}}(1+AD)}\ .$$

Consequently,

$$z_{12n-3}=\frac{A^{2n}D^{2n}}{a^{2n}d^{2n-1}(1+AD)^{2n}}\ .$$

$$t_{12n-3}=\frac{z_{12n-4}z_{12n-7}}{t_{12n-6}(-1-t_{12n-4}t_{12n-7})}\ ,$$

$$t_{12n-3}=\frac{\frac{a^{2n}d^{2n}(-1-AD)^{2n-1}}{A^{2n-1}D^{2n-1}b}\frac{A^{2n-1}D^{2n-1}b}{a^{2n-1}d^{2n-1}(-1-AD)^{2n-1}}}{\frac{A^{2n}D^{2n-1}(-1-ad)^{n-1}(-1+ad)^{n}}{(-1)^{n}a^{2n-1}d^{2n-1}(-1-2AD)^{n}}(-1-ad)}\ ,$$

then,

$$t_{12n-3}=\frac{ad}{\frac{A^{2n}D^{2n-1}(-1-ad)^{n-1}(-1+ad)^{n}}{(-1)^{n}a^{2n-1}d^{2n-1}(-1-2AD)^{n}}(-1-ad)}\ .$$

Therefore,

$$t_{12n-3}=\frac{(-1)^{n}a^{2n}d^{2n}(-1-2AD)^{n}}{A^{2n}D^{2n-1}(-1-ad)^{n}(-1+ad)^{n}}\ .$$

The proof is completed . □

Theorem 2.5. *The system of non-linear difference equation (2.3) has a periodic solution of period twelve iff* $AD = -2$, $ad = 2$ *and it will take the following form*

$$\{z_n\} = \{d, c, b, a, \frac{-AD}{c}, \frac{-ad}{b}, \frac{AD}{a}, c, -b, a, \frac{AD}{c}, \frac{-ad}{b}...\},$$

$$\{t_n\} = \left\{D, C, B, A, \tfrac{ad}{C(-1-ad)}, \tfrac{-AD}{B(-1-2AD)}, \tfrac{-ad}{A}, -C(-1-ad), B, \tfrac{A}{(-1-2AD)}, \tfrac{-ad}{(-1-ad)}, \tfrac{AD}{B}, ...\right\}$$

Proof. Suppose that a prime period twelve solution exists

$$\{z_n\} = \{d, c, b, a, \frac{-AD}{c}, \frac{-ad}{b}, \frac{AD}{a}, c, -b, a, \frac{AD}{c}, \frac{-ad}{b}...\},$$

$$\{t_n\} = \left\{D, C, B, A, \tfrac{ad}{C(-1-ad)}, \tfrac{-AD}{B(-1-2AD)}, \tfrac{-ad}{A}, -C(-1-ad), B, \tfrac{A}{(-1-2AD)}, \tfrac{-ad}{(-1-ad)}, \tfrac{AD}{B}, ...\right\}$$

of the system (2.3). Then, we see from the solution's form of system (2.3) that

$$d = \frac{A^{2n}D^{2n}}{a^{2n}d^{2n-1}(1+AD)^{2n}}\ ,\ D = \frac{(-1)^n a^{2n}d^{2n}(-1-2AD)^n}{A^{2n}D^{2n-1}(-1+ad)^n(-1-ad)^n}$$

$$c = \frac{a^{2n}d^{2n}c(1+AD)^{2n}}{A^{2n}D^{2n}}\ ,\ C = \frac{A^{2n}D^{2n}C(-1-ad)^n(-1+ad)^n}{(-1)^n a^{2n}d^{2n}(-1-2AD)^n}$$

$$b = \frac{A^{2n}D^{2n}b}{a^{2n}d^{2n}(-1-AD)^{2n}}\ ,\ B = \frac{(-1)^n a^{2n}d^{2n}B(-1-2AD)^n}{A^{2n}D^{2n}(-1+ad)^n(-1-ad)^n}$$

$$a = \frac{a^{2n+1}d^{2n}(1+AD)^{2n}}{A^{2n}D^{2n}}\ ,\ A = \frac{A^{2n+1}D^{2n}(-1-ad)^n(-1+ad)^n}{(-1)^n a^{2n}d^{2n}(-1-2AD)^n}$$

$$\frac{-AD}{c} = \frac{A^{2n+1}D^{2n+1}}{a^{2n}d^{2n}c(1+AD)^{2n+1}}\ ,\ \frac{ad}{C(-1-ad)} = \frac{(-1)^n a^{2n+1}d^{2n+1}(-1-2AD)^n}{A^{2n}D^{2n}C(-1+ad)^n(-1-ad)^{n+1}}$$

$$\frac{-ad}{b} = \frac{a^{2n+1}d^{2n+1}(-1-AD)^{2n}}{-A^{2n}D^{2n}b}\ ,\ \frac{-AD}{B(-1-2AD)} = \frac{A^{2n+1}D^{2n+1}(-1-ad)^n(-1+ad)^n}{(-1)^n a^{2n}d^{2n}B(-1-2AD)^{n+1}}$$

$$\frac{AD}{a} = \frac{A^{2n+1}D^{2n+1}}{-a^{2n+1}d^{2n}(1+AD)^{2n+1}}\ ,\ \frac{-ad}{A} = \frac{(-1)^{n+1}a^{2n+1}d^{2n+1}(-1-2AD)^n}{A^{2n+1}D^{2n}(-1+ad)^{n+1}(-1-ad)^n}$$

$$c = \frac{a^{2n+1}d^{2n+1}c(1+AD)^{2n+1}}{A^{2n+1}D^{2n+1}}\ ,\ -C(-1-ad) = \frac{A^{2n+1}D^{2n+1}C(-1-ad)^{n+1}(-1+ad)^n}{(-1)^n a^{2n+1}d^{2n+1}(-1-2AD)^n}$$

$$-b = \frac{A^{2n+1}D^{2n+1}b}{a^{2n+1}d^{2n+1}(-1-AD)^{2n+1}}\ ,\ B = \frac{(-1)^n a^{2n+1}d^{2n+1}B(-1-2AD)^{n+1}}{A^{2n+1}D^{2n+1}(-1+ad)^n(-1-ad)^{n+1}}$$

$$a = \frac{a^{2n+2}d^{2n+1}(1+AD)^{2n+1}}{A^{2n+1}D^{2n+1}}\ ,\ \frac{A}{(-1-2AD)} = \frac{A^{2n+2}D^{2n+1}(-1-ad)^n(-1+ad)^{n+1}}{(-1)^{n+1}a^{2n+1}d^{2n+1}(-1-2AD)^{n+1}}$$

$$\frac{AD}{c} = \frac{A^{2n+2}D^{2n+2}}{-a^{2n+1}d^{2n+1}c(1+AD)^{2n+2}}\ ,\ \frac{-ad}{(-1-ad)} = \frac{(-1)^{n+1}a^{2n+2}d^{2n+2}(-1-2AD)^n}{A^{2n+1}D^{2n+1}C(-1+ad)^{n+1}(-1-ad)^{n+1}}$$

$$\frac{-ad}{b} = \frac{a^{2n+2}d^{2n+2}(-1-AD)^{2n+1}}{A^{2n+1}D^{2n+1}b}\ ,\ \frac{AD}{B} = \frac{A^{2n+2}D^{2n+2}(-1-ad)^{n+1}(-1+ad)^n}{(-1)^n a^{2n+1}d^{2n+1}B(-1-2AD)^{n+1}}$$

So,

$$AD = -2,\ \ ad = 2.$$

Then, suppose that $AD = -2$, $ad = 2$. Thus, we see from the form of the solution of system(2.3) that

$$z_{12n-3} = d\ ,\ t_{12n-3} = D$$

$$z_{12n-2}=c\ ,\ t_{12n-2}=C$$

$$z_{12n-1}=b\ ,\ t_{12n-1}=B$$

$$z_{12n}=a\ ,\ t_{12n}=A$$

$$z_{12n+1}=\frac{-AD}{c}\ ,\ t_{12n+1}=\frac{ad}{C(-1-ad)}$$

$$z_{12n+2}=\frac{-ad}{b}\ ,\ t_{12n+2}=\frac{-AD}{B(-1-2AD)}$$

$$z_{12n+3}=\frac{AD}{a}\ ,\ t_{12n+3}=\frac{-ad}{A}$$

$$z_{12n+4}=c\ ,\ t_{12n+4}=-C(-1-ad)$$

$$z_{12n+5}=-b\ ,\ t_{12n+5}=B$$

$$z_{12n+6}=a\ ,\ t_{12n+6}=\frac{A}{(-1-2AD)}$$

$$z_{12n+7}=\frac{AD}{c}\ ,\ t_{12n+7}=\frac{-ad}{(-1-ad)}$$

$$z_{12n+8}=\frac{-ad}{b}\ ,\ t_{12n+8}=\frac{AD}{B}.$$

Therefore, we have a periodic solution of period twelve. The proof is complete. □

Now, our results of the system (2.3) are verified numerically. In figure (4), the behavior represents the dynamics of the system when the initial values are $z_{-3}=1$, $z_{-2}=-0.5$, $z_{-1}=2$, $z_0=2.4$, $t_{-3}=4$, $t_{-2}=-1.1$, $t_{-1}=4$ and $t_0=-0.5$. Figure (5) confirms that the solution is periodic of period twelve when **Theorem 2.5** is satisfied and the initial values are $z_{-3}=4$, $z_{-2}=-5$, $z_{-1}=7$, $z_0=0.5$, $t_{-3}=1$, $t_{-2}=-1$, $t_{-1}=4$ and $t_0=-2$.

2.4. **The Fourth System.** In this part, the solution's form and the periodicity character are investigated for the following system

$$z_{n+1}=\frac{t_nt_{n-3}}{z_{n-2}(1+t_nt_{n-3})}\ ,t_{n+1}=\frac{z_nz_{n-3}}{t_{n-2}(-1+z_nz_{n-3})} \tag{2.4}$$

where $n=0,1,2,...$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 2.6. *Suppose that $\{z_n,t_n\}$ are solutions of the system (2.4). Then for $n=0,1,2,3,..$ the solutions of system (2.4) can be formed as follows*

$$z_{12n-3}=\frac{A^{2n}D^{2n}(2ad-1)^n}{a^{2n}d^{2n-1}(AD-1)^n(AD+1)^n}\ ,\ t_{12n-3}=\frac{a^{2n}d^{2n}}{A^{2n}D^{2n-1}(ad-1)^{2n}}$$

$$z_{12n-2}=\frac{a^{2n}d^{2n}c(AD-1)^n(AD+1)^n}{A^{2n}D^{2n}(2ad-1)^n}\ ,\ t_{12n-2}=\frac{A^{2n}D^{2n}C(ad-1)^{2n}}{a^{2n}d^{2n}}$$

$$z_{12n-1}=\frac{A^{2n}D^{2n}b(2ad-1)^n}{a^{2n}d^{2n}(AD-1)^n(AD+1)^n}\ ,\ t_{12n-1}=\frac{a^{2n}d^{2n}B}{A^{2n}D^{2n}(ad-1)^{2n}}$$

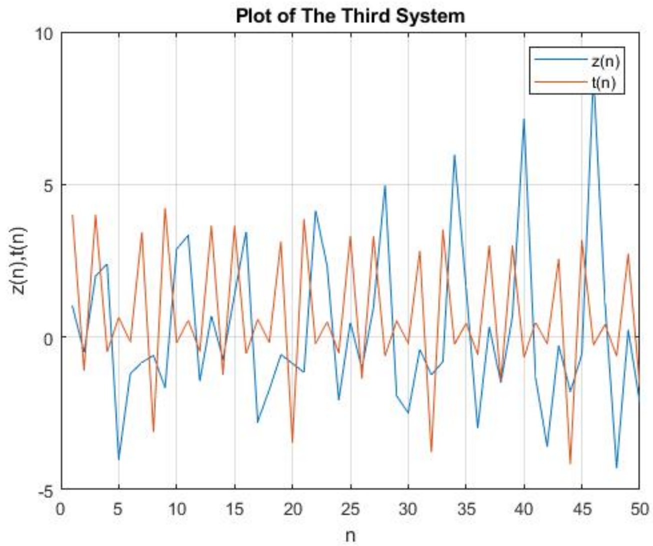

FIGURE 4. The Dynamics of The Solution of The System (2.3)

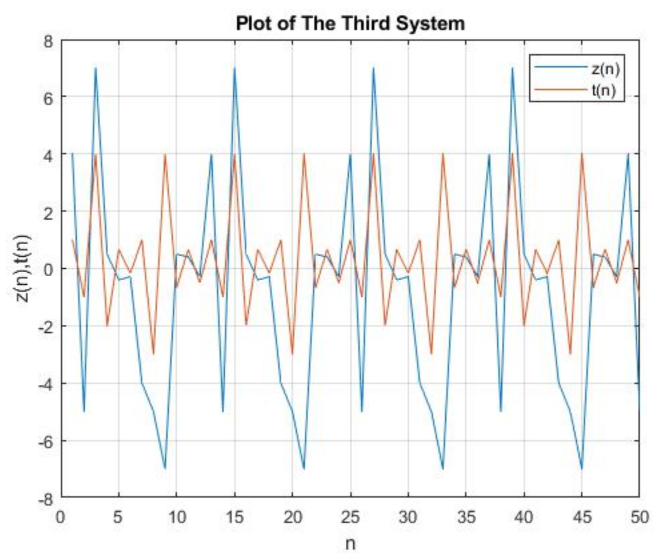

FIGURE 5. The Dynamics of The Solution of The System (2.3)

$$z_{12n} = \frac{a^{2n+1}d^{2n}(AD-1)^n(AD+1)^n}{A^{2n}D^{2n}(2ad-1)^n}\ ,\ t_{12n} = \frac{A^{2n+1}D^{2n}(ad-1)^{2n}}{a^{2n}d^{2n}}$$

$$z_{12n+1} = \frac{A^{2n+1}D^{2n+1}(2ad-1)^n}{a^{2n}d^{2n}c(AD-1)^n(AD+1)^{n+1}}\ ,\ t_{12n+1} = \frac{a^{2n+1}d^{2n+1}}{A^{2n}D^{2n}C(ad-1)^{2n+1}}$$

$$z_{12n+2} = \frac{a^{2n+1}d^{2n+1}(AD-1)^n(AD+1)^n}{A^{2n}D^{2n}b(2ad-1)^{n+1}}\ ,\ t_{12n+2} = \frac{-A^{2n+1}D^{2n+1}(ad-1)^{2n}}{a^{2n}d^{2n}B}$$

$$z_{12n+3} = \frac{A^{2n+1}D^{2n+1}(2ad-1)^n}{a^{2n+1}d^{2n}(AD-1)^{n+1}(AD+1)^n}\ ,\ t_{12n+3} = \frac{a^{2n+1}d^{2n+1}}{-A^{2n+1}D^{2n}(ad-1)^{2n+1}}$$

$$z_{12n+4}=\frac{a^{2n+1}d^{2n+1}c(AD-1)^{n}(AD+1)^{n+1}}{A^{2n+1}D^{2n+1}(2ad-1)^{n}}\ ,\ t_{12n+4}=\frac{A^{2n+1}D^{2n+1}C(ad-1)^{2n+1}}{a^{2n+1}d^{2n+1}}$$

$$z_{12n+5}=\frac{A^{2n+1}D^{2n+1}b(2ad-1)^{n+1}}{a^{2n+1}d^{2n+1}(AD-1)^{n}(AD+1)^{n+1}}\ ,\ t_{12n+5}=\frac{-a^{2n+1}d^{2n+1}B}{A^{2n+1}D^{2n+1}(ad-1)^{2n+1}}$$

$$z_{12n+6}=\frac{a^{2n+2}d^{2n+1}(AD-1)^{n+1}(AD+1)^{n}}{A^{2n+1}D^{2n+1}(2ad-1)^{n+1}}\ ,\ t_{12n+6}=\frac{A^{2n+2}D^{2n+1}(ad-1)^{2n+1}}{a^{2n+1}d^{2n+1}}$$

$$z_{12n+7}=\frac{A^{2n+2}D^{2n+2}(2ad-1)^{n}}{a^{2n+1}d^{2n+1}c(AD-1)^{n+1}(AD+1)^{n+1}}\ ,\ t_{12n+7}=\frac{-a^{2n+2}d^{2n+2}}{A^{2n+1}D^{2n+1}C(ad-1)^{2n+2}}$$

$$z_{12n+8}=\frac{a^{2n+2}d^{2n+2}(AD-1)^{n}(AD+1)^{n+1}}{A^{2n+1}D^{2n+1}b(2ad-1)^{n+1}}\ ,\ t_{12n+8}=\frac{-A^{2n+2}D^{2n+2}(ad-1)^{2n+1}}{a^{2n+1}d^{2n+1}B}$$

Proof. The results are true for $n=0$. Now for $n>0$, assume the results hold for $n-1$ and they are given as follows

$$z_{12n-15}=\frac{A^{2n-2}D^{2n-2}(2ad-1)^{n-1}}{a^{2n-2}d^{2n-3}(AD-1)^{n-1}(AD+1)^{n-1}}\ ,\ t_{12n-15}=\frac{a^{2n-2}d^{2n-2}}{A^{2n-2}D^{2n-3}(ad-1)^{2n-2}}$$

$$z_{12n-14}=\frac{a^{2n-2}d^{2n-2}c(AD-1)^{n-1}(AD+1)^{n-1}}{A^{2n-2}D^{2n-2}(2ad-1)^{n-1}}\ ,\ t_{12n-14}=\frac{A^{2n-2}D^{2n-2}C(ad-1)^{2n-2}}{a^{2n-2}d^{2n-2}}$$

$$z_{12n-13}=\frac{A^{2n-2}D^{2n-2}b(2ad-1)^{n-1}}{a^{2n-2}d^{2n-2}(AD-1)^{n-1}(AD+1)^{n-1}}\ ,\ t_{12n-13}=\frac{a^{2n-2}d^{2n-2}B}{A^{2n-2}D^{2n-2}(ad-1)^{2n-2}}$$

$$z_{12n-12}=\frac{a^{2n-1}d^{2n-2}(AD-1)^{n-1}(AD+1)^{n-1}}{A^{2n-2}D^{2n-2}(2ad-1)^{n-1}}\ ,\ t_{12n-12}=\frac{A^{2n-1}D^{2n-2}(ad-1)^{2n-2}}{a^{2n-2}d^{2n-2}}$$

$$z_{12n-11}=\frac{A^{2n-1}D^{2n-1}(2ad-1)^{n-1}}{a^{2n-2}d^{2n-2}c(AD-1)^{n-1}(AD+1)^{n}}\ ,\ t_{12n-11}=\frac{a^{2n-1}d^{2n-1}}{A^{2n-2}D^{2n-2}C(ad-1)^{2n-1}}$$

$$z_{12n-10}=\frac{a^{2n-1}d^{2n-1}(AD-1)^{n-1}(AD+1)^{n-1}}{A^{2n-2}D^{2n-2}b(2ad-1)^{n}}\ ,\ t_{12n-10}=\frac{-A^{2n-1}D^{2n-1}(ad-1)^{2n-2}}{a^{2n-2}d^{2n-2}B}$$

$$z_{12n-9}=\frac{A^{2n-1}D^{2n-1}(2ad-1)^{n-1}}{a^{2n-1}d^{2n-2}(AD-1)^{n}(AD+1)^{n-1}}\ ,\ t_{12n-9}=\frac{a^{2n-1}d^{2n-1}}{-A^{2n-1}D^{2n-2}(ad-1)^{2n-1}}$$

$$z_{12n-8}=\frac{a^{2n-1}d^{2n-1}c(AD-1)^{n-1}(AD+1)^{n}}{A^{2n-1}D^{2n-1}(2ad-1)^{n-1}}\ ,\ t_{12n-8}=\frac{A^{2n-1}D^{2n-1}C(ad-1)^{2n-1}}{a^{2n-1}d^{2n-1}}$$

$$z_{12n-7}=\frac{A^{2n-1}D^{2n-1}b(2ad-1)^{n}}{a^{2n-1}d^{2n-1}(AD-1)^{n-1}(AD+1)^{n}}\ ,\ t_{12n-7}=\frac{-a^{2n-1}d^{2n-1}B}{A^{2n-1}D^{2n-1}(ad-1)^{2n-1}}$$

$$z_{12n-6}=\frac{a^{2n}d^{2n-1}(AD-1)^{n}(AD+1)^{n-1}}{A^{2n-1}D^{2n-1}(2ad-1)^{n}}\ ,\ t_{12n-6}=\frac{A^{2n}D^{2n-1}(ad-1)^{2n-1}}{a^{2n-1}d^{2n-1}}$$

$$z_{12n-5}=\frac{A^{2n}D^{2n}(2ad-1)^{n-1}}{a^{2n-1}d^{2n-1}c(AD-1)^{n}(AD+1)^{n}}\ ,\ t_{12n-5}=\frac{-a^{2n}d^{2n}}{A^{2n-1}D^{2n-1}C(ad-1)^{2n}}$$

$$z_{12n-4}=\frac{a^{2n}d^{2n}(AD-1)^{n-1}(AD+1)^{n}}{A^{2n-1}D^{2n-1}b(2ad-1)^{n}}\ ,\ t_{12n-4}=\frac{-A^{2n}D^{2n}(ad-1)^{2n-1}}{a^{2n-1}d^{2n-1}B}$$

Now, we verify the first form.

Substituting $12n-3$ into the system of difference equations (2.4). We get

$$z_{12n-3}=\frac{t_{12n-4}t_{12n-7}}{z_{12n-6}(1+t_{12n-4}t_{12n-7})}\ ,$$

$$z_{12n-3} = \frac{\frac{-A^{2n}D^{2n}(ad-1)^{2n-1}}{a^{2n-1}d^{2n-1}B}\frac{-a^{2n-1}d^{2n-1}B}{A^{2n-1}D^{2n-1}(ad-1)^{2n-1}}}{\frac{a^{2n}d^{2n-1}(AD-1)^{n}(AD+1)^{n-1}}{A^{2n-1}D^{2n-1}(2ad-1)^{n}}(1+AD)},$$

that is,

$$z_{12n-3} = \frac{AD}{\frac{a^{2n}d^{2n-1}(AD-1)^{n}(AD+1)^{n-1}}{A^{2n-1}D^{2n-1}(2ad-1)^{n}}(1+AD)}.$$

Thus,

$$z_{12n-3} = \frac{A^{2n}D^{2n}(2ad-1)^{n}}{a^{2n}d^{2n-1}(AD-1)^{n}(AD+1)^{n}},$$

$$t_{12n-3} = \frac{\frac{a^{2n}d^{2n}(AD-1)^{n-1}(AD+1)^{n}}{A^{2n-1}D^{2n-1}b(2ad-1)^{n}}\frac{A^{2n-1}D^{2n-1}b(2ad-1)^{n}}{a^{2n-1}d^{2n-1}(AD-1)^{n-1}(AD+1)^{n}}}{\frac{A^{2n}D^{2n-1}(ad-1)^{2n-1}}{a^{2n-1}d^{2n-1}}(-1+ad)},$$

so,

$$t_{12n-3} = \frac{ad}{\frac{A^{2n}D^{2n-1}(ad-1)^{2n-1}}{a^{2n-1}d^{2n-1}}(-1+ad)},$$

Therefore,

$$t_{12n-3} = \frac{a^{2n}d^{2n}}{A^{2n}D^{2n-1}(ad-1)^{2n}}.$$

Similarly, we can prove the other forms. The proof is completed. □

Theorem 2.7. *The system of non-linear difference equations (5) has a periodic solution of period twelve iff* $AD = -2$, $ad = 2$ *and it will take the following form*

$$\{z_n\} = \left\{d, c, b, a, \frac{-AD}{c}, \frac{ad}{b(2ad-1)}, \frac{AD}{a(AD-1)}, c, b(2ad-1), a, \frac{-AD}{c(AD-1)(AD+1)}, \frac{ad}{b(2ad-1)}..., \right\}$$

$$\{t_n\} = \left\{D, C, B, A, \frac{ad}{C}, \frac{-AD}{B}, \frac{-ad}{A}, -C, B, -A, \frac{ad}{C}, \frac{AD}{B}, ..., \right\}$$

Proof. (We leave this proof to readers and it can be proven by using the same approach as **Theorem 2.5**) □

We simulate the non-linear difference equations system (2.4) numerically. Figure (6) shows the dynamics of the system when the initial values are $z_{-3} = 1$, $z_{-2} = -0.5$, $z_{-1} = 3$, $z_0 = 2.4$, $t_{-3} = 4$, $t_{-2} = -1.3$, $t_{-1} = 5$ and $t_0 = -0.5$. In figure (6), the behaviour of the system is periodic of period twelve when the initial values are $z_{-3} = 1$, $z_{-2} = -0.5$, $z_{-1} = 3$, $z_0 = 2$, $t_{-3} = 4$, $t_{-2} = -1.3$, $t_{-1} = 5$ and $t_0 = -0.5$.

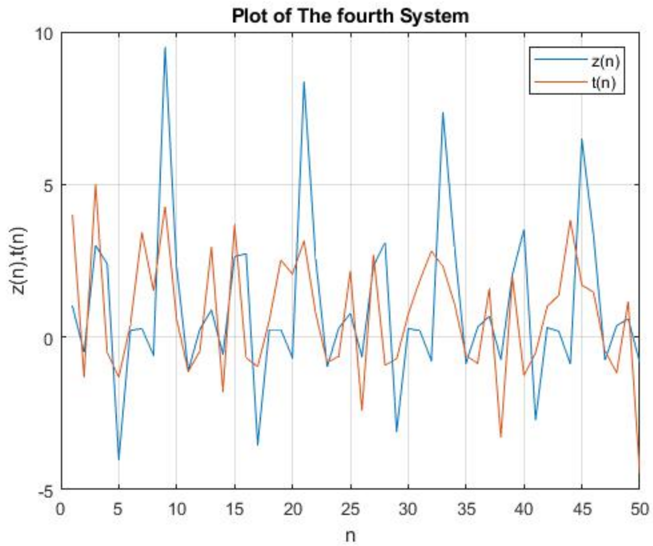

FIGURE 6. The Dynamics of The Solution of The System(2.4)

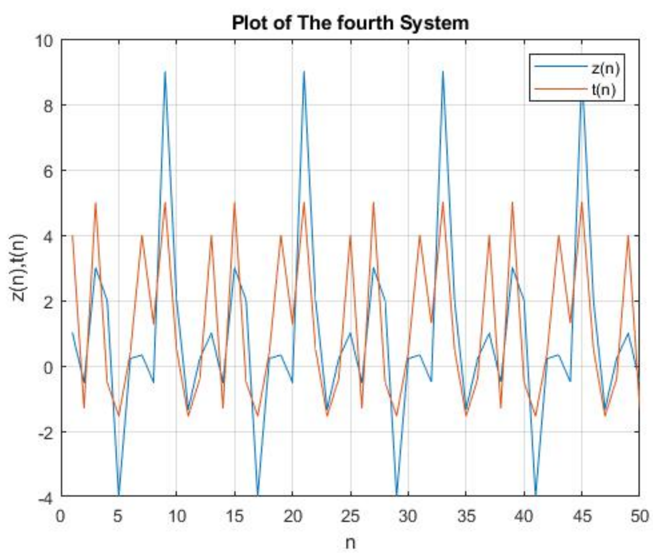

FIGURE 7. The Dynamics of The Solution of The SysThe tem (2.4)

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