

VLASOV-POISSON-FOKKER-PLANCK IN FRACTIONAL SOBOLEV-LEBESGUE SPACES

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ABSTRACT. In this paper, we are concerned with the local well-posedness of the Vlasov-Poisson-Fokker-Planck equation near vacuum in the fractional Sobolev-Lebesgue space for the large initial data. To achieve this goal, we mainly adopt the energy method. In order to obtain the energy estimate, we establish an L^2 - L^q estimate related to the electronic term, and take advantage of the commutator estimates as well.

1. INTRODUCTION

In this paper, we study the following Vlasov-Poisson-Fokker-Planck equation (VPFP) in n -dimensional space:

$$(1.1) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f - \Delta_v f = 0, \\ -\Delta_x \phi = \int_{\mathbb{R}^n} f \, dv, \\ f(0, x, v) = f_0(x, v), \end{cases}$$

where $f(t, x, v)$ denotes the distribution function of particles, $x \in \mathbb{R}^n$ is the position, $v \in \mathbb{R}^n$ is the velocity, $t > 0$ is the time, and $n \geq 5$.

In statistical physics, the VPFP system is one of key equations governing the evolution of a distribution of particles over time. Specifically, it models the distribution of particles in a plasma with respect to position and velocity affected by gravitational or electrostatic forces, and the collision effects are produced by the Brownian motion of particles. In the stellar dynamical context, one of the fundamental problems is to incorporate, in the framework of a general theory, the effect of encounters between stars. Stellar encounters under Newtonian inverse square attractions influence the motion of stars in the manner of Brownian motion [4].

The Cauchy problem for the VPFP system has been studied for several decades. In 1984, Neunzert, Pulvirenti, and Triolo [12] proved the global existence of smooth solutions for the two-dimensional case using a probabilistic method. Two years later, Degond [5] proved the global existence for the VPFP system for dimensions only one and two with initial data in

$$(1.2) \quad \mathcal{D} = \{f_0 \in W^{1,1}(\mathbb{R}^{2d}) : \langle v \rangle^\gamma (|f_0| + |Df_0|) \in L^\infty(\mathbb{R}^{2d}), \, d = 1, 2, \gamma > d\}.$$

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Submitted on Oct. 31, 2022.

2020 *Mathematics Subject Classification.* Primary 35Q84, 46E35; Secondary 47B47.

Key words and phrases. Fokker-Planck equation, Fractional Sobolev-Lebesgue spaces, Commutator estimates, L^2 - L^q estimates.

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In the higher dimension, Victory [16] studied the VPFP system with initial data satisfying

$$(1.3) \quad f_0 \in L^1(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n}) \text{ and } \int_{\mathbb{R}^{2n}} (|x|^2 + |v|^2) f_0(x, v) dx dv < \infty,$$

and Victory proved the existence of the global weak solutions under these conditions. Later, Rein and Weckler [14] proved the global existence of a classical solution in a three-dimensional case with initial condition in the following class:

$$(1.4) \quad \mathcal{D} = \{f_0 \in C_b^1(\mathbb{R}^6) : \langle v \rangle^\gamma f_0 \in L^1 \cap L^\infty \text{ and } \langle v \rangle^\gamma \nabla_{x,v} f_0 \in L^1 \cap L^\infty\}.$$

Then, Carrillo and Soler [2], proved the global existence of weak solutions for the VPFP system in three-dimensions with the initial data in $L^1(\mathbb{R}^{2n}) \cap L^p(\mathbb{R}^{2n})$, i.e. the initial data is still in L^1 -framework. They also studied the VPFP equations with measures in Morrey spaces as initial data, see [3].

All the literature mentioned above were concerned about the solutions in $L^1 \cap L^\infty$. It comes naturally to ask how about in the L^2 framework? For instance, whether there is a solution in the fractional Sobolev space H^s is an interesting question, which becomes the main theme in this paper. Besides, there is rare paper which devoted to the study of the solutions of the VPFP system in the fractional Sobolev space, which motivates us to take a step forward in this direction.

To study the well-posedness of (1.1), the main difficulty lies in estimating the electronic term $\nabla_x \phi$. Our strategy is to take advantage of the following type of commutator estimates related to the electronic term $\nabla_x \phi$:

$$\begin{aligned} & \| [D_x^s, \nabla_x \phi^k] \nabla_v f^{k+1} \|_{L_x^2} \\ & \leq \| D_x^s (\nabla_x \phi^k) \|_{L_x^q} \| \nabla_v f^{k+1} \|_{L_x^n} \\ & \quad + \| \nabla_x (\nabla_x \phi^k) \|_{L_x^\infty} \| D_x^{s-1} \nabla_v f^{k+1} \|_{L_x^2}, \end{aligned}$$

which is the basis for the estimate in the hybrid Sobolev-Lebesgue space. Combining with the L^2 - L^q estimate with respect to $\nabla_x \phi$ (see Section 3), we could then close the energy. Also, it is worth to mention that a weight w is necessary to be introduced to derive an L^2 - L^q off-diagonal estimate which plays an important role to deal with the Poisson equation near vacuum.

2. PRELIMINARIES AND MAIN THEOREM

Before we state our main theorem, we would like to set our notations and definitions first.

2.1. Notations and definitions.

- Given a locally integrable function f , the maximal function Mf is defined by

$$(2.1) \quad (Mf)(x) = \sup_{\delta > 0} \frac{1}{|B(x, \delta)|} \int_{B(x, \delta)} |f(y)| dy,$$

where $|B(x, \delta)|$ is the volume of the ball of $B(x, \delta)$ with center x and radius δ .

- Given $f \in \mathcal{S}$ Schwartz class, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

and its inverse Fourier transform is defined by $\mathcal{F}^{-1}f(x) = \hat{f}(-x)$. In this paper, we use $\mathcal{F}_x f(x, v)$ to represent the Fourier transform of x only, and ξ to represent the dual variable of x .

- $D_x^s f = \mathcal{F}^{-1} \langle \xi \rangle^s \hat{f}$, $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.
- Throughout this paper, the weight function is $w(v) = \langle v \rangle^\gamma$, $\gamma > n$.
- $\|f\|_{L_{x,v}^2(w)} = \left(\int_{\mathbb{R}^{2n}} |f|^2 w dx dv \right)^{\frac{1}{2}}$.

- Let $s > 0$, the hybrid Sobolev-Lebesgue space with weight in v is defined by

$$\tilde{H}^s(\mathbb{R}_{x,v}^{2n}) = \left\{ f \in \mathcal{S}' : \|f\|_{\tilde{H}^s(\mathbb{R}_{x,v}^{2n})} = \|D_x^s f\|_{L_{x,v}^2(w)} \leq \infty \right\},$$

where \mathcal{S}' is the dual space of Schwartz class. For the convenience, we use \tilde{H}^s for the abbreviation of $\tilde{H}^s(\mathbb{R}_{x,v}^{2n})$ whenever there is confusion arising.

- $A \preceq B$ means there exists a positive generic constant C independent of the main parameters such that $A \leq CB$. $A \sim B$ means $A \preceq B$ and $B \preceq A$.

Remark 2.1. The hybrid space \tilde{H}^s possesses the different differential and integral properties on x and v variables.

2.2. Some useful lemmas. In this part, we collect some known results of the Riesz potential [1, 15] and the boundedness of Hardy-Littlewood operator for later use.

The pointwise estimate of the Riesz potential stated below is applied to derive the off-diagonal estimate in Section 3.

Lemma 2.2. ([1]) *For any multi-index ξ with $|\xi| < \alpha < n$, there is a constant C such that for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and almost every x , we have*

$$(2.2) \quad |D^\xi(I_\alpha * f(x))| \leq CMf(x)^{\frac{|\xi|}{\alpha}} \cdot (I_\alpha * |f|(x))^{1-\frac{|\xi|}{\alpha}},$$

where $I_\alpha = \frac{\gamma_\alpha}{|x|^{n-\alpha}}$, $\gamma_\alpha = \frac{\Gamma(n-\frac{\alpha}{2})}{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}$.

Remark 2.3. In our paper, we consider $-\Delta\phi = \int_{\mathbb{R}^n} f dv =: g, n \geq 3$. Thus, in our context, I_α can be taken

$$(2.3) \quad I_2(x) = \frac{1}{(n-2)\omega_{n-1}} \cdot \frac{1}{|x|^{n-2}}, \quad \text{i.e. } \alpha = 2,$$

where $\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the $(n-1)$ -dimensional area of the unit sphere in \mathbb{R}^n . Additionally, we have the pointwise estimate

$$(2.4) \quad |D^\xi(I_2 * g(x))| \leq cMg(x)^{\frac{|\xi|}{2}} \cdot (I_2 * |g|(x))^{1-\frac{|\xi|}{2}}.$$

The boundedness of the Riesz potential in Lebesgue space is needed in our proof as well.

Lemma 2.4. ([15]) *If $-\Delta\phi = g \in L^2(\mathbb{R}^n)$, then $\phi = I_2 * g$ and*

$$(2.5) \quad \|I_2 * g\|_{L^{\tilde{q}}(\mathbb{R}^n)} \leq c\|g\|_{L^2(\mathbb{R}^n)},$$

where $n > 4$, $c = c(p, \tilde{q})$ and

$$(2.6) \quad \frac{1}{\tilde{q}} = \frac{1}{2} - \frac{2}{n}.$$

For more results of the Riesz potential and its applications in partial differential equations, see [6–8].

Now we give the boundedness of Hardy-Littlewood operator M which is defined by (2.1).

Lemma 2.5. ([10]) *Let $1 < p \leq \infty$, then the Hardy-Littlewood operator M is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. i.e.,*

$$\|Mf\|_{L^p} \leq C_{n,p}\|f\|_{L^p}, \quad \forall f \in L^p(\mathbb{R}^n).$$

The following commutator estimate is of importance in this paper, see Section 4.

Lemma 2.6. ([11]) Let $1 < p < \infty$, and let $1 < p_1, p_2, p_3, p_4 \leq \infty$ satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}.$$

Then for any $f, g \in \mathcal{S}(\mathbb{R}^n)$, the following holds

$$\|D^s(fg) - fD^s g\|_{L^p} \leq \|D^{s-1}\partial f\|_{p_1}\|g\|_{p_2} + \|\partial f\|_{p_3}\|D^{s-2}\partial g\|_{p_4}.$$

Now we are ready to state our theorem.

2.3. The statement of the main theorem.

Theorem 2.7. Suppose $f_0 \in \tilde{H}^s$, $s > \frac{n}{2}$ with $n \geq 5$, then there exists a $T_0 > 0$ such that the Cauchy problem of the Vlasov-Poisson-Fokker-Planck system (1.1) admits a unique solution in $[0, T_0] \times \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\sup_{0 \leq t \leq T_0} \mathcal{E}(f(t)) \leq 4\|f_0\|_{\tilde{H}^s}^2,$$

where

$$\mathcal{E}(f(t)) =: \|f(t)\|_{\tilde{H}^s}^2 + \int_0^t \|\nabla_v D_x^s f\|_{L_{x,v}^2(w)}^2 d\tau.$$

Remark 2.8. In the definition of space \tilde{H}^s , we only impose the differential assumption on the variable x , not on the variable v ; by contrast, the weight function $w(v)$ is only about v , not about x . So our space is different from the fractional Sobolev space, it is a hybrid Sobolev-Lebesgue space with weight.

3. L^2 - L^q ESTIMATES

In this section, we are aiming to obtain an L^2 - L^q estimates related to the electronic term $\nabla_x \phi$, which plays a fundamentally important role in the commutator estimate of the electronic term $\nabla_x \phi$.

First of all, we establish the boundedness of the solution of Laplacian equation in Lebesgue space.

Lemma 3.1. Assume $-\Delta \phi = g$, then it holds that

$$(3.1) \quad \|\nabla_x \phi\|_{L^q(\mathbb{R}^n)} \leq \|g\|_{L^2(\mathbb{R}^n)},$$

where $n > 4$ and $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$.

Proof. Note that $\nabla_x \phi = \nabla_x(I_2 * g)$ by Lemma 2.4, therefore there holds

$$\begin{aligned} \|\nabla_x \phi\|_{L^q(\mathbb{R}^n)} &= \|\nabla_x(I_2 * g)\|_{L^q(\mathbb{R}^n)} \\ &\leq \|Mg^{\frac{1}{2}} \cdot (I_2 * |g|)^{\frac{1}{2}}\|_{L^q(\mathbb{R}^n)} \\ &\leq \|(Mg)^{\frac{1}{2}}\|_{L^4(\mathbb{R}^n)} \cdot \|(I_2 * |g|)^{\frac{1}{2}}\|_{L^{q_2}(\mathbb{R}^n)} \\ &\leq \|Mg\|_{L^2(\mathbb{R}^n)}^{\frac{1}{2}} \cdot \|I_2 * |g|\|_{L^{\frac{q_2}{2}}(\mathbb{R}^n)}^{\frac{1}{2}}, \end{aligned}$$

where

$$(3.2) \quad \frac{1}{4} + \frac{1}{q_2} = \frac{1}{q},$$

and we applied Lemma 2.2 in the second line.

On the one hand, the boundedness of Hardy-Littlewood operator Lemma 2.5 yields that

$$(3.3) \quad \|Mg\|_{L^2(\mathbb{R}^n)} \leq \|g\|_{L^2(\mathbb{R}^n)}.$$

On the other hand, by the boundedness of the Riesz potential Lemma 2.4, we have

$$(3.4) \quad \|I_2 * |g|\|_{L^{\frac{q_2}{2}}(\mathbb{R}^n)} \leq \|g\|_{L^2(\mathbb{R}^n)},$$

where

$$(3.5) \quad \frac{2}{q_2} = \frac{1}{2} - \frac{2}{n}.$$

Consequently,

$$\|\nabla_x \phi\|_{L^q(\mathbb{R}^n)} \preceq \|g\|_{L^2(\mathbb{R}^n)}^{\frac{1}{2}} \cdot \|g\|_{L^2(\mathbb{R}^n)}^{\frac{1}{2}} = \|g\|_{L^2(\mathbb{R}^n)},$$

which ends the proof of this lemma. \square

Remark 3.2. This lemma explains the reason that we can not have a solution in the L^2 -framework for $n \leq 4$, in some sense.

Remark 3.3. We summarize the conditions imposed on the indices in Lemma 3.1 as follows:

$$(3.6) \quad \begin{cases} q_1 = 4, \\ q_2 = \frac{4n}{n-4}, \\ q = \frac{2n}{n-2}. \end{cases}$$

The following corollary will be used in our commutator estimate involving the electronic term $\nabla_x \phi$.

Corollary 3.4. Assume $-\Delta \phi = g$, then it holds that

$$(3.7) \quad \|D_x^s \nabla_x \phi\|_{L^q} \preceq \|D_x^s g\|_{L^2_{x,v}(w)}.$$

Proof. Observe that

$$D_x^s \nabla_x \phi = \nabla_x D_x^s \phi = D_x^s \nabla_x (I_2 * g) = \nabla_x (I_2 * D_x^s g),$$

applying Lemma 3.1 with ϕ and g replaced by $D_x^s \phi$ and $D_x^s g$ respectively yields the desired result. \square

Corollary 3.5. Take $g = \int_{\mathbb{R}^n} f dv$ in Corollary 3.4, then we have

$$\|D_x^s \nabla_x \phi\|_{L^q_x} \preceq \|f\|_{\tilde{H}^s}.$$

Proof. Hölder's inequality leads to

$$\left| \int_{\mathbb{R}^n} D_x^s f dv \right| \preceq \left(\int_{\mathbb{R}^n} |D_x^s f|^2 w dv \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} w^{-1} dv \right)^{\frac{1}{2}}.$$

Note that $w = \langle v \rangle^\gamma$ and $\gamma > n$, which implies that

$$\left(\int_{\mathbb{R}^n} w^{-1} dv \right)^{\frac{1}{2}} \leq c.$$

Thus, we ends the proof of Corollary 3.5. \square

Remark 3.6. $\nabla_x \phi$ is a function of the variable x only, while f is a function of the variables x and v . \tilde{H}^s is a hybrid Sobolev-Lebesgue space with weight depending on v only, which is defined in Section 2.1.

An L^∞ estimate is also needed in the proof of main result Theorem 2.7.

Lemma 3.7. Suppose $-\Delta \phi = \int_{\mathbb{R}^n} f dv$. If $s > \frac{n}{2}$, then

$$(3.8) \quad \sum_{|\alpha| \leq 1} \|\partial_x^\alpha \nabla_x \phi\|_{L^\infty_x} \preceq \|f\|_{\tilde{H}^s}.$$

Proof. Note that $s > \frac{n}{2}$, i.e. $s > \frac{n}{q} + 1$. For $|\alpha| \leq 1$, we have

$$\begin{aligned} \|\partial_x^\alpha \nabla_x \phi\|_{L_x^\infty} &\preceq \|D_x^{s-1} \partial_x^\alpha \nabla_x \phi\|_{L^q} \\ (3.9) \qquad \qquad \qquad &\preceq \|D_x^s f\|_{L_{x,v}^2(w)} \\ &\preceq \|f\|_{\dot{H}^s}, \end{aligned}$$

where we applied Corollary 3.5 and the assumption

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{n}.$$

□

4. PROOF OF MAIN THEOREM

To prove Theorem 2.7, we split its proof into two parts which are existence and uniqueness of the solution to (1.1). Let us start with proving the existence.

Proof of existence. In this part, we adopt the energy method and the iteration method. To do so, we need to close the energy by applying the commutator estimate. In this process, the L^2 - L^q estimate of electronic term $\nabla_x \phi$ plays a nice role.

Proof. We consider the following iterating sequence for solving the VPFP (1.1),

$$(4.1) \qquad \begin{cases} \partial_t f^{k+1} + v \cdot \nabla_x f^{k+1} + \nabla_x \phi^k \cdot \nabla_v f^{k+1} - \Delta_v f^{k+1} = 0, \\ -\Delta \phi^k = \int_{\mathbb{R}^n} f^k dv, \\ f^{k+1}(0, x, v) = f_0(x, v). \end{cases}$$

Applying D_x^s to the first equation in (4.1), we have

$$(4.2) \qquad \begin{aligned} \partial_t D_x^s f^{k+1} + v \cdot \nabla_x D_x^s f^{k+1} + [D_x^s, \nabla_x \phi^k] \nabla_v f^{k+1} + \nabla_x \phi^k \cdot \nabla_v D_x^s f^{k+1} \\ - \Delta_v D_x^s f^{k+1} = 0. \end{aligned}$$

Multiplying $(D_x^s f^{k+1})w$ on both sides of (4.2), and then integrating over $\mathbb{R}_x^n \times \mathbb{R}_v^n$ yields that

$$\begin{aligned} (4.3) \qquad &\frac{1}{2} \frac{d}{dt} \|D_x^s f^{k+1}\|_{L_{x,v}^2(w)}^2 + \underbrace{\left\langle v \cdot \nabla_x D_x^s f^{k+1}, (D_x^s f^{k+1}) \cdot w \right\rangle}_{J_1} \\ &+ \underbrace{\left\langle [D_x^s, \nabla_x \phi^k] \nabla_v f^{k+1}, (D_x^s f^{k+1}) \cdot w \right\rangle}_{J_2} \\ &+ \underbrace{\left\langle \nabla_x \phi^k \cdot \nabla_v D_x^s f^{k+1}, (D_x^s f^{k+1}) \cdot w \right\rangle}_{J_3} \\ &+ \underbrace{\left\langle -\Delta_v D_x^s f^{k+1}, (D_x^s f^{k+1}) \cdot w \right\rangle}_{J_4} = 0. \end{aligned}$$

We now estimate (4.3) term by term.

For J_1 , we have

$$(4.4) \qquad J_1 = \int_{\mathbb{R}^{2n}} v \cdot \nabla_x |D_x^s f^{k+1}|^2 w \, dx dv = 0.$$

For \mathbf{J}_2 , applying the commutator estimate Lemma 2.6 yields

$$(4.5) \quad \begin{aligned} \|[D_x^s, \nabla_x \phi^k] \nabla_v f^{k+1}\|_{L_x^2} &\leq \|D_x^s(\nabla_x \phi^k)\|_{L_x^q} \|\nabla_v f^{k+1}\|_{L_x^n} \\ &\quad + \|\nabla_x(\nabla_x \phi^k)\|_{L_x^\infty} \|D_x^{s-1} \nabla_v f^{k+1}\|_{L_x^2}, \end{aligned}$$

where $\frac{1}{n} + \frac{1}{q} = \frac{1}{2}$.

For the first term on the right-hand side of (4.5), by the embedding theorem [13],

$$\|\nabla_v f^{k+1}\|_{L_x^n} \leq \|D_x^s \nabla_v f^{k+1}\|_{L_x^2}, \quad s > \frac{n}{2};$$

and by the L^2 - L^q estimate Corollary 3.4,

$$(4.6) \quad \|D_x^s(\nabla_x \phi^k)\|_{L_x^q} \leq \|f^k\|_{\tilde{H}^s}.$$

For the second term on the right-hand side of (4.5), by Lemma 3.7, we have

$$(4.7) \quad \|\nabla_x(\nabla_x \phi^k)\|_{L_x^\infty} \leq \|f^k\|_{\tilde{H}^s}, \quad s > \frac{n}{2}.$$

Consequently,

$$(4.8) \quad \begin{aligned} J_2 &\leq \|[D_x^s, \nabla_x \phi^k] \nabla_v f^{k+1}\|_{L_{x,v}^2(w)} \cdot \|D_x^s f^{k+1}\|_{L_{x,v}^2(w)} \\ &\leq \|f^k\|_{\tilde{H}^s} \cdot \|D_x^s \nabla_v f^{k+1}\|_{L_{x,v}^2(w)} \cdot \|f^{k+1}\|_{\tilde{H}^s} \\ &\leq \epsilon \|D_x^s \nabla_v f^{k+1}\|_{L_{x,v}^2(w)}^2 + C_\epsilon \|f^k\|_{\tilde{H}^s}^2 \cdot \|f^{k+1}\|_{\tilde{H}^s}^2. \end{aligned}$$

For \mathbf{J}_3 , note that $|\nabla_v w| \leq w$, integration by parts yields

$$(4.9) \quad \begin{aligned} J_3 &\leq \int_{\mathbb{R}^{2n}} |\nabla_x \phi^k| \cdot |D_x^s f^{k+1}|^2 w \, dx dv \\ &\leq \|\nabla_x \phi^k\|_{L_x^\infty} \|D_x^s f^{k+1}\|_{L_{x,v}^2(w)}^2 \\ &\leq \|f^k\|_{\tilde{H}^s} \|f^{k+1}\|_{\tilde{H}^s}^2, \end{aligned}$$

where we applied the assumption $s > \frac{n}{2}$ in the last line.

For \mathbf{J}_4 , we have

$$(4.10) \quad J_4 \leq \|D_x^s \nabla_v f^{k+1}\|_{L_{x,v}^2(w)} - \|D_x^s f^{k+1}\|_{L_{x,v}^2(w)}^2.$$

Plugging all the estimates from J_1 to J_4 into (4.3), we obtain,

$$(4.11) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \|D_x^s f^{k+1}\|_{L_{x,v}^2(w)}^2 + \|\nabla_v D_x^s f^{k+1}\|_{L_{x,v}^2(w)}^2 \\ &\leq \|f^k\|_{\tilde{H}^s}^2 \cdot \|f^{k+1}\|_{\tilde{H}^s}^2 + \|f^k\|_{\tilde{H}^s} \cdot \|f^{k+1}\|_{\tilde{H}^s}^2 + \|f^{k+1}\|_{\tilde{H}^s}^2. \end{aligned}$$

Integrating over $[0, t]$ on both sides of (4.11), we deduce,

$$\begin{aligned} \mathcal{E}(f^{k+1}(t)) &\leq \|f_0\|_{\tilde{H}^s}^2 + Ct \sup_{0 \leq \tau \leq t} \mathcal{E}(f^{k+1}(\tau)) \cdot \sup_{0 \leq \tau \leq t} \mathcal{E}(f^k(\tau)) \\ &\quad + Ct \sup_{0 \leq \tau \leq t} (\mathcal{E}^{\frac{1}{2}}(f^k(\tau))) \cdot \sup_{0 \leq \tau \leq t} \mathcal{E}(f^{k+1}(\tau)) + Ct \sup_{0 \leq \tau \leq t} (\mathcal{E}(f^{k+1}(\tau))). \end{aligned}$$

Inductively, assume

$$\sup_{0 \leq \tau \leq T_0} \mathcal{E}(f^k(\tau)) \leq 4\|f_0\|_{\tilde{H}^s}^2,$$

then

$$(4.12) \quad \sup_{0 \leq t \leq T_0} \mathcal{E}(f^{k+1}(t)) \leq \|f_0\|_{\tilde{H}^s}^2 + CT_0 \left(4\|f_0\|_{\tilde{H}^s}^2 + 2\|f_0\|_{\tilde{H}^s} + 1 \right) \cdot \sup_{0 \leq t \leq T_0} \mathcal{E}(f^{k+1}(t)).$$

Taking T_0 sufficiently small such that

$$(4.13) \quad 1 - C \left(4\|f_0\|_{\tilde{H}^s}^2 + 2\|f_0\|_{\tilde{H}^s} + 1 \right) T_0 \geq \frac{1}{4},$$

i.e.,

$$(4.14) \quad T_0 \leq \frac{3}{4C \left(4\|f_0\|_{\tilde{H}^s}^2 + 2\|f_0\|_{\tilde{H}^s} + 1 \right)},$$

then we have

$$(4.15) \quad \sup_{0 \leq \tau \leq T_0} \mathcal{E}(f^{k+1}(\tau)) \leq 4\|f_0\|_{\tilde{H}^s}^2.$$

Inductively,

$$(4.16) \quad \sup_k \sup_{0 \leq \tau \leq T_0} \mathcal{E}(f^k(t)) \leq 4\|f_0\|_{\tilde{H}^s}^2,$$

i.e., we get a uniform-in-k estimate.

As a routine, let $k \rightarrow \infty$, we obtain the solution and complete the proof of existence. \square

Let us move on to proving the uniqueness.

Proof of uniqueness. In the second part, we apply a similar trick in the proof of existence.

Proof. Assume another solution g exists such that

$$\sup_{0 \leq \tau \leq T_0} \mathcal{E}(g(\tau)) \leq 4\|f_0\|_{\tilde{H}^s}^2,$$

taking the difference of f and g , we have

$$(4.17) \quad \begin{cases} (\partial_t + v \cdot \nabla_x + \nabla_x \phi_f \cdot \nabla_v)(f - g) + (\nabla_x \phi_f - \nabla_x \phi_g) \cdot \nabla_v g \\ - \Delta_v(f - g) = 0, \\ - \Delta_x(\phi_f - \phi_g) = \int_{\mathbb{R}^n} (f - g) dv, \\ f(0, x, v) = g(0, x, v). \end{cases}$$

Applying D_x^s to (4.17)₁, we have

$$(4.18) \quad \begin{aligned} & \partial_t D_x^s(f - g) + v \cdot \nabla_x D_x^s(f - g) + [D_x^s, \nabla_x \phi_f] \nabla_v(f - g) \\ & + \nabla_x \phi_f \cdot \nabla_v D_x^s(f - g) + [D_x^s, \nabla_x(\phi_f - \phi_g)] \nabla_v g \\ & + \nabla_x(\phi_f - \phi_g) \cdot D_x^s \nabla_v g - \Delta_v D_x^s(f - g) = 0. \end{aligned}$$

Multiplying $(D_x^s(f - g)) \cdot w$ on both sides of (4.18), and then integrating over $\mathbb{R}_x^n \times \mathbb{R}_v^n$ yields that

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|D_x^s(f - g)\|_{L_{x,v}^2(w)}^2 + \underbrace{\left\langle v \cdot \nabla_x D_x^s(f - g), (D_x^s(f - g)) \cdot w \right\rangle}_{J_5} \\
 & + \underbrace{\left\langle [D_x^s, \nabla_x \phi_f] \nabla_v(f - g), (D_x^s(f - g)) \cdot w \right\rangle}_{J_6} \\
 & + \underbrace{\left\langle \nabla_x \phi_f \cdot \nabla_v D_x^s(f - g), (D_x^s(f - g)) \cdot w \right\rangle}_{J_7} \\
 & + \underbrace{\left\langle [D_x^s, \nabla_x(\phi_f - \phi_g)] \nabla_v g, (D_x^s(f - g)) \cdot w \right\rangle}_{J_8} \\
 & + \underbrace{\left\langle \nabla_x(\phi_f - \phi_g) \cdot D_x^s \nabla_v g, (D_x^s(f - g)) \cdot w \right\rangle}_{J_9} \\
 & + \underbrace{\left\langle -\Delta_v D_x^s(f - g), (D_x^s(f - g)) \cdot w \right\rangle}_{J_{10}} = 0.
 \end{aligned}
 \tag{4.19}$$

We could repeat the estimates in the proof of existence except for some special terms. Thus, we would like to write down the estimates directly without too much details.

Obviously, $\mathbf{J}_5 = 0$.

For \mathbf{J}_6 and \mathbf{J}_8 , which are similar to the estimate of J_2 , we have

$$\begin{aligned}
 J_6 & \leq \|f\|_{\tilde{H}^s} \cdot \|D_x^s \nabla_v(f - g)\|_{L_{x,v}^2(w)} \cdot \|f - g\|_{\tilde{H}^s} \\
 & \leq \epsilon \|D_x^s \nabla_v(f - g)\|_{L_{x,v}^2(w)}^2 + C_\epsilon \|f\|_{\tilde{H}^s}^2 \|f - g\|_{\tilde{H}^s}^2,
 \end{aligned}
 \tag{4.20}$$

where Young's inequality [9] with ϵ small is applied.

Also, we have

$$\begin{aligned}
 J_8 & \leq \|f - g\|_{\tilde{H}^s} \cdot \|D_x^s \nabla_v g\|_{L_{x,v}^2(w)} \cdot \|D_x^s(f - g)\|_{L_{x,v}^2(w)} \\
 & \leq \|f - g\|_{\tilde{H}^s}^2 + \|D_x^s \nabla_v g\|_{L_{x,v}^2(w)}^2 \cdot \|f - g\|_{\tilde{H}^s}^2.
 \end{aligned}
 \tag{4.21}$$

For \mathbf{J}_7 , we have

$$\begin{aligned}
 J_7 & \leq \|\nabla_x \phi_f\|_{L^\infty} \cdot \|\nabla_v D_x^s(f - g)\|_{L_{x,v}^2(w)} \cdot \|D_x^s(f - g)\|_{L_{x,v}^2(w)} \\
 & \leq \|f\|_{\tilde{H}^s} \cdot \|\nabla_v D_x^s(f - g)\|_{L_{x,v}^2(w)} \cdot \|D_x^s(f - g)\|_{L_{x,v}^2(w)} \\
 & \leq \epsilon \|\nabla_v D_x^s(f - g)\|_{L_{x,v}^2(w)}^2 + C_\epsilon \|f - g\|_{\tilde{H}^s}^2 \cdot \|f\|_{\tilde{H}^s}^2,
 \end{aligned}
 \tag{4.22}$$

where Young's inequality [9] with ϵ small is applied once again.

For \mathbf{J}_9 , we get

$$\begin{aligned}
 J_9 & \leq \|\nabla_x(\phi_f - \phi_g)\|_{L^\infty} \cdot \|D_x^s \nabla_v g\|_{L_{x,v}^2(w)} \cdot \|D_x^s(f - g)\|_{L_{x,v}^2(w)} \\
 & \leq \|f - g\|_{\tilde{H}^s} \cdot \|D_x^s \nabla_v g\|_{L_{x,v}^2(w)} \cdot \|D_x^s(f - g)\|_{L_{x,v}^2(w)} \\
 & \leq \|f - g\|_{\tilde{H}^s}^2 + \|D_x^s \nabla_v g\|_{L_{x,v}^2(w)}^2 \cdot \|f - g\|_{\tilde{H}^s}^2.
 \end{aligned}
 \tag{4.23}$$

For \mathbf{J}_{10} , which is similar to J_4 , we obtain

$$(4.24) \quad J_{10} \succeq \|\nabla_v D_x^s(f - g)\|_{L_{x,v}^2(w)}^2 - \|D_x^s(f - g)\|_{L_{x,v}^2(w)}^2.$$

Collecting all the estimates from J_5 through J_{10} , and plugging into (4.19), we have,

$$(4.25) \quad \begin{aligned} & \frac{d}{dt} \|f - g\|_{\tilde{H}^s}^2 + \|\nabla_v D_x^s(f - g)\|_{L_{x,v}^2(w)}^2 \\ & \preceq \left(1 + \|f\|_{\tilde{H}^s}^2 + \|g\|_{\tilde{H}^s}^2 + \|D_x^s \nabla_v g\|_{L_{x,v}^2(w)}^2\right) \cdot \|f - g\|_{\tilde{H}^s}^2. \end{aligned}$$

Integrating over $[0, t]$ on both sides of (4.25), we get

$$(4.26) \quad \begin{aligned} & \|f - g\|_{\tilde{H}^s}^2 + \int_0^t \|\nabla_v D_x^s(f - g)\|_{L_{x,v}^2(w)}^2 d\tau \\ & \leq \int_0^t \left(1 + \|f\|_{\tilde{H}^s}^2 + \|D_x^s \nabla_v g\|_{L_{x,v}^2(w)}^2\right) \cdot \|f - g\|_{\tilde{H}^s}^2 d\tau. \end{aligned}$$

Recall

$$\mathcal{E}(f(t)) =: \|f(t)\|_{\tilde{H}^s}^2 + \int_0^t \|\nabla_v D_x^s f\|_{L_{x,v}^2(w)}^2 d\tau,$$

and

$$\begin{aligned} \sup_{0 \leq t \leq T_0} \mathcal{E}(f(t)) & \leq 4\|f_0\|_{\tilde{H}^s}^2, \\ \sup_{0 \leq t \leq T_0} \mathcal{E}(g(t)) & \leq 4\|f_0\|_{\tilde{H}^s}^2, \end{aligned}$$

we have

$$(4.27) \quad \begin{aligned} & \int_0^t 1 + \|f\|_{\tilde{H}^s}^2 + \|D_x^s \nabla_v g\|_{L_{x,v}^2(w)}^2 d\tau \\ & \preceq \int_0^t 1 + 4\|f_0\|_{\tilde{H}^s}^2 d\tau + \int_0^t \|D_x^s \nabla_v g\|_{L_{x,v}^2(w)}^2 d\tau \\ & \preceq T_0 \left(1 + 4\|f_0\|_{\tilde{H}^s}^2\right) + 4\|f_0\|_{\tilde{H}^s}^2 \\ & \preceq_{T_0} 1. \end{aligned}$$

By Gronwall's inequality, we have $\|f - g\|_{\tilde{H}^s}^2 = 0$, i.e., $f \equiv g$, which completes the proof of uniqueness. Thus, we end the proof of Theorem 2.7. \square

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