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APPROXIMATION OF SOLUTIONS OF SPLIT MONOTONE VARIATIONAL INCLUSION PROBLEMS AND FIXED POINT PROBLEMS

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ABSTRACT. Many researchers have incorporated an inertial term and will continue to involve it in iterative algorithms due to the fact that it speeds up the rate of convergence which is desirable in applications. In this paper, we propose a new inertial extrapolation algorithm for solving split monotone variational inclusion problems which turns out to solve fixed point problems in the framework of real Hilbert spaces. Our proposed algorithm which is the generalization of split feasibility problems among many others, does not involve the knowledge of operator norm which is sometimes difficult in practice. Furthermore, we prove under some mild assumptions that the sequence generated recursively by our algorithm converges strongly to a common solution of split monotone variational inclusion problems and fixed point of $\kappa-$demicontractive mapping. Finally, we apply our algorithm to solve other related problems, precisely linear inverse problems. Some numerical illustrations are provided to further demonstrate the efficiency and competitiveness of our algorithm.

1. INTRODUCTION

Throughout this paper, unless otherwise stated H_1 and H_2 will denote real Hilbert spaces with inner product $\langle .,. \rangle$ and associated norm $\|.\|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2 respectively while $\longrightarrow$ and $\rightharpoonup$ will respectively represent strong and weak convergence.
The split feasibility problem (SFP) introduced and studied by Censor and Elfving [1] is formulated as follows:

$$\text{find } x^* \in C \quad \text{such that} \quad y^* = Tx^* \in Q, \tag{1.1}$$

where C and Q have their usual definition and $T \in R^{N\times M}$ is a real matrix. For the past two decades, (1.1) has recieved huge attention due to variety of its applications. It is worth mentioning that the SFP is the first known model for Split Inverse Problem (SIP) (see [2,3])which is formulated as follows:

$$\text{find } x^* \in X_1 \text{ that solvss } IP_1 \text{ such that } y^* = Tx^* \in X_2 \text{ solves } IP_2, \tag{1.2}$$

where IP_1 and IP_2 are two inverse problems respectively defined on two vector spaces X_1 and X_2. It was further investigated that the SFP has been used as a model in intensity-Modulated Radiation Therapy (IMRT) treatment planning. It has a wide application in phase retrieval, medical imaging, signal processing, data compression, computerized tomography among many others (see [4–10] for details). Furthermore, due to the interest of scientists and researchers, in the study of the SFP, a lot of modifications and generalizations have been made. For instance, Split Variational Inequality Problem (SVIP) is another form of SFP which is

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more general than the SFP. The SVIP which was first introduced by Censor et al. [2] can be formulated as follows:

(1.3) $$\text{find } x^* \in C \text{ that solvss } \langle f(x^*), x - x^* \rangle \geq 0, \forall x \in C,$$

(1.4) $$\text{and such that } \ y^* = Tx^* \in Q \ \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0 \ \forall y \in Q,$$

where C, Q, H_1 and H_2 have their usual definitions,$T : H_1 \to H_2$ is a bounded linear operator while $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given operators. If we consider (1.3) and (1.4) seperately, then we observed that (1.3) is a classical Variational Inequality Problem (VIP). Therefore, (1.3) and (1.4) can be viewed as an interesting combination of the VIP and SFP.
To solve the SVIP without a product space formulation, Censor et al. [2] proposed the following iterative algorithm. Let $\lambda > 0$. Select an arbitrary starting point $x_0 \in H_1$. Given the current iterate x_k, compute

(1.5) $$x_{k+1} = P_C(I - \lambda f)(x_k + \gamma T^*(P_Q(I - \lambda g) - I)Tx_k), \quad k \geq 1,$$

where $\gamma \in (0, 1/L)$ with L being the spectral radius of the operator T^*T and T^* is an adjoint of T. Let SOL(f, C) and SOL(g, Q) denote the solution set of (1.3) and (1.4) respectively. They proved that if the solution set of (1.3)-(1.4) is nonempty, then the sequence $\{x_k\}$ generated by the algorithm (1.5) converged weakly to a solution set of problem (1.3) and (1.4).
Moreso, if in (1.3) and (1.4) $C = H_1, Q = H_2$; and choosing $x := x^* - f(x^*) \in H_1$ in (1.3) and $y = T(x^*) - g(T(x^*) \in H_2$, in (1.4) we obtain the Split Zeros Problems (SZP) for two operators $f : H_1 \to H_1$ and $g : H_2 \to H_2$, a concept introduced and studied in section 7.3 of [2]. The SZP is formulated as follows:

(1.6) $$\text{find } \ x^* \in H_1 \ \text{ such that } \ f(x^*) = 0 \ \text{ and } \ g(T(x^*)) = 0.$$

Suppose we denote a normal cone by

$$N_C(v) := \{d \in H : \langle d, y - v \rangle \leq 0, \forall y \in C\},$$

where $v \in C$ and C is a nonemty, closed and convex set and a multivalued mapping, K is defined by

$$K(v) := \begin{cases} f(v) + N_C(v), v \in C, \\ \emptyset, \quad otherwise, \end{cases}$$

where f is some given operator, then, under a certain continuity assumption on f, it was shown in [43] that K is a maximal monotone mapping and $B^{-1}(0) = SOL(f, C)$. Further advancement has been made towards Inclusion Problems. The inclusion problem (see [11] and references therein) can be formed as follows:

(1.7) $$\text{find } \ x \in H \ \text{ such that } \ 0 \in f(x) + M(x),$$

where 0 is the zero vector in H, f is a single-valued map from H to itself while $M : H \to 2^H$ is a set valued mapping. In this line of research, Moudafi [12] introduced the Split Monotone Inclusion Variational Problem (SMVIP) which generalizes so many other constrained optimization problems. It is formulated as follows:

(1.8) $$\text{find } \ x^* \in H_1 \ \text{ such that } \ 0 \in f_1(x^*) + B_1(x^*)$$

and

(1.9) $$y^* = A(x^*) \in H_2 \ \text{ such that } \ 0 \in f_2(y^*) + B_2(y^*),$$

where f_1 and f_2 are two inverse strongly monotone, B_1 and B_2 are two maximal monotone and A is a bounded linear operator. We denote the solution set of (1.8)-(1.9) by:

$$\Gamma := \{x^* \in H_1 : 0 \in f_1(x^*) + B_1(x^*) \text{ and}$$
$$y^* = A(x^*) \in H_2 \text{ such that } \quad 0 \in f_2(y^*) + B_2(y^*)\}.$$

We can view (1.8) seperately as a Variational Inclusion Problem with its solution set $(B_1 + f_1)^{-1}$ and (1.9) is a variational inclusion problem with its solution set $(B_2 + f_2)^{-1}$. It was noted that the SMVIP generalizes the split fixed point problem, the split variational inequality problem, the split zero problem and the split feasibility problem (see [1, 2, 4–6, 13])

Suppose in SMVIP (1.8)-(1.9), $f_1 \equiv 0$ and $f_2 \equiv 0$, we obtain the following Split Variational Inclusion Problem:

$$\text{find} \quad x^* \in H_1 \text{ such that } \quad 0 \in B_1(x^*) \tag{1.10}$$

and

$$y^* = A(x^*) \in H_2 \text{ such that } \quad 0 \in B_2(y^*). \tag{1.11}$$

In [12], the iterative scheme for solving problem (1.10)-(1.11) was constructed below: for a given initial value $x_0 \in H_1$, the sequence $\{x_n\}$ generated by the following algorithm is given by:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \lambda > 0, \tag{1.12}$$

where $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are the resolvent operators associated with the maximal monotones, B_1 and B_2 respectively. They obtained weak convergence for the proposed theorems for solving (1.10)-(1.11).

Recently, inspired by the work of Byrne et al. [13], and Kazmi and Rizvi [14], Shehu and Ogbuisi [15] proposed the following algorithm for SMVIP (1.8)-(1.9) and Fixed Point Problem (FPP) for strictly pseudocontractive mapping, S:

$$\begin{cases} x_0 \in H_1, \\ w_n = (1 - \alpha_n)x_n, \\ y_n = J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n S y_n, \forall n \geq 0. \end{cases} \tag{1.13}$$

where $0 < \lambda < 2\mu, 2v$ and $\gamma \in (0, \frac{1}{L})$, Ldenotes the largest enginvector for the matrix A^*A, A is abounded linear operator while A^* is the adjoint of A, f_1 and f_2 are $v, \mu-$inverse strongly monotone operators and B_1 and B_2 are maximal monotone operators. They proved under some mild conditions that the sequence $\{x_n\}$ generated by the Algorithm (1.13) converged strongly to a point $p \in \Gamma \cap F(S)$ where Γ is the solution set of (1.8) and (1.9).

In 2018, Ezeora and Izuchukwu [16] followed the idea of Moudafi [12] and constructed a new iterative scheme for approximation of a solution of split variational inclusion problem. The following algorithm was presented: given the initial values, $x_1, u \in H_1$,

$$\begin{cases} u_n = (1 - \beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - T_\gamma A u_n), \\ x_{n+1} = J_\lambda^M(I - \lambda f)y_n, n \geq 1. \end{cases} \tag{1.14}$$

where $T_\gamma := \gamma I + (1 - \gamma)S$ with $\gamma \in [\mu, 1), \{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, \dfrac{1}{\|A\|^2}), \lambda \in (0, 2\alpha)$ and $\{\beta_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$, f is an $\alpha-$inverse strongly monotone, S is μ strictly pseudocontractive mapping while M is a multi-valued maximal monotone mapping. They proved that the

sequence $\{x_n\}$ generated by (1.14) strongly converged to an element of $\Omega := \{z \in (M+f)^{-1}(0) : Az \in F(S)\} \neq \emptyset$.
In 2020, a modification of [16] was established by Izuchukwu et. al. [17]. They proposed an inertial method for solving generalized split feasibility problems over the solution set of (1.8). They presented the following algorithm: given the initial guess $x_0, x_1 \in H_1$. Let $\{x_n\}$ be a sequence generated recursively by

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ w_n = u_n - \tau_n T^*(I-S)Tu_n, \\ y_n = J^B_{\lambda_n}(I-\lambda_n A)w_n = (I+\lambda_n B)^{-1}(I-\lambda_n A)w_n, \\ z_n = y_n - \lambda_n(Ay_n - Aw_n), \\ x_{n+1} = (1-\theta_n-\beta_n)w_n + \theta_n z_n, n \geq 1, \end{cases} \tag{1.15}$$

where $0 \leq \alpha_n \leq \overline{\alpha_n}$, and

$$\overline{\alpha_n} \quad := \quad \begin{cases} min\{\frac{n-1}{n+\alpha-1}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\} \ if x_n \neq x_{n-1}, \\ \frac{n-1}{n+\alpha-1} \qquad otherwise. \end{cases} \tag{1.16}$$

and

$$\lambda_{n+1} \quad := \quad \begin{cases} min\{\mu\frac{\|w_n-y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\} \ if Aw_n \neq Ay_n, \\ \lambda_n \qquad otherwise, \end{cases} \tag{1.17}$$

T is a bounded linear operator, A is a monotone and Lipschitz continuous operator, B is a maximal monotone operator and S is a nonexpansive mapping. They proved under some assumptions that the sequence $\{x_n\}$ generated by algorithm (1.15) converged strongly to a point $p \in \Gamma = \{z \in (A+B)^{-1}(0) : Tz \in F(S)\} \neq \emptyset$.

In recent years, there has been tremendous interest in developing the fast convergence of algorithms, especially for the inertial type extrapolation method, which was first proposed by Polyak in [25]. Recently, some researchers have constructed different fast iterative algorithms by means of inertial extrapolation techniques,for example, inertial Mann algorithm [26], inertial forward–backward splitting algorithm [27, 28], inertial extragradient algorithm [29, 30], inertial projection algorithm [31, 32], and fast iterative shrinkage–thresholding algorithm (FISTA) [33].

We study problem (1.8)-(1.9) such that the solution set involves a fixed of demicontractive mapping. To be precise, Let H_1 and H_2 have their usual definition. Let A be a bounded linear operator from H_1 to H_2. Let f_1 and f_2 are $\mu, v-$ inverse strongly monotone. Let B_1 and B_2 be two multi-valued maximal monotone operators. Let $S : H_1 \to H_1$ be $\kappa-$ demicontractive mapping and $\Gamma \cap F(S) \neq \emptyset$. Our interest is to solve the following problem:

$$\text{find} \quad x^* \in H_1 \ \text{ such that } \quad 0 \in f_1(x^*) + B_1(x^*), Sx^* = x^* \tag{1.18}$$

and

$$y^* = A(x^*) \in H_2 \ \text{ such that } \quad 0 \in f_2(y^*) + B_2(y^*). \tag{1.19}$$

Remark 1.1: It turns out that if S is choosen to be an identity operator in H_1, we recover the work of Moudafi [12] which is a generalization of SVIP and SZP studied by Censor et al., [2]. It is worth noting that (see [12])

$$0 \in f_1(x^*) + B_1(x^*) \Longleftrightarrow x^* = J^{B_1}_\lambda(I - \lambda f_1)x^*$$

and

$$0 \in f_2(y^*) + B_2(y^*) \Longleftrightarrow y^* = J_\lambda^{B_2}(I - \lambda f_2)y^*.$$

See also that if $J_\lambda^{B_1} = I$ and $J_\lambda^{B_2} = I$, then problem (1.18)- (1.19) becomes just a fixed point problem for demicontractive operator. However, if $B_2 \equiv 0$, $f_2 \equiv 0$, and S is defined from H_2 to itself, then we recover the result of Ezeora and Izuchukwu [16] and Izuchukwu et al., [17]. Therefore, (1.18) - (1.19) generalizes so many other optimization problems (see [21–24]).

Remark 1.2: The major drawbacks in the work of Byrne et al. [13], Kazmi and Rizvi [14], Shehu and Ogbuisi [15], Ezeora and Izuchukwu [16] is that the stepsize used heavily relied in the operator norm which is difficult to estimate and in many cases, impossible in practice. The **Remark 3.4 (a-b)** of Izuchukwu et al. [17] is very interesting feature that truly improved the results of [13–16]. The stepsize λ_n is constructed such that it is generated at each iteration and hence, does not depend on the Lipschtiz constant L of the operator A. However, we gently remark that the control sequence τ_n is dependent on the norm of the bounded linear operator T. This is a major drawback in the announced result of Izuchukwu et al. [17].

Motivated and inspired by the excellent work of Byrne et al. [13], Kazmi and Rizvi [14], Shehu and Ogbuisi [15], Ezeora and Izuchukwu [16] and Izuchukwu et al. [17], we propose a new inertial iterative scheme for finding a solution of SMVIP (1.8)-(1.9). Thus, our contribution in this paper should include the following:

1) In order to improve on the rate of convergence, we incorperate inertial extrapolation term $(\alpha_n(x_n - x_{n-1}))$ in our proposed algorithm which is desirable in applications.

2) Our inertial term neither does it involve computation of the norm difference between x_n and x_{n-1} nor requires that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$.

3) We construct our iterative scheme in such a way that is does not depend on the operator norm of the bounded linear operator as in the case of [13–17] which is one of the major improvements in the above mentioned research papers.

4) Our choice of operator S is more general than the use in [13–17] (see remark 2.1 below). It is well known that the class of decontractive mapping is more general than that of strictly pseudocontractive mapping and nonexpansive mappings.

5)We also present some numerical illustrations of the proposed method in comparison with Algorithms (1.13), (1.14) and (1.15) to further show the efficiency of our scheme.

6) We apply our algorithm to solve related inverse problem. To be explicit, we solve a linear inverse problem (LIP).

The rest of the paper is organized as follows: In section two, we state without proofs the relevant lemmas that will be helpful to achieving our result, some definitions are also stated. Section three is the algorithm that we propose in this paper and some assumptions that help to established strong convergence. The convergence analysis is discussed in section four. In section five, we apply our algorithm to solve linear inverse problems. Section six deals with numerical illustration for comparison of our algorithm with others in this research direction, follow by result and discussion. The conclusion of our work is presented in section seven.

2. Preliminaries

Let $T : H \to H$ be a nonlinear map. A point $x \in H$ is called the fixed point of T if $Tx = x$. The set of fixed point of T is denoted by $F(T) := \{x \in H : Tx = x\}$.

Definition 2.1. The operator T is said to be:
i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|; \forall x, y \in H,$$

ii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, , \forall x, y \in H;$$

iii) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \forall x \in H; p \in F(T),$$

iv) strictly quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| < \|x - p\| \forall x \in H, p \in F(T)$,
iii) strictly pseudocontractive if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(x - y) - (Tx - Ty)\|^2 \quad \forall x, y, \in H;$$

iv) demicontractive if $F(T) \neq \emptyset$ and there exits $\kappa \in [0, 1)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \kappa\|x - Tx\|^2 \quad \forall x \in H; p \in F(T).$$

v) $\alpha-$inverse strongly monotone (ism), if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \leq \alpha\|Tx - Ty\|^2 \ \ \forall x, y \in H;$$

vi) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x, y \in H;$$

vii) L-ipschitz continuous, if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \forall x, y \in H.$$

Remark 2.1: Clearly, demicontractive map (iv) is more general than (i)-(iii). Also, $\alpha-$inverse strongly monotone operators are monotone. It is well known that every $\alpha-$inverse strongly monotone operator is $\frac{1}{\alpha}-$Lipschitz coninuous.

Example 1 [47]: Let H be the real line $C = [-1, 1]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{2}{3}xsin(\frac{1}{x}), if x \neq 0, \\ 0, \quad if \ \ x = 0. \end{cases} \tag{2.1}$$

Then, T is demicontractive but not strictly pseudocontractive.
Example 2 [48] Let H be the real line and $C = [-2, 1]$. Define $T : C \to C$ by $Tx = -x^2 - x$. Then, T is demicontractivewhich is not quasi-nonexpansive.
If T is a multivalued map, that is $T : H \to 2^H$, then T is called monotone if

$$\langle x - y, u - v\rangle \geq 0, \ \forall x, y \in H, u \in T(x), v \in T(y),$$

and T is called maximal monotone, if the graph

$$G(T) := \{(x, y) \in H \times H : y \in T(x)\}$$

of T is not properly contained in the graph of any other monotone operator. In other words, T is maximal monotone if and only if for $(x, u) \in H \times H$, $\langle x-y, u-v\rangle \geq 0, \ \forall (y, v) \in G(T)$ implies $u \in T(x)$. The resolvent operator J_λ^T associated with a multivalued map T and λ is a mapping $J_\lambda^T : H \to 2^H$ defined by

$$J_\lambda^T(x) = (I + \lambda T)^{-1}x, x \in H, \lambda > 0.$$

In [34], it was shown that the resolvent operator J_λ^T is single-valued, nonexpansive and 1-inverse strongly monotone and the solution of (1.7) is a fixed point of $J_\lambda^T(x)(I - \lambda f), \lambda > 0$ (for example, see [35]). Suppose

f is $\alpha-$ inverse strongly monotone with $0 < \lambda < 2\alpha$,then clearly $J^T_\lambda(x)(I-\lambda f)$ is nonexpansive and not only nonexpansive, it is firmly nonexpansive. Also $(f_1+B_1)^{-1} := \{z \in H : 0 \in (f_1+B_1)(z)\}$ is closed and convex.

The T is said to be averaged (see, [44] for details) if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T := (1-\beta)I + IS,$$

where $\beta \in (0,1)$ and $S : H \to H$ is a nonexpansive mapping. It is known that every averaged mapping is nonexpansive and every firmly nonexpansive mapping is averaged. Therefore, since the resolvent of maximal monotone operators are firmly nonexpansive, it is clear that they are averaged (see, [45]).

Recall that for a nonempty closed and convex subset C of$H$, the metric projection denoted by P_C is a map from C to H which assigns for each $x \in H$ a unique point $P_C(x) \in C$ such that

$$\|x - P_C(x)\| = inf\{\|x-y\| : y \in C\}.$$

The P_C is characterized by the following inequality

$$\langle x - P_C(x), z - P_C(x)\rangle \leq 0, \quad \forall z \in C.$$

The following results will be very useful in our work: Let H be a real Hilbert space. Then for all $x, y \in H$, the following hold:

Lemma 2.1

(i) $2\langle x,y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2$;

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;

(iii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$, for some $\alpha \in (0,1)$.

Lemma 2.2 [36] (Demicloseness principle) Let $T : C \to C$ be a demicontractive mapping. Then, $I - T$ is demiclosed at 0 that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.

Lemma 2.3 [35]: Let H be a real Hilbert space, $A : H \to H$ be a maximal monotone and Lipschitz continuous operator and $B : H \to 2^H$ be a maximal monotone operator. Then, the operator $(A+B) : H \to 2^H$ is maximal monotone.

Lemma 2.4 [38]: Let $\{a_n\}$ be a sequence of non-negative real numbers,$\{\beta_n\}$ be a sequence in $(0,1)$ and d_n be a sequence of real numbers such that

$$a_{n+1} \leq (1-\beta_n)a_n + \beta_n d_n, n \geq 0.$$

If 1) $\limsup_{n\to\infty} d_n \leq$ or $\sum_{n=0}^{\infty} \|\beta_n d_n\| < \infty$ and

2) $\sum_{n=1}^{\infty} \beta_n = \infty$,

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 [39]: Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following condition:

$$a_{n+1} \leq (1-\beta_n)a_n + \beta_n d_n + \gamma_n + \eta_n, n \geq 1,$$

where $\{\beta_n\}$ be a sequence in $(0,1)$ and γ_n and $\eta_n$are sequences of real numbers. Assume that

1) $\sum_{n=1}^{\infty} \eta_n < \infty$ and

2) $\gamma_n \leq \beta_n M$ for some $M \geq 0$.

Then, $\{a_n\}$ is a bounded sequence.

3. THE PROPOSED ALGORITHM

We present in this section our proposed iterative scheme, assumptions and its features. However, we highlight the advantages it has over others in this research direction.

Assumption 3.1:

a) $A : H_1 \to H_2$ is a bounded linear operator such that $A \neq 0$ with $A^* : H_2 \to H_1$ its adjoint. Also, $S : H_1 \to H_1$ is a demicontractive mapping such that $I - S$ is demiclosed at zero.

b) $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are $\mu, v-$inverse strongly monotone respectively.

c) $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two multivalued maximal monotone operators.

d)The solution set

$$\Gamma := \{x^* \in H_1 : 0 \in f_1(x^*) + B_1(x^*) \text{ and}$$

$$y^* = Ax^* \in H_2 \text{ such that } \quad 0 \in f_2(y^*) + B_2(y^*)\} \cap F(T) = \Gamma \cap F(S) \neq \emptyset.$$

Assumption 3.2: The control sequences $\{\theta_n\}, \{\beta_n\}, \{\alpha_n\}$ and $\{\epsilon_n\}$ satisfy the following conditions:

a) $\{\beta_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$,

b) $\{\theta_n\} \subset (a, 1 - \beta_n)$ for some $a > 0$,

c) $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$.

Algorithm 3.3: The inertial extrapolation algorithm for SMVIP and FPP

Step 0: Choose sequences $\{\theta_n\}, \{\theta_n\}, \{\alpha_n\}$ and $\{\varepsilon_n\}$ such that the conditions from Assumption 3.2 hold. Let $\lambda > 0, \alpha = 3$ and $x_0, x_1 \in H_1$ be arbitrarily chosen. Set $n := 1$.

Iterative steps: Step 1. Given the current iterates x_{n-1} and x_n and choose α_n such that $0 \le \alpha_n \le \overline{\alpha_n}$, where

$$\overline{\alpha_n} \quad := \quad \begin{cases} min\{\frac{n-1}{n+\alpha-1}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\} \ if x_n - x_{n-1} > 0, \\ \frac{n-1}{n+\alpha-1} \qquad otherwise. \end{cases} \tag{3.1}$$

and compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}). \tag{3.2}$$

Step 2: compute

$$y_n = J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma_n A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n), \tag{3.3}$$

where the step size γ_n is chosen such that for small enough $\epsilon > 0$, $\gamma_n \in (\epsilon, \frac{\|(T-I)Aw_n\|^2}{\|A^*(T-I)Aw_n\|^2} - \epsilon)$ provided $TAw_n \neq Aw_n$

Step 3: compute

$$x_{n+1} = (1 - \theta_n - \beta_n)y_n + \theta_n Sy_n. \tag{3.4}$$

Set $n := n + 1$ and go back to **Step 1.**

Remark 3.4: (I) The choice of choosing $\alpha = 3$ in the inertial factor plays an important role in the rate of convergence of our proposed algorithm.

4. Convergence Analysis

Lemma 4.1: Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 such that Assumption 3.1 and 3.2 hold. Then $\{x_n\}$ is bounded.

For conveniences, we shall denote $U := J_\lambda^{B_1}(I-\lambda f_1)$ and $T := J_\lambda^{B_2}(I-\lambda f_2.)$.

Proof: Let $p \in \Gamma \cap F(S)$, we have $p = J_\lambda^{B_1}p, Ap = J_\lambda^{B_2}(Ap)$ and $Sp = p$. Now, from (3.1) of Algorithm 3.3, we see that

$\alpha_n\|x_n - x_{n-1}\| \leq \varepsilon_n, \forall n \in N$

so that

$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq \frac{\varepsilon_n}{\beta_n} \to 0$ as $n \to \infty$.

Hence, there exists $M_1 \geq 0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq M_1. \;\; \forall n \in N. \tag{4.1}$$

Next, for all $p \in \Gamma \cap F(S)$, we get that

$$\begin{aligned}
\|w_n - p\| &\leq \|x_n - p\| + \alpha_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\| + \beta_n\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \\
&\leq \|x_n - p\| + \beta_n M_1. &(4.2)
\end{aligned}$$

Observe that

$$\begin{aligned}
\gamma_n\langle w_n - p, A^*(T-I)Aw_n\rangle &= \gamma_n\langle Aw_n - Ap, (T-I)Aw_n\rangle \\
&= \gamma_n\langle Aw_n - Ap + (T-I)Aw_n - (T-I)Aw_n, (T-I)Aw_n\rangle \\
&= \gamma_n\langle TAw_n - Ap, (T-I)Aw_n\rangle - \gamma_n\|(T-I)Aw_n\|^2 \\
&= \frac{1}{2}[\gamma_n\|TAw_n - Ap\|^2 + \gamma_n\|(T-I)Aw_n\|^2 - \gamma_n\|Aw_n - Ap\|^2] \\
&- \gamma_n\|(T-I)Aw_n\|^2 \\
&\leq \frac{1}{2}[\gamma_n\|Aw_n - Ap\|^2 + \gamma_n\|(T-I)Aw_n\|^2 - \gamma_n\|Aw_n - Ap\|^2] \\
&- \gamma_n\|(T-I)Aw_n\|^2 \\
&= -\frac{1}{2}\gamma_n\|(T-I)Aw_n\|^2. &(4.3)
\end{aligned}$$

Now, for all $p \in \Gamma \cap F(S)$ and with the condition on γ_n we get,

$$\begin{aligned}
\|y_n - p\|^2 &= \|U(w_n + \gamma_n(T-I)Aw_n) - P\|^2 \\
&= \|U(w_n + \gamma_n(T-I)Aw_n) - U(P)\|^2 \\
&\leq \|w_n + \gamma_n(T-I)Aw_n) - P\|^2 \\
&= \|w_n - p\|^2 + \gamma_n^2\|A^*(T-I)Aw_n\|^2 + 2\gamma_n\langle w_n - p, A^*(T-I)Aw_n\rangle \\
&\leq \|w_n - p\|^2 + \gamma_n^2\|A^*(T-I)Aw_n\|^2 - \gamma_n\|(T-I)Aw_n\|^2 \\
&= \|w_n - p\|^2 - [\gamma_n\|(T-I)Aw_n\|^2 - \gamma_n\|A^*(T-I)Aw_n\|^2 \\
&\leq \|w_n - p\|^2. &(4.4)
\end{aligned}$$

See also that

$$\begin{aligned}
\|(1-\theta_n-\beta_n)(y_n - p) + \theta_n(Sy_n - p)\|^2 &= (1-\theta_n-\beta_n)^2\|y_n - p\|^2 + \theta_n^2\|Sy_n - p\|^2 \\
&+ 2\theta_n(1-\theta_n-\beta_n)\langle y_n - p, Sy_n - p\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\theta_n-\beta_n)^2\|y_n-p\|^2+\theta_n^2\|Sy_n-Sp\|^2\\
&+ 2\theta_n(1-\theta_n-\beta_n)\|y_n-p\|\|Sy_n-Sp\|\\
&= [(1-\theta_n-\beta_n)^2+\theta_n^2+2\theta_n(1-\theta_n-\beta_n)]\|y_n-p\|^2\\
&\leq (1-\beta_n)^2\|y_n-p\|^2\\
&\leq (1-\beta_n)^2\|w_n-p\|^2. \qquad (4.5)
\end{aligned}$$

It follows from (4.2), (3.4) of step 3 and (4.4) that,

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\theta_n-\beta_n)y_n+\theta_n Sy_n-p\|\\
&= \|(1-\theta_n-\beta_n)(y_n-p)+\theta_n(Sy_n-p)-\beta_n p\|\\
&\leq \|(1-\theta_n-\beta_n)(y_n-p)+\theta_n(Sy_n-p)\|+\beta_n\|p\|\\
&\leq (1-\beta_n)\|w_n-p\|+\beta_n\|p\|\\
&\leq (1-\beta_n)[\|x_n-p\|+\beta_n M]+\beta_n\|p\|\\
&= (1-\beta_n)\|x_n-p\|+(1-\beta_n)\beta_n M+\beta_n\|p\|\\
&\leq (1-\beta_n)\|x_n-p\|+\beta_n(M+\|p\|)\\
&= (1-\beta_n)\|x_n-p\|+\beta_n M_o, for some M_0\geq 0\\
&\leq max\{\|x_n-p\|, M_0\}\\
&\vdots\\
&\leq max\{\|x_1-p\|, M_0\}. \qquad (4.6)
\end{aligned}$$

Following the estimate and Lemma 2.5, we obtain that $\{x_n\}$ is bounded. We we deduce from the proof that $\{w_n\}, \{y_n\}, \{Sy_n\}$ and $\{Sw_n\}$ are all bounded sequences. This completes the proof of boundedness.

Lemma 4.2: Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 such that Assumption 3.1 and 3.2 hold. If there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to a point $q\in H_1$ and $\lim_{n\to\infty}\|y_n-Sy_n\| = 0 = \|w_n-x_n\|$ then, $q\in\Gamma\cap F(S)$.

Proof: To establish this result, we consider the following cases:

Case 1: Assume $\{\|x_n-p\|\}$ is monotone non-increasing sequence, then $\{\|x_n-p\|\}$ is convergent. Clearly, we obtain that

$$\lim_{n\to\infty}(\|x_n-p\|-\|x_{n+1}-p\|)=0.$$

We get from step 1 of the Algorithm 3.3 that

$$\begin{aligned}
\|w_n-x_n\| &= \alpha_n\|x_n-x_{n-1}\|\\
&= \beta_n\frac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|\to 0 \text{ as } n\to\infty. \qquad (4.7)
\end{aligned}$$

Next, we show that $\lim_{n\to\infty}\|y_n-Sy_n\|=0$.

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\theta_n-\beta_n)(y_n-p)+\theta_n(Sy_n-p)-\beta_n p\|^2\\
&= \|(1-\theta_n-\beta_n)(y_n-p)+\theta_n(Sy_n-p)\|^2+\beta_n^2\|p\|\\
&\quad -2\beta_n\langle(1-\theta_n-\beta_n)(y_n-p)+\theta_n(Sy_n-p), p\rangle\\
&= \|(1-\theta_n-\beta_n)(y_n-p)+\theta_n(Sy_n-p)\|^2+\beta_n^2\|p\|^2\\
&\quad +2\beta_n\langle(1-\theta_n-\beta_n)(y_n-p)+\theta_n(p-Sy_n), p\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \|(1-\theta_n-\beta_n)(y_n-p)+\theta_n(Sy_n-p)\|^2+\beta_n^2\|p\|^2\\
&\quad +2\beta_n M_2 \text{ for some } \; M_2>0\\
&= (1-\theta_n-\beta_n)^2\|y_n-p\|^2+\theta_n^2\|Sy_n-p\|^2+2\theta_n(1-\theta_n-\beta_n)\langle y_n-p,Sy_n-p\rangle\\
&\quad +(\beta_n\|p\|)^2+2\beta_n M_2\\
&\leq (1-\theta_n-\beta_n)^2\|y_n-p\|^2+\theta_n^2\|Sy_n-p\|^2+2\theta_n(1-\theta_n-\beta_n)\|y_n-p\|\|Sy_n-p\|\\
&\quad +(\beta_n\|p\|)^2+2\beta_n M_2\\
&\leq (1-\theta_n-\beta_n)^2\|y_n-p\|^2+\theta_n^2\|Sy_n-p\|^2+\theta_n(1-\theta_n-\beta_n)[\|y_n-p\|^2+\|Sy_n-p\|^2]\\
&\quad +(\beta_n\|p\|)^2+2\beta_n M_2\\
&= [(1-\theta_n-\beta_n)^2+\theta_n(1-\theta_n-\beta_n)]\|y_n-p\|^2+[\theta_n^2+\theta_n(1-\theta_n-\beta_n)]\|Sy_n-p\|^2\\
&\quad +(\beta_n\|p\|)^2+2\beta_n M_2\\
&\leq [(1-\theta_n-\beta_n)^2+\theta_n(1-\theta_n-\beta_n)]\|y_n-p\|^2+[\theta_n(1-\beta_n][\|y_n-p\|^2+\kappa\|y_n-Sy_n\|^2\\
&\quad +(\beta_n\|p\|)^2+2\beta_n M_2\\
&= [(1-\theta_n-\beta_n)^2+\theta_n(1-\theta_n-\beta_n)+\theta_n(1-\beta_n)]\|y_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2\\
&\quad +(\beta_n\|p\|)^2+2\beta_n M_2\\
&\leq (1-\beta_n)^2\|w_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+(\beta_n\|p\|)^2+2\beta_n M_2\\
&\leq (1-\beta_n)\|w_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+(\beta_n\|p\|)^2+2\beta_n M_2\\
&\leq \|w_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+(\beta_n\|p\|)^2+2\beta_n M_2\\
&\leq \|x_n-p\|^2+\beta_n M_1-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+(\beta_n\|p\|)^2+2\beta_n M_2\\
&= \|x_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+\beta_n[M_1+\beta\|p\|^2+2M_2]. \qquad (4.8)
\end{aligned}
$$

From (4.8) and condition on β_n, we get

$$\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2 \quad\leq\quad \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n[M_1+\beta\|p\|^2+2M_2]\to 0. \tag{4.9}$$

Therefore, from (4.9) we have that

$$\lim_{n\to\infty}\|y_n-Sy_n\|=0. \tag{4.10}$$

From step 2 of algorithm 3.3, we know that

$$\gamma_n\|A^*(T-I)Aw_n\|^2<\|(T-I)Aw_n\|^2-\epsilon\|A^*(T-I)Aw_n\|^2. \tag{4.11}$$

Now, using (4.4),(4.8) and (4.11) we get

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\beta_n)^2\|y_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+(\beta_n\|p\|)^2+2\beta_n M_2\\
&\leq (1-\beta_n)\|y_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+\beta_n^2\|p\|^2+2\beta_n M_2\\
&\leq \|y_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Ty_n\|^2+\beta_n^2\|p\|^2+2\beta_n M_2\\
&\leq \|w_n-p\|^2+\gamma_n[\gamma_n\|A^*(T-I)Aw_n\|^2-\|(T-I)Aw_n\|^2]-\kappa\theta_n(\beta_n-1)\|y_n-Ty_n\|^2\\
&\quad +\beta_n^2\|p\|^2+2\beta_n M_2\\
&< \|x_n-p\|^2+\beta_n M_1-\epsilon\gamma_n\|A^*(T-I)Aw_n\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2\\
&\quad +\beta_n^2\|p\|^2+2\beta_n M_2\\
&< \|x_n-p\|^2+\beta_n M_1-\epsilon\gamma_n\|A^*(T-I)Aw_n\|^2+\beta_n^2\|p\|^2+2\beta_n M_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\epsilon\gamma_n\|A^*(T-I)Aw_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_nM_1+\beta_n^2\|p\|^2+2\beta_nM_2\\
&= \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n[M_1+\beta\|p\|^2+2M_2]\to 0, n\to\infty. \qquad (4.12)
\end{aligned}
$$

Thus,

$$
\lim_{n\to\infty}\|A^*(T-I)Aw_n\|=0. \qquad (4.13)
$$

Consequently, from (4.11) and (4.12)

$$
\begin{aligned}
\gamma_n\|(T-I)Aw_n\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\gamma_n^2\|A^*(T-I)Aw_n-p\|^2\\
+\beta_n[M_1+\beta\|p\|^2+2M_2]\to 0, n\to\infty. \qquad (4.14)
\end{aligned}
$$

Therefore,

$$
\lim_{n\to\infty}\|(T-I)Aw_n\|=0. \qquad (4.15)
$$

Next, using the nonexpansiveness of the resolvent operator U,we show that $\lim_{n\to\infty}\|y_n-w_n\|=0$

$$
\begin{aligned}
\|y_n-p\|^2 &= \|U(w_n+\gamma_n(T-I)Aw_n)-U(P)\|^2\\
&\leq \langle U(w_n+\gamma_nA^*(T-I)Aw_n)-U(p), w_n+\gamma_nA^*(T-I)Aw_n-p\rangle\\
&\leq \langle y_n-p, w_n+\gamma_nA^*(T-I)Aw_n-p\rangle\\
&= \frac{1}{2}[\|y_n-p\|^2+\|w_n+\gamma_nA^*(T-I)Aw_n-p\|^2\\
&\quad -\|y_n-w_n-\gamma_nA^*(T-I)Aw_n\|^2]\\
&= \frac{1}{2}[\|y_n-p\|^2+\|w_n-p\|^2+\gamma_n^2\|A^*(T-I)Aw_n\|^2+2\gamma_n\langle w_n-p, A^*(T-I)Aw_n\rangle\\
&\quad -\|y_n-w_n\|^2-\gamma_n^2\|A^*(T-I)Aw_n\|^2+2\gamma_n\langle y_n, A^*(T-I)Aw_n\rangle]\\
&\leq \frac{1}{2}[\|y_n-p\|^2+\|w_n-p\|^2-\|y_n-w_n\|^2+2\gamma_n\|w_n-p\|\|A^*(T-I)Aw_n\|\\
&\quad +2\gamma_n\|y_n-w_n\|\|A^*(T-I)Aw_n\|]\\
&\leq \frac{1}{2}[\|y_n-p\|^2+\|w_n-p\|^2-\|y_n-w_n\|^2\\
&\quad +2\gamma_n\|A^*(T-I)Aw_n\|[\|w_n-p\|+\|y_n-y_n\|]]\\
&\leq \|w_n-p\|^2-\|y_n-w_n\|+2\gamma_n\|A^*(T-I)Aw_n\|[\|w_n-p\|+\|y_n-w_n\|]. \qquad (4.16)
\end{aligned}
$$

Using (4.16) and (4.12), we estimate that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\beta_n)\|y_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+\beta_n^2\|p\|^2+2\beta_nM_2\\
&\leq \|y_n-p\|^2-\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+\beta_n^2\|p\|^2+2\beta_nM_2\\
&\leq \|w_n-p\|^2-\|y_n-w_n\|+2\gamma_n\|A^*(T-I)Aw_n\|[\|w_n-p\|+\|y_n-w_n\|]\\
&\quad -\kappa\theta_n(\beta_n-1)\|y_n-Sy_n\|^2+\beta_n^2\|p\|^2+2\beta_nM_2\\
&\leq \|w_n-p\|^2-\|y_n-w_n\|+2\gamma_n\|A^*(T-I)Aw_n\|[\|w_n-p\|+\|y_n-w_n\|]\\
&\quad +\beta_n^2\|p\|^2+2\beta_nM_2\\
&\leq \|x_n-p\|^2+\beta_nM_1-\|y_n-w_n\|+2\gamma_n\|A^*(T-I)Aw_n\|[\|w_n-p\|+\|y_n-w_n\|]\\
&\quad +\beta_n^2\|p\|^2+2\beta_nM_2. \qquad (4.17)
\end{aligned}
$$

We therefore obtain from (4.17) that

$$\begin{aligned} \|y_n - w_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\gamma_n \|A^*(T-I)Aw_n\|[\|w_n - p\| + \|y_n - w_n\|] \\ &\quad + \beta_n [M_1 + \beta \|p\|^2 + 2M_2] \to 0, n \to \infty. \end{aligned} \tag{4.18}$$

Hence, it follows from (4.18) that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{4.19}$$

Consequently, from(4.7) and (4.19) we get that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.20}$$

Using (4.10) and (4.20), we estimate that

$$\lim_{n\to\infty} \|Sy_n - x_n\| = 0. \tag{4.21}$$

Again, using (4.10) and (4.19) we obtain that

$$\lim_{n\to\infty} \|Sy_n - w_n\| = 0. \tag{4.22}$$

Finally, we show that $\{x_n\}$ is asymptotically regular, that is, $\|x_{n+1} - x_n\| \to 0$.

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\theta_n-\beta_n)(y_n - x_n) + \theta_n(Sy_n - x_n) - \beta_n x_n\| \\ &\leq \|(1-\theta_n-\beta_n)(y_n - x_n) + \theta_n(Sy_n - x_n)\| + \beta_n\|x_n\| \\ &\leq (1-\theta_n-\beta_n)\|y_n - x_n\| + \theta_n\|Sy_n - x_n\| + \beta_n\|x_n\| \to 0. \end{aligned} \tag{4.23}$$

Therefore, using the estimates (4.20) (4.21) and the condition on β_n, i.e., Assumption 3.2 (a) we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.24}$$

Denote $u_n = w_n + \gamma_n A^*(T-I)Aw_n$, so that

$$\|u_n - w_n\|^2 = \gamma_n^2 \|A^*(T-I)Aw_n\|^2 \to 0. \tag{4.25}$$

Combining (4.19) and (4.25), we conclude that

$$\|y_n - u_n\| \to 0, n \to \infty. \tag{4.26}$$

It follows from the boundedness of $\{y_n\}$ that there exists $\{y_{n_j}\}$ of $\{y_n\}$ that converges. Without loss of generality, we may assume that $y_{n_j} \rightharpoonup q$ as $j \to \infty$. Using Limma 2.2, (4.10) and the fact that $I - S$ is demiclosed at zero, we obtain that $p \in F(S.)$ Consequently, $\{x_n\}$ and $\{w_n\}$ converge weakly to the point p.

Next, we show that $q \in (B_1 + f_1)^{-1}$. From **Remark 2.1** and by Lemma 2.3, we deduce that $B_1 + f_1$is maximal monotone. Ultilizing the definition and property of maximal monotone, let $(v, h) \in (B_1 + f_1)$ be arbitrary. It follows that $h - f_1 v \in B_1(v)$.
From the fact that $y_n = Uu_n = J_\lambda^{B_1}(I - \lambda f_1)u_n$, we obtain that

$$(I - \lambda f_1)u_n \in (I + \lambda B_1)y_n.$$

Hence,

$$\frac{1}{\lambda}(u_n - \lambda f_1 u_n - y_n) \in B_1(y_n).$$

Considering the fact that $(B_1 + f_1)$ is a maximal monotone, we get that

$$\langle v - y_n, h - f_1 v - \frac{1}{\lambda}(u_n - \lambda f_1 u_n - y_n)\rangle \geq 0.$$

We obtain from this last inequality that

$$\begin{aligned}
\langle v - y_n, h\rangle \geq \langle v - y_n, f_1 v + \frac{1}{\lambda}(u_n - \lambda f_1 u_n - y_n)\rangle &= \langle v - y_n, f_1 v - f_1 y_n + f_1 y_n - f_1 u_n + \frac{1}{\lambda}(u_n - y_n)\rangle \\
&= \langle v - y_n, f_1 v - f_1 y_n + f_1 y_n - f_1 u_n + \frac{1}{\lambda}(u_n - y_n)\rangle \\
&\geq 0 \quad 0 + \langle v - y_n, f_1 y_n - f_1 u_n\rangle \\
&\quad + \langle v - y_n, \frac{1}{\lambda}(u_n - y_n)\rangle. \qquad (4.27)
\end{aligned}$$

It follows from (4.26), and the condition on f_1 (see remark 2.1), we get

$$\lim_{n\to\infty} \|f_1 y_n - f_1 u_n\| = 0.$$

Since $\{y_n\}$ is weakly convergent to a point p, we get (4.27) that

$$\lim_{n\to\infty} \langle v - y_n, h\rangle = \langle v - p, h\rangle \geq 0.$$

Thus, by the maximal monotonicity of $B_1 + f_1$, we get that $0 \in (B_1 + f_1)p \Rightarrow p \in (B_1 + f_1)^{-1}$.
In similiar argument we see that for $(\mu, v) \in G(B_2 + f_2)$ implies $z - f_2\mu \in B_2\mu$. Let

$$Ay_n = J_\lambda^{B_2}(I - \lambda f_2)Au_n.$$

That is,

$$\frac{1}{\lambda}(Au_n - \lambda f_2 u_n - Ay_n) \in B_2 Ay_n$$

.

We obtain from maximal monotonicity of $B_2 + f_2$

$$\langle \mu - Ay_n, z - f_2\mu - \frac{1}{\lambda}(Au_n - \lambda f_2 Au_n - Ay_n)\rangle \geq 0.$$

Using the fact that A is a bounded linear operator and (4.19),we obtain $Aw \rightharpoonup Ap$, Lemma 2.2 and from (4.15), we get that

$$0 \in f_2 Ap + B_2(Ap).$$

Implies that $Ap \in (B_2 + f_2)^{-1}$. Therefore, $Ap \in \Gamma \cap F(S)$ as required and this completes the proof of weak convergence.

Theorem 4.3: Let $\{x_n\}$ be a sequence generated by the Algorithm 3.3 under Assumption 3.1 and 3.2. Then, $\{x_n\}$ converges strongly to $p \in \Gamma \cap F(S)$ where

$$\|p\| = min\{\|z\| : z \in \Gamma \cap F(S)\}.$$

Proof: See that

$$\begin{aligned}
\|(1-\theta_n)y_n + \theta_n S y_n - p\|^2 &= \|(1-\theta_n)y_n - p) + \theta_n(Sy_n - p)\|^2 \\
&= (1-\theta_n)^2\|y_n - p\|^2 + \theta_n^2\|Sy_n - p\|^2 + 2\theta_n(1-\theta_n)\langle y_n - p, Sy_n - p\rangle \\
&\leq (1-\theta_n)^2\|y_n - p\|^2 + \theta_n^2\|Sy_n - p\|^2 + 2\theta_n(1-\theta_n)\|y_n - p\|.\|Sy_n - p\| \\
&\leq (1-\theta_n)^2\|y_n - p\|^2 + \theta_n^2\|Sy_n - p\|^2 + \theta_n(1-\theta_n)\|y_n - p\|^2 \\
&\quad + \theta_n(1-\theta_n)\|Sy_n - p\|^2 \\
&= [(1-\theta_n)^2 + \theta_n(1-\theta_n)]\|y_n - p\|^2 + [\theta_n^2 + \theta_n(1-\theta_n)]\|Sy_n - p\|^2
\end{aligned}$$

$$
\begin{aligned}
&= (1-\theta_n)\|y_n - p\|^2 + \theta_n\|Sy_n - Sp\|^2 \\
&\leq (1-\theta_n)\|y_n - p\|^2 + \theta_n[\|y_n - p\|^2 \\
&= \|y_n - p\|^2 \\
&\leq \|w_n - p\|^2 \\
&= \|x_n - p\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq \|x_n - p\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - p\|.\|x_n - x_{n-1}\| \\
&= \|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_n\|x_n - x_{n-1}\|] \\
&\leq \|x_n - p\|^2 + 3\alpha_n\|x_n - x_{n-1}\|M_3, \qquad (4.28)
\end{aligned}
$$

where $M_3 = sup\{\|x_n - p\|, \|x_n - x_{n-1}\|\}$.
Further more, using (4.28) and step 3 of the Algorithm, we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\beta_n)[(1-\theta_n)y_n + \theta_n Sy_n - p] - [\beta_n\theta_n(y_n - Sy_n) + \beta_n p\|^2] \\
&\leq (1-\beta_n)^2\|(1-\theta_n)y_n + \theta_n Sy_n - p\|^2 - 2\langle\beta_n\theta_n(y_n - Sy_n) + \beta_n p, x_{n-1} - p\rangle \\
&\leq (1-\beta_n)\|(1-\theta_n)y_n + \theta_n Sy_n - p\|^2 + 2\langle\beta_n\theta_n(y_n - Sy_n), p - x_{n+1}\rangle \\
&\quad +2\beta_n\langle p, p - x_{n+1}\rangle \\
&\leq (1-\beta_n)\|(1-\theta_n)y_n + \theta_n Sy_n - p\|^2 + 2\beta_n\theta_n\|y_n - Sy_n\|.\|x_{n+1} - p\| \\
&\quad +2\beta_n\langle p, p - x_{n+1}\rangle \\
&\leq (1-\beta_n)[\|x_n - p\|^2 + 3\alpha_n\|x_n - x_{n-1}\|M_3] + 2\beta_n\theta_n\|y_n - Sy_n\|.\|x_{n+1} - p\| \\
&\quad +2\beta_n\langle p, p - x_{n+1}\rangle \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n(3\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|M_3 + 2\theta_n\|y_n - Sy_n\|.\|x_{n+1} - p\| \\
&\quad +2\langle p, p - x_{n+1}\rangle) \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n d_n, \qquad (4.29)
\end{aligned}
$$

where $d_n = (3\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|M_3 + 2\theta_n\|y_n - Sy_n\|.\|x_{n+1} - p\| + 2\langle p, p - x_{n+1}\rangle)$.
We know from Lemma 4.1 that $\{x_n\}$ is bounded. Thus, there exists a subsequnce $\{x_{nj}\}$ of $\{x_n\}$ that weakly converges to a point $q \in H_1$ such that

$$\limsup_{n\to\infty}\langle p, p - x_{n_j}\rangle = \lim_{n\to\infty}\langle p, p - x_{n_j}\rangle = \langle p, p - q\rangle \leq 0. \qquad (4.30)$$

It follows from (4.30) that

$$\limsup_{n\to\infty}\langle p, p - x_{n_j+1}\rangle = \langle p, p - q\rangle \leq 0. \qquad (4.31)$$

The fact that $\limsup_{n\to\infty} d_n \leq 0$ frollows from (4.1), (4.10) and (4.31). Therefore, we obtain from the concluding part of Lemma 2.2 that $\lim_{n\to\infty}\|x_n - p\| = 0$. Hence, $\{x_n\}$ strongly converges to $p \in P_{\Gamma\cap F(S)}0$.
Case 2: Suppose that $\{\|x_n - p\|\}$ is not monotone decreasing sequence. Denote $\Omega_n = \|x_n - p\|^2$ and let $\tau : N \to N$ be a mapping for all $n \geq n_0$(for sufficiently large n_0) defined by:

$$\tau(n) := \ \max\{k \in N : k \leq n, \Omega_k \leq \Omega_{k+1}\}.$$

Then, it is easy to see that τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Omega_{\tau(n)} \leq \Omega_{\tau(n)+1}, \text{ for } n \geq n_o.$$

It follows from (4.8) that

$$\begin{aligned} 0 &\leq \|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)} - p\|^2 \\ &\leq \beta_n M_1 + \beta_n^2 \|p\|^2 + 2\beta_n M_2 - \kappa\theta_n(\beta_n - 1)\|y_n - Sy_n\|^2. \end{aligned} \tag{4.32}$$

This implies that

$$\kappa\theta_{\tau(n)}(\beta_{\tau(n)} - 1)\|y_{\tau(n)} - Sy_{\tau(n)}\|^2 \leq \beta_{\tau(n)} M_1 + \beta_{\tau(n)}^2 \|p\|^2 + \beta_{\tau(n)} M_2 \to 0.$$

Using the same argument as above (4.7) -(4.27), as in case one above, we deduce that $\{x_{\tau(n)}\}, \{y_{\tau(n)}\}$ and $\{w_{\tau(n)}\}$ are all weakly convergent to $p \in \Gamma \cap F(S)$. Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - p\|^2 - \|x_{\tau(n)} - p\|^2 \\ &\leq \beta_{\tau(n)} M_1 + \beta_{\tau(n)}^2 \|p\|^2 + 2\beta_{\tau(n)} M_2 - \|x_{\tau(n)} - p\|^2 \\ &= \beta_{\tau(n)}[M_1 + \beta_{\tau(n)} + 2\beta_{\tau(n)}] - \|x_{\tau(n)} - p\|^2. \end{aligned} \tag{4.33}$$

Thus,

$$\|x_{\tau(n)} - p\|^2 \leq \beta_{\tau(n)}[M_1 + \beta_{\tau(n)} + 2\beta_{\tau(n)}] \to 0. \tag{4.34}$$

Hence,

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\|^2 = 0.$$

It follows that

$$\lim_{n\to\infty} \Omega_{\tau(n)} = \lim_{n\to\infty} \Omega_{\tau(n)+1}.$$

Furthermore, for $n \geq n_0$, we see that $\Omega_{\tau(n)} \leq \Omega_{\tau(n)+1}$ if $\tau(n) < n$,
Since, $\Omega_j \geq \Omega_{j+1}$ for $\tau(n+1) \leq j \leq n$. Consequently, $\forall n \geq n_0$,

$$0 \leq \Omega_n \leq \max \{\Omega_{\tau(n)}, \Omega_{\tau(n)+1}\} = \Omega_{\tau(n)+1}.$$

Therefore,

$$\lim_{n\to\infty} \Omega_n = 0.$$

We conclude that $\{x_n\}, \{y_n\}$ and $\{w_n\}$ converge strongly to $p \in \Gamma \cap F(S) \ \ \forall n \geq n_0$. This completes the proof of Theorem 4.3.

We obtain the following corollares are the immediate consequences of Theorem 4.3.

Corollary 4.4: Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ with its adjoont A^*. Let $T : H_2 \to H_2$ be a nonexpansive map. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be $\upsilon-$ and $\mu-$ inverse strongly monotone respectively. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two multivalued maximal monotone operators. Let $\{x_n\}$ be an iterate sequence generated from the following algorithm,

$$\begin{cases} x_1, x_0, H_1, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma_n A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n), \\ x_{n+1} = (1 - \theta_n - \beta_n)y_n + \theta_n Sy_n, n \geq 0. \end{cases} \tag{4.35}$$

If the assumptions 3.1 and 3.2 hold, then the sequence $\{x_n\}$ strongly converges to the solution set of $\Gamma \cap F(T)$.

Assuming, $\theta_n \equiv 0$, and we have a linear combination of $\{x_n\}$, we obtain the following corollary.

Corollary 4.5 Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ with its adjoont A. Let $T : H_2 \to H_2$ be $\kappa-$ strictly pseudocontractive mapping. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be $\upsilon-$ and

$\mu-$ inverse strongly monotone respectively. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two multivalued maximal monotone operators. Let $\{x_n\}$ be an iterate generated from the following algorithm,

$$\begin{cases} x_1, x_0, H_1, \\ w_n = (1-\alpha_n)x_n, \\ y_n = J_\lambda^{B_1}(I-\lambda f_1)(w_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, n \geq 0, \end{cases} \tag{4.36}$$

If the assumptions 3.1 and 3.2 hold, then the sequence $\{x_n\}$ strongly converges to the solution set of $\Gamma \cap F(T)$.

Furthermore, if we are interested to study SFP of (1.1), and fixed point problem, we can set $C \equiv H_1, Q \equiv H_2, f_1 \equiv 0 \equiv f_2, B_1 \equiv 0 \equiv B_2$. We construct the following algorithm for the SFP.

Corollary 4.6 Let C and Q be two nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2. Let A be a bounded linear operator with A^* its adjoint. Assume the solution set Γ of (1.1) is nonempty. Let $\{x_n\}$ a sequence generated by the foolowing algorithm:

$$\begin{cases} x_0, x_1 \in H_1, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n + \gamma_n A^*(P_Q - I)Aw_n), \\ x_{n+1} = (1-\theta_n-\beta_n)y_n + \theta_n S y_n, n \geq 0. \end{cases} \tag{4.37}$$

where $\gamma_n \in (\epsilon, \frac{\|(P_Q-I)Aw_n\|^2}{\|A^*(P_Q-I)Aw_n\|^2} - \epsilon)$. If the assumptions 3.1 and 3.2 hold, then the sequence $\{x_n\}$ strongly converges to the solution set of $\Gamma \cap F(T)$.

5. Applications

In this section, we consider applying our algorithm to linear inverse problem.
Linear Inverse Problems (LIP) arises in many applications such as signal processing and image reconstrutions, astrophysics, statistical inference, optics among many others. A basic LIP is of the form:

$$Ax = b + w, \tag{5.1}$$

where $A \in R^{m\times n}$ and $b \in R^m$ are known, w is an unknown noise vector,and x is the unknown signal/image to be estimated (see [42]). In the case of image bluring problems, $b \in R^m$ represent the blurred image while x is the unknown true image, supposing to have the same dimension as b. The operator A is taken to be blur operator in which typically in this instance of spatially invariant, blurs represent two-dimensional convolution operator. The classical approach to problem (5.1) is the least square (LS) approach in which the estimator is chosen to minimize the data error as follows:

$$LS : \overline{x}_{LS} = \text{ argmin}\|Ax - b\|^2. \tag{5.2}$$

Taking this idea to our work, let $f_1, f_2 : H_1 \to R$ be convex and continously differentiable functions and $g_1, g_2 : H_1 \to R$ be a convex and lower semicontinous function. The Split Linear Inverse Problem can be formulated as follows:

$$\begin{aligned} &\text{find } \ x^* \in H_1 \text{ such that } f_1(x^*) + g_1(x^*) = \min_{x\in H_1}[f_1(x^*) + g_1(x^*)] \\ &\text{and } Ax^* \in H_2 \text{ such that } \ \ f_2(x^*) + g_2(Ax^*) = \min_{Ax^*\in H_2}[f_2(Ax^*) + g_2(Ax^*)], \end{aligned} \tag{5.3}$$

where A is a bounded linear operator defined on H_1 and S is a demicontractive mapping defined on H_2. Denote Ω the solution set of (5.3) and fixed point of S by $F(S)$. Then the required solution set is denoted by $\Omega \cap F(S)$. It is well known that if f_1, f_2 are convex and continously differentaible, then the gradient ∇f_1 of f_1 is $\frac{1}{v}-$ Lipschitz continous. Further, it is $v-$ inverse strongly monotone. Also, ∇f_2 of f_2 is $\frac{1}{\mu}-$Lipschitz continous, hence $\mu-$inverse strongly monotone. It is also known that ∂g_1 and ∂g_2 are maximal monotone (see [43]). However,

$$f_1(x^*) + g_1(x^*) = \min_{x \in H_1} [f_1(x^*) + g_1(x^*)] \Leftrightarrow 0 \in \nabla f_1(x^*) + \partial g_1(x^*),$$

and

$$f_2(Ax^*) + g_2(Ax^*) = \min_{x \in H_2} [f_2(Ax^*) + g_2(Ax^*)] \Leftrightarrow 0 \in \nabla f_2(Ax^*) + \partial g_2(Ax^*).$$

Setting $f_1 = \nabla f_1, f_2 = \nabla f_2$ and $\partial g_1 = B_1, \partial g_2 = B_2$ in algorithm 3.3, we obtain the following algorithm for LIP.

Algorithm 5.1

$$\begin{cases} x_0, x_1 \in H_1, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J_\lambda^{\partial g_1}(I - \lambda \nabla f_1)(w_n + \gamma_n A^*(J^{\partial g_2}(I - \lambda \nabla f_2) - I)Aw_n), \\ x_{n+1} = (1 - \theta_n - \beta_n)y_n + \theta_n S y_n, n \geq 1. \end{cases} \tag{5.4}$$

Let $\{x_n\}$ be a recursive sequence generated by the above algorithm, under some mild conditions, the sequence strongly converges to the solution set $\Omega \cap F(S)$.

6. Numerical Illustrations

In this section, we present computational experiment and compare the scheme we proposed in section three with existing methods; precisely, the efficiency was tested with, (1.14), (1.13) and (1.15). All the codes were written in MATLAB R2018a. All the computations were perfomed on personal computer with Intel(R) Core (TM) i5-4300U CPU at 1.90Ghz 2.49GHz with 8.00 Gb-RAM and 64-OS.

The vectors $x_0, x_1 \in H$ and $\gamma > 0$ were randomly selected. We choose $\alpha_n = \overline{\alpha_n} = \frac{n-1}{n+\alpha-1}$, with choices of α ranging from 2,3,6,and 10 $\beta_n = \frac{1}{5n+5}, \theta_n = 1 - \beta_n, \epsilon_n = \theta_n/n^2$. Since Shehu and Ogbuisi [15] chosed a stepsize that is dependent of the operator norm, we shall take $\gamma_n = \frac{1}{\|A\|^2}$ while in our algorithm, our stepsize γ_n is generated at each iteration.

In many applied problems in physical sciences and engineering, finding the minimum norm is very important. In control theory for example, minimum norm problem is used for the cases where isolated point constraints appear at immediate times and makes numerical results simple. In a Hilbert space setting precisely, when minimum norm is formulated, the existence, uniqueness and characterization of optimal controls are particularly very simple. In an abtract thinking, minimum norm problem can be formulated as follows:

$$\text{find } \ x^* \in H, \ \text{ with the property that } \ \|x^*\| = \min \ \{\|x\| : x \in H\}, \tag{6.1}$$

where H is a real Hilbert space. It is commonly known that in the case of variational inequality problem, (6.1) is equivalent to:

$$\text{find } \ x^* \in H, \ \text{ such that } \ \langle x^*, x^* - x \rangle \leq 0, \forall x \in H. \tag{6.2}$$

Let $H_1 = H_2 = L_2([0,1])$ be endowed with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \forall x, y \in L_2([0,1]), t \in [0,1]$$

and

$$\|x\| = \sqrt{\int_0^1 |x(t)|^2 dt}, \ \ \forall x, y \in L_2([0,1]), t \in [0,1].$$

Let $A : L_2([0,1]) \to L_2([0,1])$ be given by

$$Ax(s) = \int_0^1 V(s,t)x(t)dt, \forall x \in L_2([0,1]),$$

where $V : [0,1] \times [0,1] \to R$ is bounded. However, the adjoint of A which is A^* is also defined by

$$A^*x(s) = \int_0^1 V(t,s)x(t)dt, \ \forall x \in L_2([0,1]).$$

Let $\|.\|_{L^2} : L_2([a,b]) \to \mathbb{R}, C = \{x \in L_2 : \langle a, x\rangle = b\}$, for some $a \in L_2 - \{0\}$ and $Q = \{x \in L_2 : \langle a, x\rangle \geq b\}$ for some $a \in L_2 - \{0\}, b \in \mathbb{R}$. Then x^* minimizes $\|.\|_{L_2} + \delta_C$ if and only if $0 \in \partial(\|.\|_{L_2} + \delta_C)(x^*)$ and Ax^* minimizes $\|.\| + \delta_Q$ if and only if $0 \in \partial(\|.\|_{L_2} + \delta_Q)(Ax^*)$, where δ_C[defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise] and δ_Q stands for indicator function of C snd Q respectively and $\partial\phi$ for the subdifferential of $\phi[\partial\phi(x) := \{u \in H : \phi(y) + \langle u, y - x\rangle, \forall y \in H\}$. If in (1.8) and (1.9) we set $B_1 = \partial(\|.\|_{L_2} + \delta_C), B_2 = \partial(\|.\|_{L_2} + \delta_Q)$ with $f_1 = f_2 = 0$,we obtain the following Split Minimization Problem (SMP):

(6.3) $$\text{find } \ x^* \in C \ \text{ such that } \ x^* = argmin\{\|x\|_{L_2} : x \in C\},$$

and

(6.4) $$\text{find } \ y^* = Ax^* \in Q \ \text{ solves } \ y^* = argmin\{\|x\|_{L_2} : x \in Q\}.$$

Let Θ be a solution set of (6.3) and (6.4) and $\Theta \neq \emptyset$. Then, the solution to problem (6.3) and (6.4) is a minimum-norm solution. It is clearly seen from this example that Alhorithm 3.3 generalizes the SMP (see, e.g., [12]).

Suppose we define a function $h : R \to (-\infty, +\infty]$ by

(6.5) $$h(x) = \begin{cases} = -In(x) + x & \text{if} \ \ x > 0, \\ \infty, & otherwise. \end{cases}$$

Hence, h is a proper, lower semicontinous and convex function. $B := \partial h$ is maximal monotone. From [46], we obtain the resolvent of B by $J_1^B x = (I + B)^{-1}x = \frac{1}{2}(x - 1 + \sqrt{(x-1)^2} + 4)$.

In the light of [17], we shall consider the following cases:

Case 1: Take $x_1(t) = t^2 + 1, x_0(t) = e^t, \gamma_1 = 0.5$

Case 2: Take $x_1(t) = t^2 + 1, x_0(t) = e^t, \gamma_1 = 2$

Case 3: Take $x_1(t) = sin(t) + 2t, x_0(t) = t + e^t, \gamma_1 = 0.5$

Case 4: Take $x_1(t) = sin(t) + 2t, x_0(t) = t + e^t, \gamma_1 = 2$

The following tables and figures are the outputs generated from our Matlab codes. Error ($\|x_{n-1} - x_n\|^2$) and number of iteration (n) are considered as vertical and horizontal axis respectively.

Table 6.1: Numerical results comparing Alg. 3.3 at difference levels of α

		Alg.3.3 $(\alpha = 1)$	Alg.3.3 $(\alpha = 2)$	Al.3.3 $(\alpha = 3)$
Case 1	CPU time (sec)	1.245	1.056	0.067
	No. of iteration	11	8	4
Case 2	CPU time (sec)	1.516	2.035	0.467
	No. of iteration	11	8	4
Case 3	CPU time (sec)	1.574	2.446	1.047
	No. of iteration	12	9	5
Case 4	CPU time (sec)	1.571	2.449	1.467
	No. of iteration	12	9	5

Table 6.2: Numerical results comparing Alg. 3.3, Alg. 1.13 and Alg. 1.14

		Alg.3.3 $(\alpha = 3)$	Alg.1.13	Alg.1.14
Case 1	CPU time (sec)	0.067	3.156	5.467
	No. of iteration	4	12	13
Case 2	CPU time (sec)	0.467	3.367	5.467
	No. of iteration	4	13	14
Case 3	CPU time (sec)	1.047	4.446	5.047
	No. of iteration	4	15	15
Case 4	CPU time (sec)	1.467	6.009	6.467
	No. of iteration	4	16	16

Table 6.3: Numerical results comparing Alg. 3.3, and Alg. 1.15 at $(\alpha = 1, 2, 3)$

		Alg.3.3 $(\alpha = 3)$		Alg.1.13 $(\alpha = 3)$	
Case 1	CPU time (sec) No. of iteration	0.067	4	2.006	9
Case 2	CPU time (sec) No. of iteration	0.467	4	2.047	9
Case 3	CPU time (sec) No. of iteration	1.047	4	2.097	9
Case 4	CPU time (sec) No. of iteration	1.467	4	2.579	9

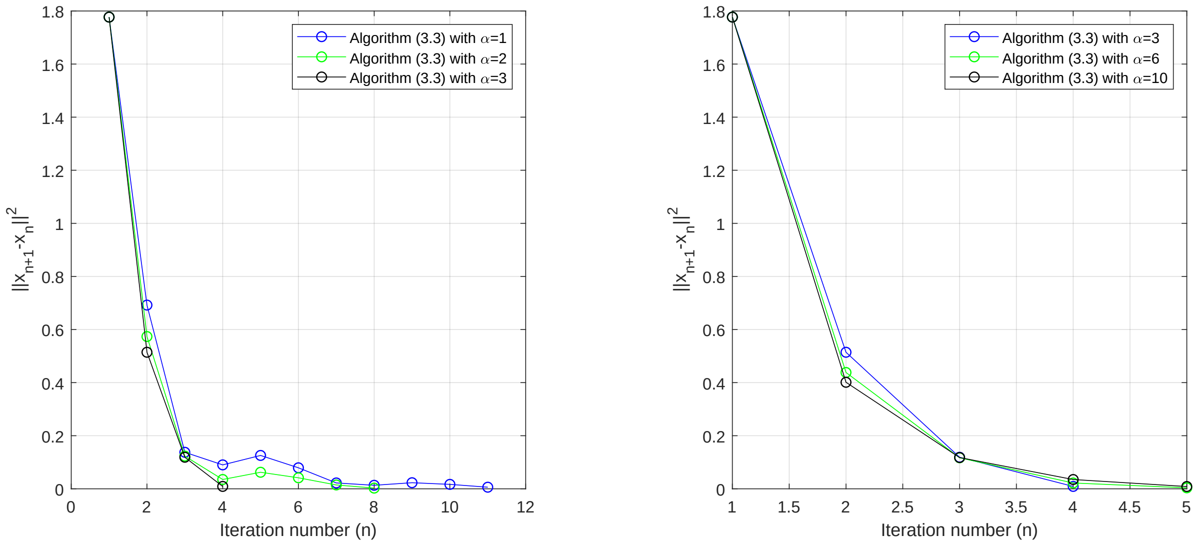

FIGURE 1. Comparing our algorithm (**Alg. 3.3**) at different choices of $'\alpha'$ values

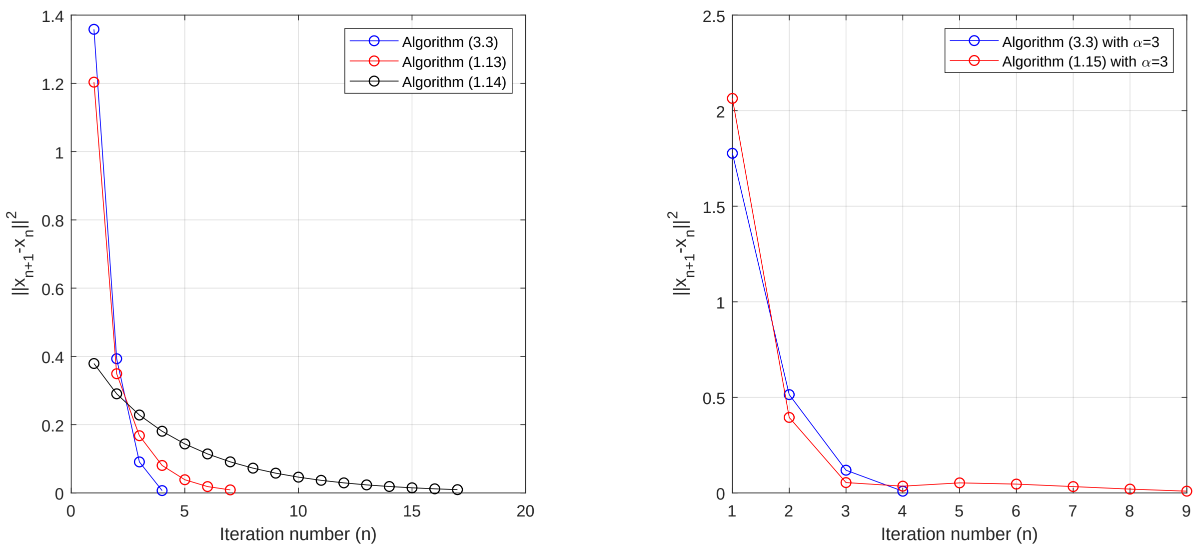

FIGURE 2. Comparing our algorithm (**Alg. 3.3**) not involving $'\alpha'$ values and Algorithm (**Alg. 3.3**) and (**Alg. 1.15**) at the same level of $'\alpha = 3'$ values

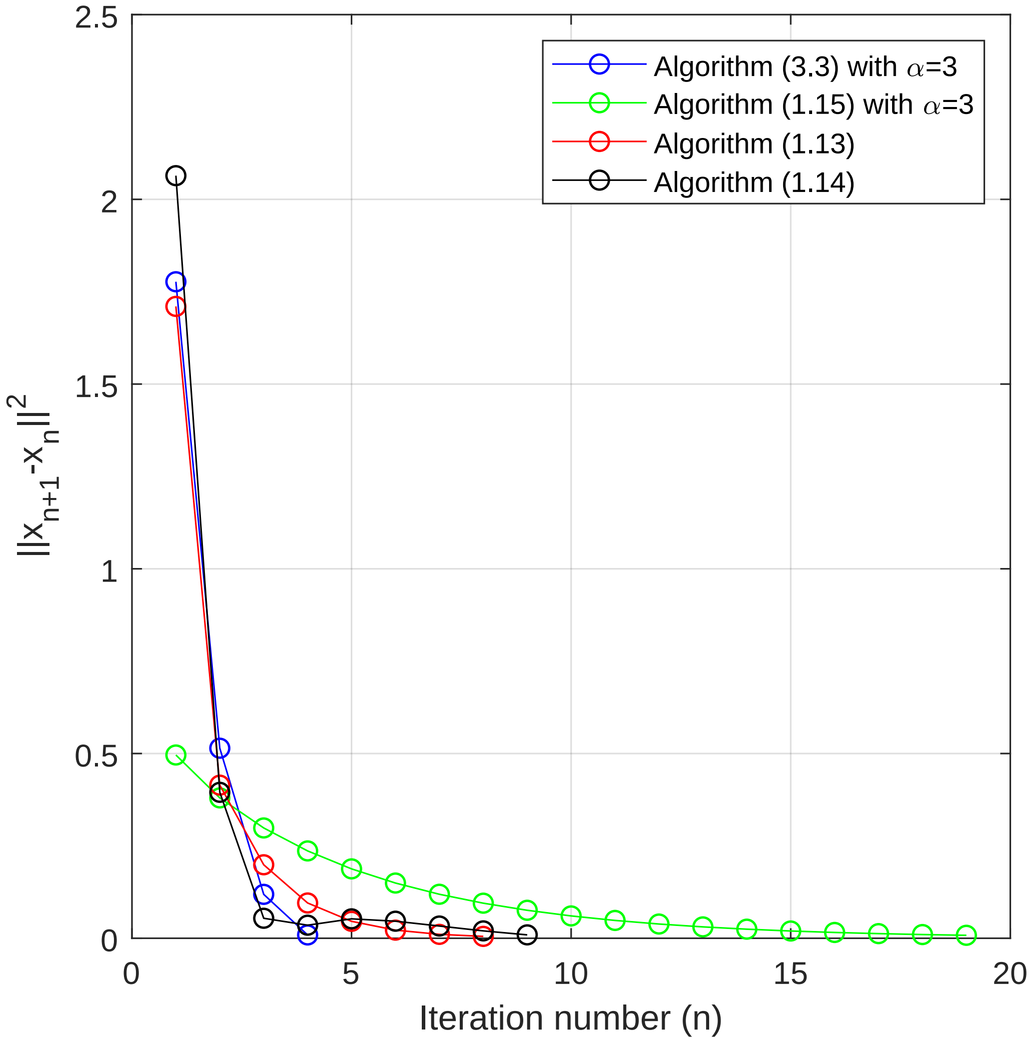

FIGURE 3. Comparing our algorithm (**Alg. 3.3**) and (**Alg. 1.15**) at $'\alpha = 3'$ and others (**Alg. 1.13, and Alg. 1.14**) that do not invlove an $'\alpha'$ term.

7. Results and Discussion

In Table 6.1, we compare our algorithm at different choices of $'\alpha'$ and observe that our algorithm converges at just fourth iteration whenever $'\alpha = 3'$ and converges far otherwise provided that α is positive. This also can be seen in **Figure 1** above.
Table 6.2 is comparison of the rate of convergence between our Algorithm 3.3, (1.13) and (1.14) which the stepsizes rely on the operator norm of the bounbed linear map. Apart from the fact that we introduce an inertial term, we also remove the condition and our Algorithm performs extremely well than others. **Figure 2** are another evidence.
In Table 6.3 we compare our Algorithm 3.3 with Algorithm (1.15) where both Algorithms have a similiar inertial terms. We tested them at the same level of $\alpha = 3$ (see [17], remark 4.3 (c) and Table 1). We observe that at $\alpha = 3$, Algorithm 3.3 converges at 4th iteration while Algorithm (1.15) converges at 9th iteration (see Figure 4 above). Nevertheless, we concord with their result and on their **remark 4.3 (c).**
In **Figure 3** above, we compare our scheme with rest as we pointed out. It is observed that our Algorithm converges faster and at a fewer iteration than others. In nutshell, both in theory and in practice, our algorithm has advantages over others. Hence, it an improvement when compared with others in the literature.

8. Conclusion

In the framework of real Hilbert spaces, the new inertial extrapolation method for solving split monotone variational inclusion problem is constructed and a strong convergence of the proposed iterative scheme is established. Under some mild conditions, which are not limited to the fact that, the step size does not requiring the knowledge of operator norm or trying to have a rough estimate of it. The most general class of operators, the demicontractive operator is considered which really makes the work more general than many others in the same direction. Our algorithm not only finds a solution to the split monotone variational inclusion problem but also solves a fixed point problem which arises in so many areas of engineering and sciences. Furthermore, we apply our algorithm to linear inverse problems. A numerical examples were provided to demostrate how effective and compective our algorithm is over others in this direction (see [13, 14, 16, 17]) among many others.

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