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DYNAMICS AND GLOBAL STABILITY OF SECOND ORDER NONLINEAR DIFFERENCE EQUATION

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ABSTRACT. In this paper, we study some qualitative behavior of the solutions of the following rational difference equation with order two

$$x_{n+1}=ax_n+bx_{n-1}+\frac{cx_n+dx_{n-1}}{e+fx_{n-1}},$$

where the parameters $a,\ b,\ c,\ d,\ e$ and f are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers.

1. INTRODUCTION

Difference equations precribe real life situations in probability theory, electrical network, physics, sociology, etc. Then the difference equations got solution of more complicated problems. There has been many papers about the global behavior and stability of rational difference equations, see [1]-[52].
Agarwal and Elsayed [1] investigated the global stability and gave the solutions of some special cases of the difference equation

$$x_{n+1}=a+\frac{dx_{n-1}x_{n-k}}{b-cx_{n-s}}.$$

Ahmed and Eshtewy [2] investigated the global attractivity of the difference equation

$$x_{n+1}=\frac{A-Bx_{n-2}}{C+Dx_{n-1}}.$$

Elsayed [10] studied the global stability, and periodicity character of the following recursive sequence

$$x_{n+1}=ax_{n-l}+\frac{bx_{n-l}}{cx_{n-l}-dx_{n-k}}.$$

Elsayed and El-Dessoky [16] investigated the global convergence, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1}=ax_{n-s}+\frac{bx_{n-l}+cx_{n-k}}{dx_{n-l}+ex_{n-k}}.$$

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Ibrahim [26] has got the closed form expressions of some higher order nonlinear rational partial difference equation in the form

$$x_{n,m} = \frac{x_{n-r,m-r}}{\Psi + \prod_{i=1}^{r} x_{n-i,m-i}}.$$

Moaaz and Abdelrahman [35] investigated the qualitative behavior of the solution of the rational difference equations

$$x_{n+1} = \frac{bx_{n-k}}{\alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i\neq j}^{k} x_{n-i}}.$$

Obaid et al. [37] investigated the global attractivity and periodic character of the following fourth order difference equation

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2} + dx_{n-3}}{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}.$$

Saleh et al. [40] studied the dynamical of a nonlinear rational difference equation of a higher order

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{bx_n + cx_{n-k}}.$$

Our goal in this article is study some properties and dynamics of the solution of the difference equation

(1) $$x_{n+1} = ax_n + bx_{n-1} + \frac{cx_n + dx_{n-1}}{e + fx_{n-1}},$$

where the parameters $a,\ b,\ c,\ d,\ e$ and f are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers.

2. Some Basic Properties and Definitions

"Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k},\ x_{-k+1},\ \dots, x_0 \in I$, the difference equation

(2) $$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ...,$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 2.1. (Equilibrium point)

A point $\bar{x} \in I$ is called an equilibrium point of Equation (2) if

$$\bar{x} = F(\bar{x}, \bar{x}, ..., \bar{x}).$$

That is, $x_n = \bar{x}\ for\ n \geq 0$, is a solution of Equation (2), or equivalently, $\bar{x}$ is a fixed point of F.

Definition 2.2. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 2.3. (Stability)

(i) The equilibrium point $\bar{x}$ of Equation (2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon \; for \; all \; n \geq -k.$$

(ii) The equilibrium point $\bar{x}$ of Equation (2) is locally asymptotically stable if $\bar{x}$ is locally stable solution of Equation (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + ... + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iii) The equilibrium point $\bar{x}$ of Equation (2) is global attractor if for all x_{-k}, x_{-k+1}, $...$, $x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) The equilibrium point $\bar{x}$ of Equation (2) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Equation (2).

(v) The equilibrium point $\bar{x}$ of Equation (2) is unstable if is not locally stable.

The linearized equation of Equation (2) about the equilibrium point $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\bar{x}, \bar{x}, ..., \bar{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [33] *Assume that* $p_i \in R,\ i = 1, 2, ...$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1, \tag{4}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \quad n = 0, 1, ... \tag{5}$$

Theorem B [33] Let $g : [a, b]^{k+1} \to [a, b]$, be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ... \tag{6}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq k+1$; the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{k+1}), \quad M = g(M_1, M_2, ..., M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, ..., k+1$, we set

$$\begin{aligned} m_i &= \begin{Bmatrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{Bmatrix}, \\ Mi &= \begin{Bmatrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{Bmatrix}. \end{aligned}$$

Then there exists exactly one equilibrium point $\bar{x}$ of Equation (6), and every solution of Equation (6) converges to $\bar{x}$."

3. Local Stability of Equation (1)

In this section, we investigate the local stability character of the solutions of Equation (1).

Equation (1) has two equilibrium points and given by the relation

$$\bar{x} = a\bar{x} + b\bar{x} + \frac{c\bar{x} + d\bar{x}}{e + f\bar{x}}.$$

Then

$$\begin{aligned} \bar{x}_1 &= 0. \\ \bar{x}_2 &= \frac{c+d}{[1-(a+b)]f} - \frac{e}{f}. \end{aligned}$$

Let $f : (0,\infty)^2 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v) = au + bv + \frac{cu + dv}{e + fv}.$$

Therefore it follows that

$$\begin{aligned} \frac{\partial f(u,v)}{\partial u} &= a + \frac{c}{e + fv}, \\ \frac{\partial f(u,v)}{\partial v} &= b + \frac{de - cfu}{(e + fv)^2}. \end{aligned}$$

Theorem 3.1. *Assume that*

$$(a+b)e < e - (c+d)\,.$$

Then the first equilibrium point ($\bar{x}_1 = 0$) of Equation (1) is locally asymptotically stable.

Proof. Let $f : (0,\infty)^2 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v) = au + bv + \frac{cu + dv}{e + fv}.$$

Then, we see at the first point $\bar{x}_1 = 0$, that

$$\begin{aligned} \frac{\partial f(\bar{x},\bar{x})}{\partial u} &= a + \frac{c}{e}, \\ \frac{\partial f(\bar{x},\bar{x})}{\partial v} &= b + \frac{d}{e}. \end{aligned}$$

Then, the linearized equation of Equation (1) about $\bar{x}_1 = 0$ is

$$y_{n+1} - \left(a + \frac{c}{e}\right) y_n - \left(b + \frac{d}{e}\right) y_{n-1} = 0. \tag{7}$$

The characteristic equation of Eq.(7) is

$$\lambda^2 - \left[a + \frac{c}{e}\right] \lambda - \left[b + \frac{d}{e}\right] = 0,$$

and hence it follows from Theorem A, that the equilibrium point is locally asymptotically stable if

$$\left|a + \frac{c}{e}\right| + \left|b + \frac{d}{e}\right| < 1,$$

or

$$(a+b)e < e - (c+d)\,.$$

This completes the proof. □

Theorem 3.2. *Assume that*

$$c < eA < c + d, \qquad or \quad eA < c < d + eA,$$

since $A = 1 - (a + b)$*. Then the second equilibrium point* $\left(\bar{x}_2 = \frac{c+d}{[1-(a+b)]f} - \frac{e}{f}\right)$ *of Equation (1) is locally asymptotically stable.*

Proof. Let $f : (0, \infty)^2 \longrightarrow (0, \infty)$ be a function defined by

$$f(u, v) = au + bv + \frac{cu + dv}{e + fv}.$$

Then, we see at the point $\bar{x}_2 = \frac{c+d}{Af} - \frac{e}{f}$, (since $A = 1 - (a + b)$) that

$$\begin{aligned} \frac{\partial f(\bar{x}, \bar{x})}{\partial u} &= a + \frac{c(1 - (a + b))}{c + d} = a + \frac{cA}{c + d}, \\ \frac{\partial f(\bar{x}, \bar{x})}{\partial v} &= b + \frac{(eA - c)A}{c + d}. \end{aligned}$$

Then, the linearized equation of Equation (1) about $\bar{x}_2$ is

$$y_{n+1} - \left(a + \frac{cA}{c + d}\right) y_n - \left(b + \frac{(eA - c)A}{c + d}\right) y_{n-1} = 0. \tag{8}$$

It follows by Theorem A that Equation (8) is asymptotically stable if

$$\left|a + \frac{cA}{c + d}\right| + \left|b + \frac{(eA - c)A}{c + d}\right| < 1.$$

We consider three cases

Case (I):- $A > 0,\ eA > c$, we see that

$$a + \frac{cA}{c + d} + b + \frac{(eA - c)A}{c + d} < 1.$$

Then

$$a + b + \frac{eA^2}{c + d} < 1 \quad \Rightarrow \quad \frac{eA^2}{c + d} < A \quad \Rightarrow eA < c + d.$$

Case (II):- $A > 0,\ eA < c$, we see that

$$a + \frac{cA}{c + d} + b + \frac{(c - eA)A}{c + d} < 1.$$

$$\frac{(2c - eA)A}{c + d} < 1 - a - b = A.$$

$$\frac{2c - eA}{c + d} < 1.$$

Then

$$2c - eA < c + d.$$

$$c < d + eA.$$

Case (III):- $A < 0$, we suppose ($A = -B = a + b - 1$) and we get

$$a + \frac{cB}{c + d} + b + \frac{(eB + c)B}{c + d} < 1.$$

$$\frac{(eB + 2c)B}{c + d} < A.$$

which is contradictions, thus A should be positive. This completes the proof. $\square$

4. Global Attractivity of the Equilibrium Point of Equation (1)

In this section we deals the global attractivity character of solutions of Equation (1).

Theorem 4.1. *The equilibrium point $\bar{x}$ is a global attractor of equation (1) if one of the following conditions holds:*

$$(i)\ \ (1+b-a)e+d>c.$$

$$(ii)\ \ a+b>1.$$

Proof. Let r, s be nonnegative real numbers and assume that

$$f:[r,s]^2 \to [r,s]$$

be a function defined by

$$f(u,v)=au+bv+\frac{cu+dv}{e+fv}.$$

It is easy to see that the function $f(u,v)$ is increasing in u but it is unknown in v. So, we consider two cases:

Case 1: Assume that $f(u,v)$ is decreasing in v.

Suppose that (m, M) is a solution of the system

$$M=f(M,m) \quad and \quad m=f(m,M).$$

Then from Equation (1), we see that

$$M=aM+bm+\frac{cM+dm}{e+fm}, \quad m=am+bM+\frac{cm+dM}{e+fM},$$

or

$$M[1-a]=bm+\frac{cM+dm}{e+fm},\ m[1-a]=bM+\frac{cm+dM}{e+fM},$$

then

$$\begin{aligned} M[1-a](e+fm) &= bm(e+fm)+cM+dm, \\ m[1-a](e+fM) &= bM(e+fM)+cm+dM, \end{aligned}$$

Subtracting this two equations, we obtain

$$(M-m)\left\{(1+b-a)e+be-c+d+bf(M+m)\right\}=0,$$

under the condition $(1+b-a)e+d>c$, we see that

$$M=m.$$

It follows from Theorem B that $\bar{x}$ is a global attractor of Equation (1).

Case 2: Assume that $f(u,v)$ is increasing in v.

Suppose that (m, M) is a solution of the system

$$M=f(M,M) \quad and \quad m=f(m,m).$$

Then from Equation (1), we see that

$$M=aM+bM+\frac{cM+dM}{e+fM},\ m=am+bm+\frac{cm+dm}{e+fm},$$

or

$$M[1-(a+b)]=\frac{cM+dM}{e+fM}, \qquad m[1-(a+b)]=\frac{cm+dm}{e+fm},$$

then

$$\begin{aligned} M^2f[1-(a+b)]+Me[1-(a+b)] &= cM+dM, \\ m^2f[1-(a+b)]+me[1-(a+b)] &= cm+dm. \end{aligned}$$

Subtracting this two equations we get

$$(M-m)\left\{f(M+m)[1-(a+b)]+e(1-a-b)-c-d\right\}=0.$$

under the condition $(a+b)>1$, we see that

$$M=m.$$

It follows from Theorem B that $\bar{x}$ is a global attractor of Equation (1) and then the proof is completed. □

5. Boundedness of Solutions of Equation (1)

Here we deal with the boundedness nature of the solutions of Equation (1).

Theorem 5.1. *Every solution of Equation (1) is bounded if* $a+b+\frac{c+d}{e}<1$.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$\begin{aligned} x_{n+1} &= ax_n+bx_{n-1}+\frac{cx_n+dx_{n-1}}{e+fx_{n-1}} \\ &\leq ax_n+bx_{n-1}+\frac{cx_n+dx_{n-1}}{e}=(a+\frac{c}{e})x_n+(b+\frac{d}{e})x_{n-1}. \end{aligned}$$

By using a comparison, we can write the right hand side as follows

$$y_{n+1}=(a+\frac{c}{e})y_n+(b+\frac{d}{e})y_{n-1}.$$

This equation is locally asymptotically stable if $\left|a+\frac{c}{e}\right|+\left|b+\frac{d}{e}\right|<1$, and converges to the equilibrium points. Thus the solution is bounded. □

Theorem 5.2. *Equation (1) has unbounded solution if* $a>1$*or* $b>1$.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$x_{n+1}=ax_n+bx_{n-1}+\frac{cx_n+dx_{n-1}}{e+fx_{n-1}}>ax_n \ \text{(or we can write } x_{n+1}>bx_{n-1})$$

By using a comparison, we can write the right hand side as follows

$$y_{n+1}=ay_n \Longrightarrow y_n=a^n y_0 \Longrightarrow \lim_{n\to\infty} y_n=\infty.$$

$$y_{n+1}=by_{n-1} \Longrightarrow y_n=(b^{\frac{n}{2}}y_0 \ \ or \ \ b^{\frac{n+1}{2}}y_{-1}) \Longrightarrow \lim_{n\to\infty} y_n=\infty.$$

So the solutions are unbounded solutions. □

6. Existence of Periodic Solutions

In this section we study the existence of periodic solutions of Equation (1). The following theorem states the necessary and sufficient conditions that this equation has periodic solution of prime period two.

Theorem 6.1. *Equation (1) has a prime period two solutions if and only if*

$$(d-c-e(B+a))^2>\frac{4B}{a+B}(c+ae)(c+eB+ea-d), \quad B=1-b.$$

Proof. First suppose that there exists a prime period two solution $\ldots, p, q, p, q, \ldots$, of Equation (1). We will prove that Condition holds.

We see from Equation (1) that

$$p=aq+bp+\frac{cq+dp}{e+fp},$$

and

$$q = ap + bq + \frac{cp + dq}{e + fq}.$$

Then

$$pBe + fBp^2 - aeq - afpq = cq + dp, \tag{9}$$

and

$$qBe + fBq^2 - aep - afpq = cp + dq. \tag{10}$$

Subtracting (9) from (10) gives

$$(p - q)\{Be + fB(p + q) + ae\} = (d - c)(p - q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{d - c - eB - ea}{fB}. \tag{11}$$

Again, adding (9) and (10) yields

$$(p + q)\{eB - ea - c - d\} + fB(p + q)^2 - 2fBpq = 2afpq. \tag{12}$$

It follows by (11), (12) and the relation $p^2 + q^2 + 2pq = (p + q)^2$ that

$$pqf(a + B) = (p + q)(-c - ea).$$

Thus

$$pq = \frac{(d - c - eB - ea)(-c - ea)}{f^2B(a + B)}. \tag{13}$$

Now it is clear from Equations (11) and (13) that p and q are the two distinct roots of the quadratic equation

$$\lambda^2 - \frac{d - c - eB - ea}{fB}\lambda + \frac{(d - c - eB - ea)(-c - ea)}{f^2B(a + B)} = 0,$$

or

$$f^2B(a + B)\lambda^2 - f(a + B)(d - c - eB - ea)\lambda + (d - c - eB - ea)(-c - ea) = 0, \tag{14}$$

and so

$$[f(a + B)(d - c - eB - ea)]^2 - 4f^2B(a + B)(d - c - eB - ea)(-c - ea) > 0.$$

$$[(d - c - eB - ea)]^2 > \frac{4B}{a + B}(d - c - eB - ea)(-c - ea).$$

Therefore the condition holds.

Second suppose that the condition is true. We will show that Equation (1) has a prime period two solution.

Assume that

$$p = \frac{-f(a + B)(c + eB + ea - d) + \zeta}{2f^2B(a + B)},$$

and

$$q = \frac{-f(a + B)(c + eB + ea - d) - \zeta}{2f^2B(a + B)},$$

where

$$\zeta = \sqrt{f^2(a + B)^2(d - c - eB - ea)^2 - 4f^2B(a + B)(d - c - eB - ea)(-c - ea)}.$$

We see from the condition that

$$[(d - c - eB - ea)]^2 > \frac{4B}{a + B}(d - c - eB - ea)(-c - ea),$$

which equivalents to

$$(a+B)[(d-c-eB-ea)]^2-4B(d-c-eB-ea)(-c-ea)>0.$$

Therefore$p$ and q are distinct real numbers.

Set

$$x_{-1}=p, \quad x_0=q.$$

We wish to show that

$$x_1=p, \ \ and \ \ x_2=q.$$

It follows from Equation (1) that

$$\begin{aligned}
x_1 &= aq+bp+\frac{cq+dp}{e+fp} \\
&= \tfrac{-af(a+B)(c+eB+ea-d)-bf(a+B)(c+eB+ea-d)+b\zeta-a\zeta}{2f^2B(a+B)} \\
&\quad +\frac{\tfrac{-cf(a+B)(c+eB+ea-d)-df(a+B)(c+eB+ea-d)+d\zeta-c\zeta}{2f^2B(a+B)}}{e+\tfrac{-f(a+B)(c+eB+ea-d)+\zeta}{2fB(a+B)}} \\
&= \tfrac{-(a+b)f(a+B)(c+eB+ea-d)+(b-a)\zeta}{2f^2B(a+B)} \\
&\quad +\left[\begin{array}{c} \tfrac{-cf(a+B)(c+eB+ea-d)-df(a+B)(c+eB+ea-d)+d\zeta-c\zeta}{2f^2B(a+B)} \\ \times\tfrac{2fB(a+B)}{e2fB(a+B)-f(a+B)(c+eB+ea-d)+\zeta} \end{array}\right] \\
&= \frac{-(a+b)f(a+B)(c+eB+ea-d)+(b-a)\zeta}{2f^2B(a+B)}+W,
\end{aligned}$$

where

$$W=\frac{-(c+d)f(a+B)(c+eB+ea-d)+(d-c)\zeta}{e2f^2B(a+B)-f^2(a+B)(c+eB+ea-d)+f\zeta}.$$

By simple computation we can see that

$$x_1=p.$$

Similarly as before one can easily show that

$$x_2=q.$$

Then it follows by induction that

$$x_{2n}=q \quad \text{and} \quad x_{2n+1}=p \quad \text{for all} \quad n\geq -1.$$

Thus Equation (1) has the positive prime period two solution

$$...,p,q,p,q,...,$$

where p and q are the distinct roots of the quadratic equation (14) and the proof is complete. □

7. Numerical Examples

For confirming the results of this article, we consider numerical examples which represent different types of solutions to Equation (1).

Example 7.1. We consider numerical example for the difference equation (1) when we take the constants and the initial conditions as follows: $x_{-1}=2.5,\ x_0=3,\ a=.7,\ b=.5,\ c=.2,\ d=.1,\ e=.3,\ f=.5$. See Figure 1.

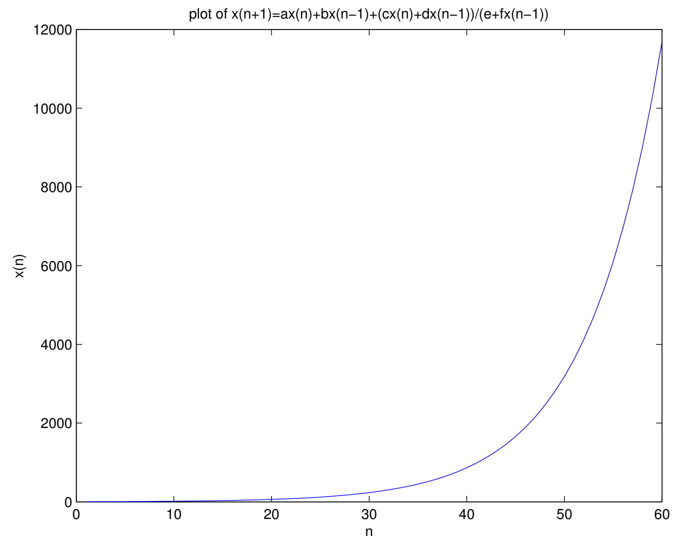

FIGURE 1.

Example 7.2. See Figure (2) when we take Equation (1) with $x_{-1} = 8,\ x_0 = 3,\ a = .6,\ b = .2,\ c = 3,\ d = 4,\ e = 7,\ f = 5.$

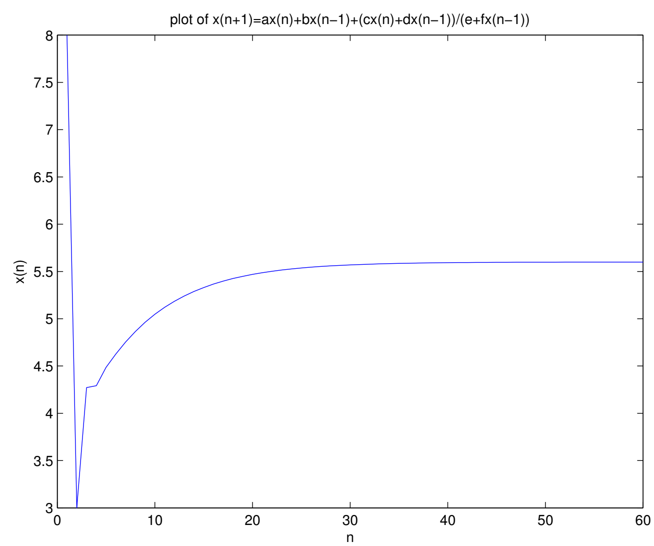

FIGURE 2.

Example 7.3. Figure (3) shows the behavior of the solution of the difference equation (1) when we put $x_{-1} = 8,\ x_0 = 3,\ a = .3,\ b = .5,\ c = 3,\ d = 1,\ e = 8,\ f = 5.$

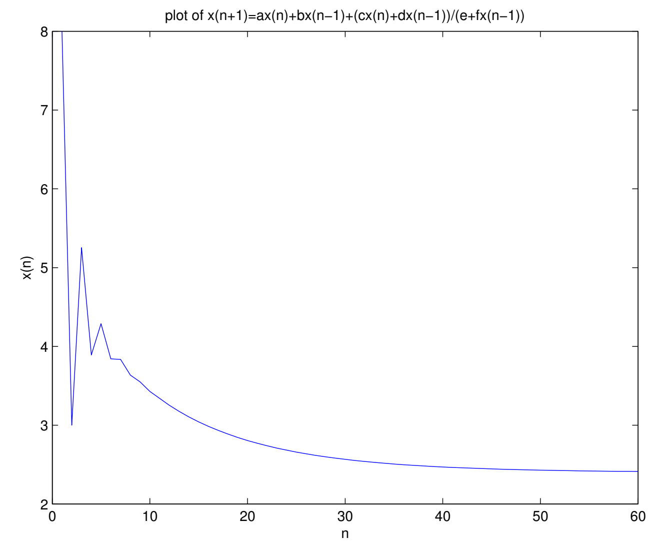

FIGURE 3.

Example 7.4. We assume $x_{-1} = 7,\ x_0 = 12,\ a = 1.1,\ b = .02,\ c = .3,\ d = .4,\ e = 2,\ f = .5$. See Figure 4.

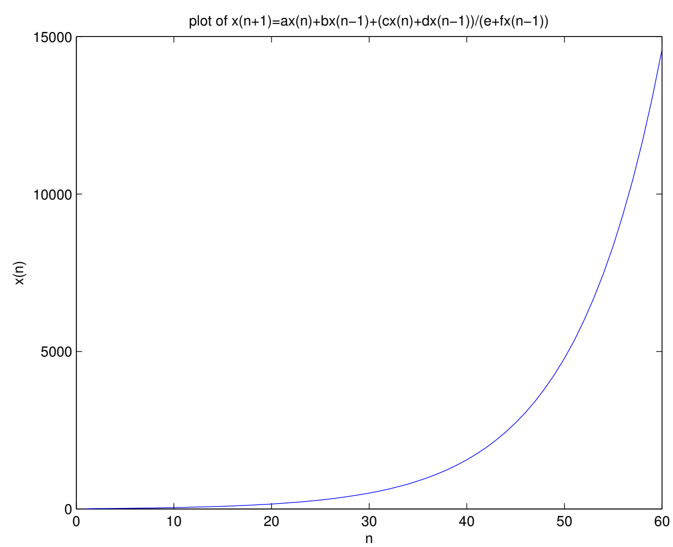

FIGURE 4.

Example 7.5. Figure (5) shows the solution of Equation (1) when $x_{-1} = 7,\ x_0 = 12,\ a = 0.01,\ b = 1.02,\ c = .3,\ d = .4,\ e = 2,\ f = .5.$

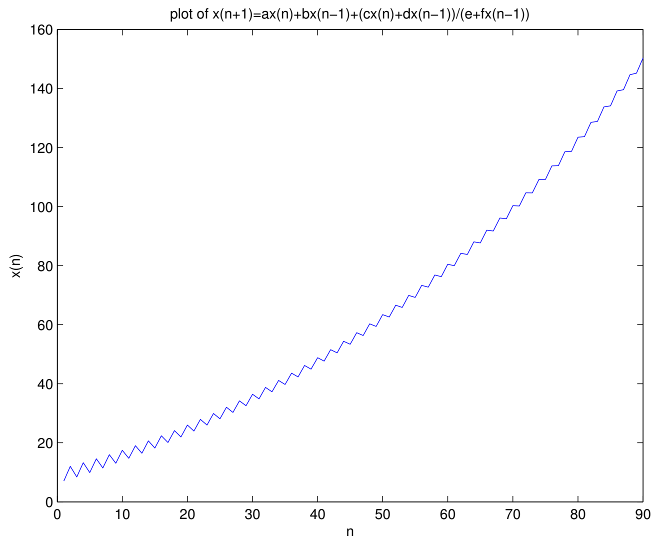

FIGURE 5.

Example 7.6. Figure (6) shows the period two solution of Equation (1) when $x_{-1} = p,\ x_0 = q,\ a = .2,\ b = .15,\ c = .123,\ d = 14,\ e = 6,\ f = .5$, since p and q as in the previous theorem.

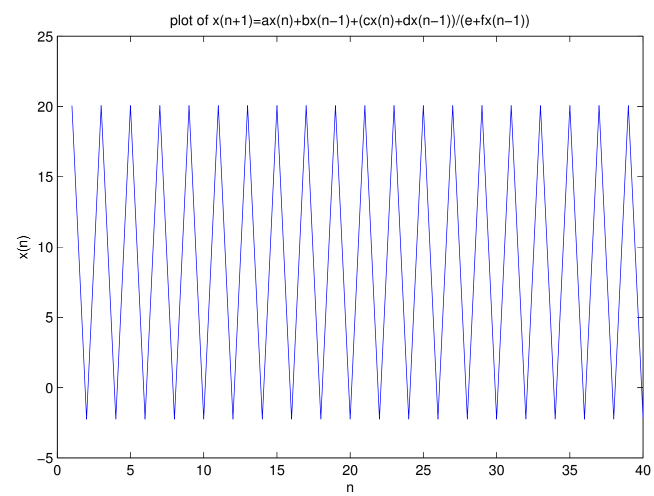

FIGURE 6.

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