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EXTENSION OF HARDY-TYPE INEQUALITIES ON TIME SCALES

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ABSTRACT. The concept of superquadratic functions of several variables with general kernels in the setting of time scales play a fundamental role in mathematical analysis. In this article, we derive some new results related to Hardy-type inequalities through Jensen inequality for multivariate superquadratic functions on time scales.

1. INTRODUCTION

In mathematical analysis an important theorem for an integral inequality is stated as follows.

Theorem 1.1. *Let $p > 1$ be a constant, $f(x)$ be a nonnegative measurable function in the interval $(0, \infty)$ and $F(x) = \int_0^x f(t)dt$, then*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^l(x)dx,$$

where equality holds if and only if $f \equiv 0$ and the constant $((p-1)/p)^p$ is the best possible but never achieved.

This theorem was first proved by Hardy in 1920 [6,7], and since then various versions and their proofs have appeared in literature, showing their applications in several areas of mathematics. Theorem 3.2 or its generalization to multivariate functions [5] may be regarded as a model for a class of inequalities with which the following deals.

Theorem 1.2. *For constants $p > 1$, $r \neq 1$, measurable function $f(x) \geq 0$ and $F(x)$ defined by*

$$F(x) = \begin{cases} \int_0^x f(t)dt, \ (r > 1), \\ \\ \int_0^\infty f(t)dt, \ (r < 1), \end{cases}$$

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it holds that

$$\int_0^\infty x^{-r}F^p(x)dx \leq \int_0^\infty x^{-r}(xf^p)(x)dx.$$

Equality holds if $f \equiv 0$ and the constant is the best possible.

Abramovich et al [1] introduced the concept of superquadratic functions in one variable as a generalization of the class of convex functions. Superquadratic functions of several variables with general kernels to the case with arbitrary time scales played a fundamental role in mathematical analysis. In this paper, we derive some new results related to Hardy-type inequalities through Jensen inequality on time scales.

A time scale $\mathbb{T}$ means any nonempty closed subset of $\mathbb{R}$. The two most popular examples of time scales are the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$. Since the time scale $\mathbb{T}$ may or may not be connected, the concept of jump operator is usually introduced. For detail discussion on time scale the readers can find [2,5,10] for examples. The idea of time scales was introduced by Stefan Hilger in his PhD thesis [8] in 1988. Note that in the theory of time scales, delta derivative is the usual derivative if $\mathbb{T} = \mathbb{R}$ and the forward difference if $\mathbb{T} = \mathbb{Z}$, while the delta integral is the usual integral if $\mathbb{T} = \mathbb{R}$ and a sum if $\mathbb{T} = \mathbb{Z}$. Hence, the following definition:

Definition 1.3. Let $\mathbb{T}$ be a time scale such that $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \ \forall\, t \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \ \forall\, t \in \mathbb{T}.$$

The point t is said to be right scattered if $\sigma(t) > t$, respectively left scattered for $\rho(t) < t$. Point that are right scattered and left scattered at the same time are called isolated. The point t is called right dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$ respectively left dense if $t > \inf\mathbb{T}$ and $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \longrightarrow [0, \infty)$ is given by

$$\mu(t) = \sigma(t) - t \ \forall t \in \mathbb{T}.$$

A mapping $f : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be rd-continuous if (i) f is continuous at each right-dense point on naximal point of $\mathbb{T}$, and (ii) at each left dense point $t \in \mathbb{T}$, $\lim_{s \longrightarrow t^-} g(s) = g(t^-)$ exists.
The set of all rd-continuous functions from $\mathbb{T} \longrightarrow \mathbb{R}$ is usually denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.
Let $\mathbb{T}$ be a time scale and $[a, b) \subset \mathbb{R}$. The Lebesque integral associated with the measure μ on $[a, b)$ is called the Lebesque $\Delta-$integral. If $f : [a, b) \longrightarrow \mathbb{R}$, the corresponding $\Delta-$integral of f over $[a, b)$ will be denoted $\int_a^b f(t)\Delta t$. If $\mathbb{T}$ is a time scale and interval $[a, b) \subset \mathbb{T}$ consists of isolated points, then

$$\int_a^b f(t)\Delta t \leq \sum_{t\in[a,b)} (\sigma(t) - t)f(t).$$

Anwar et al [2] obtained the Jensen inequality for convex functions in several variables on time scale and established some of its basic properties for multivariate convex functions on an arbitrary time scale. Specifically, they established the following result.

Theorem 1.4. *([2]) Let $(\Omega_1, \sum_1, \mu_\Delta)$ and $(\Omega_2, \sum_2, \lambda_\Delta)$ be two time scale measure spaces. Suppose that $U \subset \mathbb{R}^n$ is a closed convex set and $\phi \in C(U, \mathbb{R})$ is convex. Moreover, let $k : \Omega_1, \Omega_2 \longrightarrow \mathbb{R}$ be non negative, such that $k(x, .)$ is $\lambda_\Delta-$ integrable. Then,*

$$\phi\left(\frac{\int_{\Omega_2} k(x,y)f(y)\Delta y}{\int_{\Omega_2} k(x,y)\Delta y}\right) \leq \frac{\int_{\Omega_2} k(x,y)\phi(f(y))\Delta y}{\int_{\Omega_2} k(x,y)\Delta y} \tag{1.1}$$

holds for all functions $f : \Omega_2 \longrightarrow U$*, where* $f_j(y)$ *are* μ_Δ *-integrable for all* $j \in \{1, 2.3...n\}$ *and* $\int_{\Omega_2} k(x,y)f(y)$ *denotes the* n*-tuple*

$$\left(\int_{\Omega_2} k(x,y)f_1(y)\Delta(y), \int_{\Omega_2} k(x,y)f_2(y)\Delta(y), \cdots, \int_{\Omega_2} k(x,y)f_n(y)\Delta(y)\right).$$

Donchev et al [4] employed the above result to derive the following Hardy-type inequality involving multivariate convex functions on time scales:

Theorem 1.5. *([4]) Let* Ω_1, Ω_2 *be as defined by Theorem 1.4. If* $K : \Omega_1 \longrightarrow \mathbb{R}$ *is defined by* $K(x) := \int_{\Omega_2} k(x,y)\Delta y < \infty, x \in \Omega_1$*. Moreover, let* $\zeta : \Omega_1 \longrightarrow \mathbb{R}$ *and the weight* $w(y)$ *be defined by*

$$w(y) = \int_{\Omega_2} \left(\frac{k(x,y)\zeta(x)}{K(x)}\right) \Delta x, \quad y \in \Omega_2.$$

Then for each convex function ϕ*,*

$$\int_{\Omega_1} \zeta(x)\phi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\Delta y\right)\Delta x \le \int_{\Omega_1} w(y)\phi(f(y))\Delta y \tag{1.2}$$

holds for all λ_Δ*-integrable functions* $f : \Omega_1 \longrightarrow \mathbb{R}^n$ *such that* $f(\Omega_2) \subset U$*.*

In their work, Oguntuase and Persson [10], Fabelurin and Oguntuase [5] (see also Oguntuase ea al [9, 12]) obtained Hardy-type inequalities on time scales using the concept of superquadratic functions. In particular, the following result was obtained.

Theorem 1.6. *([10] Let* $(\Omega_1, \sum_1, \mu_{\Delta_1})$ *and* $(\Omega_2, \sum_2, \mu_{\Delta_2})$ *be two time scale measure spaces with a* $\sigma-$ *finite measures and* $u : \Omega_n \longrightarrow \mathbb{R}$ *and* $k : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ *non-negative functions such that* $k(x,\cdot)$ *is* $\mu\Delta_2$*-integrable for* $x \in \Omega_1$*. Furthermore suppose that* $K : \Omega_1 \longrightarrow \mathbb{R}$ *is defined by*

$$K(x) := \int_{\Omega_2} k(x,y)\Delta\mu_2(y) > 0, \ x \in \Omega_1$$

and

$$v(y) := \int_{\Omega_1} \left(\frac{k(x,y)u(x)}{K(x)}\right)\Delta\mu_1(x) < \infty, \quad y \in \Omega_2.$$

If $\phi : [a,\infty) \longrightarrow \mathbb{R}(a \ge 0)$ *is a non negative superquadratic function, then the inequality*

$$\begin{aligned}
&\int_{\Omega_1} u(x)\phi(A_k f(x))\Delta\mu_1(x)+ \\
&\int_{\Omega_2}\int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}\phi\left((|f(y) - A_k f(x)|)\right)\Delta\mu_1(x)\Delta\mu_2(y) \\
&\le \int_{\Omega_1} u(x)\phi(f(x))\Delta\mu_2(x)
\end{aligned} \tag{1.3}$$

holds for all non negative $\Delta\mu_2$*-integrable function* $f : \Omega_2 \longrightarrow \mathbb{R}$ *and for* $A_k f : \Omega_1 \longrightarrow \mathbb{R}$

$$A_k f(x) = \frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(x)\Delta\mu_2(y), \qquad x \in \Omega_1$$

If ϕ *is subquadratic, then the inequality sign in (1.3) is reversed.*

Motivated by Fabelurin and Oguntuase [5] and Oguntuase et al [10, 11], we are concerned with a systematic and uniform treatment of some analogues and extensions of Hardy's inequality for integrals using the above highlighted tools as applied in the references cited. The rest of the paper is devoted to statement and proofs of the results of this paper.

2. Definitions, Some Useful Tools and Preliminary Results

in this section, we discussed some needed terminologies, proofs that will be of help in the course of proving our main results.

Definition 2.1. Let (Ω, M, μ_Δ) and $(\Lambda, l, \lambda_\Delta)$ be two finite dimensional time scale measure spaces. If $f : \Omega \times \Lambda \longrightarrow \mathbb{R}$ is $\mu_\Delta \times \lambda_\Delta-$ integrable function and define the function $\varphi(y) = \int_\Omega f(x,y)\Delta x$ and $\psi(x) = \int_\Lambda f(x,y)\Delta y$ for almost everywhere $y \in \Lambda and x \in \Omega$ and φ is $\lambda_\Delta-$integrable on Λ, ψ, and $\mu_\Delta-$integrable on Ω, then

$$\int_\Omega \Delta x \int_\Lambda f(x,y)\Delta y = \int_\Lambda \Delta y \int_\Omega f(x,y)\Delta x. \tag{2.1}$$

This is called Fubini theorem.

Definition 2.2. A function is $\varphi : [0,\infty) \longrightarrow \mathbb{R}$ said to be superquadratic provided for each $x \geq 0$, there exist a constant $C_x \in \mathbb{R}$ such that

$$y^{\frac{p+r}{q}} - x^{\frac{p+r}{q}} - (|y-x|)^{\frac{p+r}{q}} - C_x(y-x) \geq 0 \tag{2.2}$$

for all $y \in [0,\infty)$. φ is subquadratic if $-\varphi$ is superquadratic.
The function $\varphi = x^{\frac{p+r}{q}},\ x \in (0,\infty)$ is superquadratic for all p $x \geq 1$.

Definition 2.3. Let $\varphi : [0,\infty) \longrightarrow \mathbb{R}$ be a superquadratic function with C_x as in Definition 2.1. Then
(a) $\varphi(0) \leq 0$
(b) If $\varphi(0) = \varphi'(0) = 0$, then $C_x = \varphi'(0)$ whenever $\varphi(0)$ is differentiable at $x > 0$.
(c) If $\varphi(0) \geq 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.

Corollary 2.4. *Let $\varphi : [0.\infty) \longrightarrow \mathbb{R}$ is said to be differentiable function if $\varphi(0) = \varphi'(0)$. Then*

$$y^{\frac{p+r}{q}} - 1^{\frac{p+r}{q}} - (|y-1|)^{\frac{p+r}{q}} - (|y-1|)^{\frac{p+r}{q}} = \begin{cases} \text{if } \varphi \text{ is superquadratic } \geq 0 \\ \text{if } \varphi \text{ is subquadratic } \leq 0 \end{cases} \tag{2.3}$$

holds for all $y \leq 0$. If $\varphi(x) = x^{\frac{p+r}{q}}$. The, the inequality (2.4) holds for all y if and only if $p = 2$

Adaptation of refined Jensen's inequality for subquadratic and superquadratic functions play essential pivot in the proof our main results which are as follows:

Corollary 2.5. *Let $(\Omega, \sum, \mu)$ be a probability measure space. Then, the inequality*

$$\left(\int_\Omega f(x)d\mu(x)\right)^{\frac{p+r}{q}} \leq \int_\Omega (f(x))^{\frac{p+r}{q}} d\mu(x) - \int_\Omega \left(\left|f(x) - \int_\Omega f(y)d\mu(y)\right|\right)^{\frac{p+r}{q}} d\mu(x) \tag{2.4}$$

holds for all probability measures μ and all non negative μ-integrable functions f if and only if φ is superquadratic. However, (2.3) holds in the reversed direction if and only if φ is subquadratic.

Corollary 2.6. *Let $h > 0$; then $h^q - (q+n)h - q - n - 1 \geq 0$ if $q + n \geq 2$ and $h^q - (q+n)h - q - n - 1 \leq 0$ if $0 \leq q + n \leq 1$.*
for all $h > 0$. Equality holds if and only if $h = 1$.
Immediate consequence of Corollary 2.1 and Corollary 2.2 produces the improvement of the well known Bernoulli's inequality in Corollary 2.3 which will be needed in the proof.

Corollary 2.7. *Let $\varphi \in C((c,d),\mathbb{R})$ be convex. Then for each $y \in (c,d)$,there exist $B_x \in \mathbb{R}$ such that*

$$(y)^{\frac{p+r}{q}} - (x)^{\frac{p+r}{q}} \geq B_x(y-x) \text{ for all } y \in (c,d) \tag{2.5}$$

If φ is strictly convex, then the inequality sign $\geq$ in should is replace by $>$.
(2.1) can be written as

$$\begin{aligned}&(C_k\left[\zeta_i(y)+\zeta_k(y)\right])^{\frac{p+r}{q}} - (x)^{\frac{p+r}{q}} \geq B_x\left(C_k\left[\zeta_i(y)+\zeta_k(y)\right]-x\right)\\ &= \sum_{k=i=1}^{n} B_x\left(C_k\zeta_i(y)+C_k\zeta_k(y)-x\right).\end{aligned} \tag{2.6}$$

If φ is continuous and $\varphi^o f$ is $\Delta\mu(y)-$integrable. Then, by integrating the above inequality with respect to $\Delta\mu(s)$ over the set Λ becomes:

$$\begin{aligned}&\int_\Lambda \kappa(x,y)\left(C_k\left[\zeta_i(y)+\zeta_k(y)\right]\right)^{\frac{p+r}{q}} - (x)^{\frac{p+r}{q}}\Delta\mu_2(y) \geq \int_\Lambda \kappa(x,y)B_x\left(C_k\left[\zeta_i(y)+\zeta_k(y)\right]-x\right)\Delta\mu_2(y)\\ &= \sum_{k=i=1}^{n=m} B_x\left[C_k\zeta_i(y)+C_k\zeta_k(y)-x\right].\end{aligned} \tag{2.7}$$

Corollary 2.8. *Let $a,b \in \mathbb{T}$. Suppose $\eta : [a,b]_{\mathbb{T}^k} \longrightarrow [0,\infty)$ is rd-continuous and $\varphi : [0,\infty) \longrightarrow \mathbb{R}$ is a continuous, convex and superquadratic function. Then,*

$$\left(\frac{1}{b-a}\int_a^b \eta(x)\Delta x\right)^{\frac{p+r}{q}} \leq \frac{1}{b-a}\int_a^b \left((\eta(x))^{\frac{p+r}{q}} - \left|\eta(x)^{\frac{p+r}{q}} - \left(\frac{1}{b-a}\int_a^b \eta(y)\Delta y\right)^{\frac{p+r}{q}}\right|\right)\Delta x. \tag{2.8}$$

We can now state and proof the preliminary results of this paper.
Read as follows:

Corollary 2.9. *Let $\zeta, \kappa \in C_{rd}([\Lambda],(0,\infty)$ with $\int_\Lambda \kappa(x,y)\zeta(y)\Delta(y)$ and $\int_\Lambda \frac{\kappa(x,y)}{f(y)}\Delta(y)$ are finite, then*

$$\frac{\int_\Lambda \frac{\kappa(x,y)}{(\zeta_i(y)+\zeta_k(y))}\log\left(\zeta_i(y)+\zeta_k(y)\right)\Delta(y)}{\int_\Lambda \frac{\kappa((x,y)}{(\zeta_i(y)+\zeta_k(y))}\Delta(y)} \leq \frac{\int_\Lambda \kappa(x,y)\left(\zeta_i(y)+\zeta_k(x)\right)\log\left(\zeta_i(y)+\zeta_x(y)\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\left(\zeta_i(y)+\zeta_k(y)\right)\Delta(y)}. \tag{2.9}$$

Proof. Since $\varphi(\upsilon) = -\log(\upsilon)$ is strictly convex, it follows from Jensen inequality that

$$\left(\frac{\int_\Lambda \kappa(x,y)\frac{1}{(\zeta_i(y)+\zeta_k(y))^{\frac{p+r}{q}}}\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}\right) \leq \frac{\int_\Lambda \kappa(x,y)\left(\frac{1}{(\zeta_i(y)+\zeta_k(y))}\right)^{\frac{p+r}{q}}\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}$$

which can be written as

$$-\log\left(\frac{\int_\Lambda \kappa(x,y)\frac{1}{(\zeta_i(r)+\zeta_k(y))}\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}\right) \leq -\frac{\int_\Lambda \kappa(x,y)\log\left(\frac{1}{(\zeta_i(y)+\zeta_k(y))}\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}$$

and by simplifying further

$$\log\left(\frac{\int_\Lambda \kappa(x,y)\Delta(y)}{\int_\Lambda \kappa(x,y)\frac{1}{(\zeta_i(y)+\zeta_k(y))}\Delta(y)}\right) \leq \frac{\int_\Lambda \kappa(x,y)\log\left(\zeta_i(y)+\zeta_k(y)\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)},$$

that is,

$$\left(\frac{\int_\Lambda \kappa(x,y)\Delta(y)}{\int_\Lambda \kappa(x,y)\frac{1}{(\zeta_i(y)+\zeta_k(y))}\Delta(y)}\right) \leq e^{\left(\frac{\int_\Lambda \kappa(x,y)\log\left(\left(\zeta_i(y)+\zeta_k(y)\right)\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}\right)}.$$

Hence

$$\left(\frac{\int_\Lambda \kappa(x,y)\left((\zeta_i(y)+\zeta_k(y))\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}\right) = \frac{\int_\Lambda \kappa(x,y)\Delta(y)}{\int_\Lambda \frac{\kappa(x,y)(\zeta_i(y)+\zeta_k(y))}{(\zeta_i(y)+\zeta_k(y))}\Delta(y)} \leq e^{\left(\frac{\int_\Lambda \kappa(x,y)\log\left((\zeta_i(y)+\zeta_k(y))\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}\right)}.$$

It follows from Jensen inequality with respect to strictly convex function of the exponential that

$$\begin{aligned} & e^{\left(\frac{\int_\Lambda \kappa(x,y)\log\left((\zeta_i(y)+\zeta_k(y))\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}\right)} < \frac{\int_\Lambda \frac{\kappa(x,y)}{(\zeta_i(y)+\zeta_k(y))} e^{\log(\zeta_i(y)+\zeta_k(y))}\Delta(y)}{\int_\Lambda \frac{\kappa(x,y)}{(\zeta_i(y)+\zeta_k(y))}\Delta(y)} \\ & \left(\frac{\int_\Lambda \kappa(x,y)\left((\zeta_i(y)+\zeta_k(y))\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}\right) = x. \end{aligned} \tag{2.10}$$

. □

We state below theorem to proof our first result in this section, it reads as follows.

Theorem 2.10. *Let $\Lambda \in \mathbb{T}, \zeta, \kappa \in C_{rd}[a,b], (c,d)$ and $\kappa(x,y) \in C([a,b], \mathbb{R}$ with*

$$\int_\Lambda \kappa(x,y)\Delta(y) > 0$$

. If $\varphi \in C([a,b], \mathbb{R}$ is convex then,

$$\left(\frac{\int_\Lambda \kappa(x,y) C_k\left[\zeta_i(y)+\zeta_k(y)\right]\Delta(s)}{\int_\Lambda \kappa(x,y)\Delta(y)}\right)^{\frac{p+r}{q}} \leq \frac{\int_\Lambda \kappa(x,y)\left(C_k\left[\zeta_i(y)+\zeta_k(y)\right]\right)^{\frac{p+r}{q}}\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}.$$

If φ is strictly convex, then the inequality sign $\leq$ would be replaced by $<$.

Proof. Since φ is convex, it follows from (2.2) that for each $x \in (c,d)$ there exists $B_x \in \mathbb{R}$ such that the condition of the lemma holds. Let

$$x = \frac{\int_\Lambda \kappa(x,y)\left(C_k\left[\zeta_i(y)+\zeta_k(y)\right]\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta\mu_2(y)},$$

substituting for x in (2.7) leads to

$$\begin{aligned} & \int_\Lambda \kappa(k,x)(C_k\left[\zeta_i(y)+\zeta_k(y)\right])^{\frac{p+r}{q}}\Delta\mu_2(s) - \left(\int_\Lambda \kappa(x,y)\Delta(y)\right)(x)^{\frac{p+r}{q}} \\ & = \int_\Lambda \kappa(x,y)C_k(\zeta_i(y))\Delta(y) + \int_\Lambda \kappa(x,y)C_k(\zeta_k(y))\Delta(y) - \left(\int_\Lambda \kappa(x,y)\Delta(y)\right)x \\ & = \int_\Lambda \kappa(x,y)C_k(\zeta_i(y))\Delta(y) + \int_\Lambda \kappa(x,y)C_k(\zeta_k(y))\Delta(y) \\ & - \left(\int_\Lambda \kappa(x,y)\Delta(y)\right)\frac{\int_\Lambda \kappa(x,y)C_k(\zeta_i(y))\Delta(y) + \int_\Lambda \kappa(x,y)C_k(\zeta_k(y))\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)} \\ & = \int_\Lambda \kappa(x,y)\Delta\mu_2(y)\left[([C_k(\zeta_i(r)) + C_k\zeta_k(y))])^{\frac{p+r}{q}} - (x)^{\frac{p+r}{q}}\right] \\ & \geq \sum_{i=k=1}^{m,n} B_t\left[([C_k(\zeta_i(y)) + C_k(\zeta_k(y))]) - x\right] = 0. \end{aligned} \tag{2.11}$$

. □

The following threorem which is equivalent to Jensen inequality on time scales will be applied to prove the next results of this paper.

Theorem 2.11. *Let $(\Omega, \sum, \mu\Delta(y))$ and $(\Lambda, \sum, \Delta(y))$ be two time scales measure spaces with a $\sigma-$finite measures. Suppose that $U \subset \mathbb{R}^n$ is a closed convex set and $\varphi \in C(U, \mathbb{R})$ is superquadratic and $\zeta(\Lambda) \subset U$. Let $\varphi : \kappa_n \longrightarrow \mathbb{R}$ be continuous and superquadratic. $\kappa : \Omega \times \Lambda \longrightarrow \mathbb{R}$ be non negative such that $\kappa(x, .)$ is $\mu\Delta-$integrable. Then the inequality*

$$\left(\frac{\int_\Lambda \kappa(x,y)\left([\zeta_i(y)+\zeta_k(y)-x]\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta y}\right)^{\frac{p+r}{q}} \leq \frac{\int_\Lambda \kappa(x,y)\left(C_k\left((y)^{\frac{p+r}{q}}-(x)^{\frac{p+r}{q}}\right)\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}$$

holds for all function $\varphi : \Lambda \longrightarrow \kappa_n$. If φ superquadratic, then the inequality is reversed.

Proof. Let $\kappa(x, y)$ and $[\zeta_i(y) + \zeta_k(y)]$ be $\Delta(y)-$integrable. Then, for each fixed $x \in \Omega$, the function

$$K(y) = \int_\Lambda \kappa(x,y)\Delta(y)$$

$$B_x\left([\zeta_i(y)+\zeta_k(y)-x]\right) = \frac{1}{K(y)}\int_\Lambda \kappa(x,y)((y)^{\frac{p+r}{q}}-(x)^{\frac{p+r}{q}})\Delta(y).$$

Hence

$$\begin{aligned}(C_k\left[\zeta_i(y)+\zeta_k(y)\right])^{\frac{p+r}{q}}-(\tau)^{\frac{p+r}{q}} &\geq B_x\left(C_k\left[\zeta_i(y)+\zeta_k(y)\right]-x\right)\\ &=\sum_{k,i}^{n} B_x\left(C_k\zeta_i(y)+C_k\zeta_k(y)-x\right)=0,\end{aligned} \tag{2.12}$$

that is,

$$\begin{aligned}&(C_k\left[\zeta_i(y)+\zeta_k(y)-x\right])^{\frac{p+r}{q}} \leq \frac{1}{\int_\Lambda \kappa(x,y)\Delta(y)} \times \int_\Lambda \kappa(x,y)\Delta\mu_2((y)^{\frac{p+r}{q}}-(x)^{\frac{p+r}{q}})\\ &=\int_\Lambda \frac{\kappa(x,y)}{\int_\Lambda \kappa(x,y)\Delta\mu_2(y)}((y)^{\frac{p+r}{q}}-(x))^{\frac{p+r}{q}}\Delta(y),\end{aligned} \tag{2.13}$$

which gives the inequality as desired. □

We remark that in (2.13), if $k = \kappa, r = x$ and $C_k\zeta_i(y) = 1$, then we have the result in [5].

3. Our preliminary main Results

To establish our main results, we need the following results taken from [10]. For convenience of readers and to avoid distraction we only give the statements here. For the proof see [3, 10, 11].

The following theorem which is equivalent to Jensen inequality on time scales will be applied to prove the next results of this paper.

In this section, some more general class of Hardy's inequalities involving the general Hardy integral operator are discussed.

We start with this result.

Theorem 3.1. *Let $(\Omega, \sum, \Delta(y))$ and $(\Lambda, \sum, \Delta(y))$ be two time scales measure spaces with a $\sigma-$finite measures. Suppose that $U \subset \mathbb{R}^n$ is a closed convex set and $\varphi \in C(U, \mathbb{R})$ is superquadratic and $f(\Lambda) \subset U$. Let $\varphi : \kappa_n \longrightarrow \mathbb{R}$ be continuous and superquadratic. $\kappa : \Omega \times \Lambda \longrightarrow \mathbb{R}$ be non negative such that $\kappa(x, .)$ is $\Delta(y)-$integrable. Then the inequality*

$$\varphi\left(\frac{\int_\Lambda \kappa(x,y)\left([\zeta_i(y)+\zeta_k(y)-x]\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}\right) \leq \frac{\int_\Lambda \kappa(x,y)\left(C_k\left(y^{\frac{p+r}{q}}-x^{\frac{p+r}{q}}\right)\right)\Delta(y)}{\int_\Lambda \kappa(x,y)\Delta(y)}$$

holds for all function $\varphi : \Lambda \longrightarrow \kappa_n$. If φ superquadratic, then the inequality is reversed.

Proof. Let $\kappa(x,y)$ and $[\zeta_i(y)+\zeta_k(y)]$ be $\Delta(y)-$integrable. Then, for each fixed $x \in \Omega$, the function

$$K(y) = \int_\Lambda \kappa(x,y)\Delta(y) \tag{3.1}$$

$$C_k\left([\zeta_i(y)+\zeta_k(y)-x]\right)^{\frac{p+r}{q}} = \frac{1}{K(y)}\int_\Lambda \kappa(x,y)((y)^{\frac{p+r}{q}} - (x)^{\frac{p+r}{q}})\Delta(y). \tag{3.2}$$

combining (3.1) and (3.2), yields

$$\begin{aligned}(C_k\left[\zeta_i(y)+\zeta_k(y)-x\right])^{\frac{p+r}{q}} &\leq \frac{1}{\int_\Lambda \kappa(x,y)\Delta(y)} \times \int_\Lambda \kappa(x,y)\Delta(y)(\varphi(y)-\varphi(x))\Delta(y)\\ &= \int_\Lambda \frac{\kappa(x,y)}{\int_\Lambda \kappa(x,y)\Delta(y)}(y^{\frac{p+r}{q}} - x^{\frac{p+r}{q}})\Delta(y).\end{aligned} \tag{3.3}$$

which gives the inequality as desired. □

We remark that in (2.13), if $k=\kappa, r=x$ and $C_k\zeta_i(r)=1$, then we have the result in [5].

Theorem 3.2. *Let $(\Omega, \sum_1, \Delta\mu_1)$ and $(\Lambda, \sum_2, \Delta\mu_2)$ be two time scale measure spaces with a $\sigma-$finite measures. If $u : \Lambda_n \longrightarrow \mathbb{R}$ and $k : \Lambda_1 \times \Lambda_2 \longrightarrow \mathbb{R}$ are non-negative functions such that $k(x,:)$ is $\mu\Delta_2$-integrable for $r \in \Lambda_1$. Furthermore, suppose that $K : \Lambda_1 \longrightarrow \mathbb{R}$ is defined by*

$$K(x) := \int_{\Lambda_2} k(x,y)\Delta\mu_2(y) > 0,\ x \in \Lambda_1$$

$$v(y) := w(y)\int_\Lambda \frac{k(x,y)u(x)}{K(x)}\Delta\mu_1(x) < \infty,\ y \in \Lambda_2.$$

Suppose $\varphi : [a,\infty) \longrightarrow \mathbb{R}(a \geq 0)$ is a non-negative superquadratic function, then the inequality:

$$\begin{aligned}&\int_{\Lambda_1} w(x)u(y)(C_k\zeta(y))^{\frac{p+r}{q}}\Delta\mu_1(y) +\\ &\int_{\Lambda_2} w(x)\int_\Omega u(y)\frac{\kappa(x,y)}{K(x)}\left(C_k\left[\zeta_i(y)+\zeta_k(y)\right]^{\frac{p+r}{q}} - x^{\frac{p+r}{q}}\right)\Delta\mu_1(y)\Delta\mu_2(x)\\ &\leq \int_\Lambda w(x)u(y)(\zeta(y))^{\frac{p+r}{q}}\Delta\mu_2(y)\end{aligned} \tag{3.4}$$

holds for all non-negative $\Delta\mu_2$-integrable function $\zeta : \Lambda_2 \longrightarrow \mathbb{R}$ and for $C_k\zeta : \Lambda \longrightarrow \mathbb{R}$

$$C_k\zeta(x) = \frac{1}{K(y)}\int_{\Lambda_2} k(r,y)\zeta(y)\Delta\mu_2(y),\ y \in \Lambda.$$

If φ is subquadratic, then the inequality sign in (3.4) is reversed.

Proof. Combining (2.1), (2.4), (2.8) and (3.3) gives

$$\begin{aligned}&\int_\Lambda w(x)u(y)(C_k(\zeta(y))^{\frac{p+r}{q}}\Delta\mu_1(y)\\ &= \int_\Lambda w(x)u(y)\left(\int_{\Lambda_2}\frac{1}{K(x)}\kappa(x,y)\zeta(y)\Delta\mu_2(x)\right)^{\frac{p+r}{q}}\Delta\mu_1(y)\\ &\leq \int_\Omega \frac{\kappa(x,y)w(x)}{K(x)}\left(\int_\Lambda u(y)(\zeta(y))^{\frac{p+r}{q}}\Delta\mu_2(y)\right)\Delta\mu_1(x)\\ &- \int_\Omega \frac{w(x)\kappa(x,y)}{K(x)}\int_\Lambda u(y)(C_k(\zeta(y))^{\frac{p+r}{q}}\Delta\mu_2(y)\Delta\mu_1(x)\\ &= \int_\Omega w(x)(\zeta(y))^{\frac{p+r}{q}}\left(\int_{\Lambda_1}\frac{u(y)\kappa(x,y)}{K(x)}\Delta\mu_1(y)\right)\Delta\mu_2(x)\\ &- \int_\Omega w(x)\int_\Lambda \frac{u(y)\kappa(x,y)}{K(x)}C_k\left(y^{\frac{p+r}{q}} - x^{\frac{p+r}{q}}\right)\Delta\mu_2(y)\Delta\mu_1(x).\end{aligned} \tag{3.5}$$

and more estimation yields

$$
\begin{aligned}
&\int_\Lambda w(x)u(y)(C_k(\zeta(y))^{\frac{p+r}{q}}\Delta\mu_1(y) \\
&\le \int_\Omega (\zeta(y))^{\frac{p+r}{q}} u(y)\Delta\mu_2(y)\int_\Lambda \frac{\kappa(x,y)w(x)}{K(x)}\Delta\mu_1(x) \\
&- \int_\Omega\int_\Lambda \frac{u(y)\kappa(x,y)}{K(x)} C_k\left(y^{\frac{p+r}{q}} - x^{\frac{p+r}{q}}\right)\Delta\mu_2(y)w(x)\Delta\mu_1(x). \\
&= \int_\Lambda w(y)\int_\Omega \frac{k(x,y)u(x)}{K(x)}\Delta\mu_1(x)u(y)(C_k(\zeta(y))^{\frac{p+r}{q}}\Delta\mu_2(y) \\
&- \int_\Omega w(x)\int_\Lambda \frac{u(y)\kappa(x,y)}{K(x)} C_k(\varphi(y)-\varphi(x))\Delta\mu_2(y)\Delta\mu_1(x).
\end{aligned} \tag{3.6}
$$

from which (3.6) follows. The inequality above is indeed the result in [9].

4. Our main Results

The main results read as follows:

Theorem 4.1. *Let $(\Lambda, \sum_1, \Delta(x))$ and $(\Omega, \sum_2, \Delta(y))$ be two measure spaces on time scale measure with $\sigma-$ finite measures and Let $\zeta : \Lambda \longrightarrow \mathbb{R}$ and $k : \Lambda\times\Omega \longrightarrow \mathbb{R}$ are non-negative functions such that $k(x,y)$ is Δ_2-integrable for $x \in \Omega$. Furthermore suppose that $K : \Omega \longrightarrow \mathbb{R}$ is defined by*

$$K(x) := \int_\Omega k(x,y)\Delta(y) > 0,\ x \in \Lambda$$

and

$$\eta(y) := \left(w(x)\int_\Lambda \left(\frac{k(x,y)}{K(x)}\right)^s \Delta(x)\right)^{\frac{1}{s}} < \infty, \quad y \in \Omega.$$

where $s \ge 1$.
(1) If $s \ge 2$ and $\varphi : [0,\infty) \longrightarrow \mathbb{R}$ be non negative superquadratic function, then the inequality.

$$
\begin{aligned}
&\int_\Lambda w(x)(C_k\zeta(x))^{\frac{s(p+r)}{q}}\Delta(x) \\
&+ s\int_\Omega\int_\Lambda w(x)\frac{k(x,y)}{K(x)}\left((|\zeta(y)-C_k\zeta(x)|)\right)^s\Delta(x)\Delta(y) \\
&\le \left(\int_\Omega \eta(y)(\zeta(y))^{\left(\frac{p+r}{q}\right)}\Delta(y)\right)^s - \int_\Lambda w(x)\int_\Omega \frac{k(x,y)}{K(x)}\left((|\zeta(y)-C_k\zeta(x)|)^{\frac{p+r}{q}}\Delta(y)\right)^s
\end{aligned} \tag{4.1}
$$

holds for all non negative $\Delta\mu_2$-integrable function $f : \Omega \longrightarrow \mathbb{R}$ and for $C_k\zeta : \Lambda \longrightarrow \mathbb{R}$

$$C_k\zeta(x) = \frac{1}{K(x)}\int_\Omega k(x,y)\zeta(x)\Delta(y), \quad x \in \Lambda$$

(2) If $1 < s \le 2$ and φ is subquadratic, then the inequality sign in (4.1) is reversed.

Proof. combining (2.4), Corollary 2.6 if the power is raised to s and Minkowski integral inequality [25]. We obtain the following inequality

$$
\begin{aligned}
&\left(\frac{1}{K(x)}\int_\Omega k(x,y)\zeta(y)\Delta(y)\right)^{\frac{p+r}{q}} \\
&+ \frac{1}{K(x)}\int_\Omega k(x,y)\left(\zeta(y) - \frac{1}{K(x)}\int_\Omega (k,y)\zeta(y)\Delta(y)\right)^{\frac{p+r}{q}}\Delta(y) \\
&\le \frac{1}{K(x)}\int_\Omega k(x,y)(\zeta(y))^{\frac{p+r}{q}}\Delta(y)
\end{aligned} \tag{4.2}
$$

Multiplying power of both sides of (4.2) with s, we get

$$\begin{aligned}
&\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{\frac{s(p+r)}{q}} \\
&+ s\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{s} \times \int_\Omega \frac{k(x,y)}{K(x)}\left(\zeta(y) - \int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{\frac{p+r}{q}}\Delta(y) \\
&+ \left(\int_\Omega \frac{k(x,y)}{K(x)}\left(\zeta(y) - \int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{\frac{p+r}{q}}\Delta(y)\right)^{s} \\
&\leq \left(\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{\frac{p+r}{q}} + \int_\Omega \frac{k(x,y)}{K(x)}\left(\zeta(y) - \int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta\mu_2(y)\right)^{\frac{p+r}{q}}\Delta(y)\right)^{s} \\
&= \left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)^{\frac{p+r}{q}}\Delta(y)\right)^{s}
\end{aligned} \tag{4.3}$$

Further calculation and when multiplying both of (4.3) with $w(x)$ yields

$$\begin{aligned}
&\left[w(x)\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{s\left(\frac{p+r}{q}\right)} + s\, w(x)\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{s}\right] \\
&\times \left[\int_\Omega \frac{k(x,y)}{K(x)}\left(\zeta(y) - \int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{\frac{p+r}{q}}\right]^{s} \\
&+ \left(\int_\Omega w(x)\frac{k(x,y)}{K(x)}\left(\zeta(y) - \int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{\frac{p+r}{q}}\Delta(y)\right)^{s} \\
&\leq \int_\Lambda w(x)\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)^{\frac{p+r}{q}}\Delta(y)\right)^{s}
\end{aligned} \tag{4.4}$$

Integrate left and right hand sides of (4.4) with the respect to $\Delta(x)$, we have

$$\begin{aligned}
&\int_\Lambda \left[w(x)\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{s\left(\frac{p+r}{q}\right)} + s\, w(x)\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{s}\right]\Delta(x) \\
&\times \left[\int_\Omega \frac{k(x,y)}{K(x)}\left(\zeta(y) - \int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{\frac{p+r}{q}}\Delta(y)\right]^{s} \\
&+ \int_\Lambda \left(\int_\Omega w(x)\frac{k(x,y)}{K(x)}\left(\zeta(y) - \int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)\Delta(y)\right)^{\frac{p+r}{q}}\Delta(y)\right)^{s}\Delta(x) \\
&\leq \int_\Lambda \int_\Lambda w(x)\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)^{\frac{p+r}{q}}\Delta(y)\right)^{s}\Delta(x)
\end{aligned} \tag{4.5}$$

and apply Minkowski integral inequality (4.5), yields

$$\left(\int_\Lambda w(x)\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta(y)^{\frac{p+r}{q}}\Delta(y)\right)^{s}\Delta(x)\right)^{\frac{1}{s}} \leq \int_\Omega \zeta(y)^{\frac{p+r}{q}}\left(\int_\Lambda w(x)\left(\frac{k(x,y)}{K(x)}\Delta(x)\right)^{s}\right)^{\frac{1}{s}}\Delta(y) \tag{4.6}$$

We also consider the following inequality as part of our result

$$\int_\Lambda w(x)\left(\int_\Omega \frac{k(x,y)}{K(x)}\zeta((y))^{\left(\frac{p+r}{q}\right)}\right)^{s}\Delta(x) \leq \left(\int_\Omega \zeta(y)^{\frac{p+r}{q}}\left(\int_\Lambda w(x)\left(\frac{k(x,y)}{K(x)}\right)^{s}\Delta(x)\right)^{\frac{1}{s}}\Delta(y)\right)^{s} \tag{4.7}$$

$$\begin{cases} \eta(y) = \left(\int_\Lambda w(x)\left(\frac{k(x,y)}{K(x)}\Delta(y)\right)^{s}\Delta(x)\right)^{\frac{1}{s}} \\ C_k\zeta(y) = \frac{1}{K(x)}\int_\Omega k(x,y)\zeta(y)\Delta(y)\ x \in \Lambda \end{cases} \tag{4.8}$$

Relating inequalities (2.1), (2.1), (4.5), (4.7) and (4.8). We obtain (4.1), we have

$$\begin{aligned}
&\int_\Lambda w(x)(C_k\zeta(x))^{\frac{s(p+r)}{q}}\Delta(x)\\
&+s\int_\Omega\int_\Lambda w(x)\frac{k(x,y)}{K(x)}\left((|\zeta(y)-C_k\zeta(x)|)\right)^s\Delta(x)\Delta(y)\\
&\le\left(\int_\Omega \eta(y)(\zeta(y))^{\frac{p+r}{q}}\Delta(y)\right)^s-\int_\Lambda w(x)\int_\Omega\frac{k(x,y)}{K(x)}\left((|\zeta(y)-C_k\zeta(x)|)^{\frac{p+r}{q}}\Delta(y)\right)^s
\end{aligned}\tag{4.9}$$

this completes the proof the theorem.

(2) adopting same method, all the corollaries and the Theorems of (1) also, but change in signs we have inequality (1) reversed.

If we define $\varphi(x)=x^{\frac{p}{q}}$ is a non-negative superquadratic function in (4.1) becomes

$$\begin{aligned}
&\int_\Lambda w(x)(C_k\zeta(x))^{\frac{sp}{q}}\Delta(x)\\
&+s\int_\Omega\int_\Lambda w(x)\frac{k(x,y)}{K(x)}\left((|\zeta(y)-C_k\zeta(x)|)\right)^s\Delta(x)\Delta(y)\\
&\le\left(\int_\Omega \eta(y)(\zeta(y))^{\frac{p}{q}}\Delta(y)\right)^s-\int_\Lambda w(x)\int_\Omega\frac{k(x,y)}{K(x)}\left((|\zeta(y)-C_k\zeta(x)|)^{\frac{p}{q}}\Delta(y)\right)^s
\end{aligned}\tag{4.10}$$

. Inequality (4.10) holds for any $\frac{p}{q}\ge 2$. The above inequality holds in the reverse direction when $1<\frac{p}{q}\le 2$.

The following observations were noticed

$$\begin{aligned}
&\int_\Lambda w(x)(C_k\zeta(x))^{\frac{sp}{q}}\Delta(x)+s\int_\Omega\int_\Lambda w(x)\frac{k(x,y)}{K(x)}\left((|\zeta(y)-C_k\zeta(x)|)\right)^s\Delta(x)\Delta(y)\\
&\le\left(\int_\Omega \eta(y)(\zeta(y))^{\frac{p}{q}}\Delta(x)\right)^s \text{ which is a classic Hardy's inequality}
\end{aligned}$$

Further estimate gives a refinement of (1.3)

$$\begin{aligned}
&\int_\Lambda w(x)(C_k\zeta(x))^{\frac{sp}{q}}\Delta(x)+s\int_\Omega\int_\Lambda w(x)\frac{k(x,y)}{K(x)}\left((|\zeta(y)-C_k\zeta(x)|)\right)^s\Delta(x)\Delta(y)\\
&\le\left(\int_\Omega \eta(y)(\zeta(y))^{\frac{p}{q}}\Delta(y)\right)^{\frac{p}{q}}\\
&\le\int_\Lambda w(x)(C_k\zeta(x))^{\frac{sp}{q}}\Delta(x)+s\int_\Omega\int_\Lambda w(x)\frac{k(x,y)}{K(x)}\left((|\zeta(y)-C_k\zeta(x)|)\right)^s\Delta(x)\Delta(y)\\
&+\left(\int_\Omega \eta(y)(\zeta(y))^{\frac{p}{q}}\Delta(y)\right)^{\frac{p}{q}}
\end{aligned}$$

Conclusion:

We succeeded in using concept of superquadratic functions to achieve a class of Hardy-type inequalities containing a more wide-ranging Hardy operator. However, indeed the results accomplish are new, refine, and improve.

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