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EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS UNDER WEAK TOPOLOGY FEATURES

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ABSTRACT. Using Krasnoselskii type fixed point theorem under the weak topology, we establish some sufficient conditions to ensure the existence of the weak solutions for kinds of initial value problems of fractional differential equations, involving Riemann-Liouville fractional derivative.

1. INTRODUCTION

The great recent interest of fractional differential equations have been the principle reason of the intensive development of the theory of fractional calculus and fractional differential equations. Various applications of such contributions in many scientific disciplines such as physics, chemistry, biology, engineering, viscoelasticity, signal processing, electrotechnical, electrochemistry and controllability was developed in [1, 3, 4, 12, 13].

In this paper, we are interested in the qualitative theory of a kind of fractional differential equations in the field of initial value problems. More precisely, let us consider the following Riemann-Liouville type fractional differential equation with initial conditions (IVP)

$$\begin{cases} D^{\alpha}u(t) = h\left(t\right) f(t, u\left(t\right)) + g\left(t, Hu\left(t\right)\right), t \in I := [0, T],\ T > 0 \\ \lim\limits_{t\to 0+} t^{2-\alpha}u(t) = \lim\limits_{t\to 0+} t^{2-\alpha}u^{'}(t) = 0, \end{cases} \tag{1.1}$$

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where u belongs to $L^1(I,E)$, the space of Lebesgue integrable functions on I with values in a finite dimensional Banach space $(E,\|\cdot\|)$, which is endowed with the norm

$$\|u\|_{L^1} = \int_0^T \|u(t)\|\,dt,$$

and D^α is the left Riemann Liouville derivative of order α, $1<\alpha\leq 2$. Here, $f(.,.)$ and $g(.,.)$ are nonlinear functions, $h(.)$ is a measurable function and H is a bounded linear operator from $L^1(I,E)$ into itself.

This type of fractional differential equations has been studied by many authors. The most of investigated papers deals with the existence, uniqueness and stability of solutions using fixed point theorems in Banach spaces (see for example the paper [3]). On the other hand, to the best of our knowledge, the use of fixed point theorems under the weak topology in the study of fractional differential equations is still not sufficiently generalized. However although, fixed point theory with weak topology has been investigated in several papers and monographs for integral equations to prove the existence of solutions [11,14] and the monograph [9] and the references therein.

The aim of this work is to study of existence of solutions for a nonlinear boundary value problem (1.1) involving the left Riemann-Liouville fractional derivative. For this end, we transform problem (1.1) to an integral equation.

By combining the theory of fixed point under weak topology point with the De Blasi measure of weak noncompactness and the theory of fractional differential equations, we give sufficient conditions on the functions f and g to prove that IVP (1.1) has at least one integrable solution. For this purpose, we give some preliminary concepts and lemmas around fractional calculus theory and weak topoloy. Then, we transform IVP (1.1) into Volterra type integral equation employing some useful definitions and lemmas of fractional integral and derivative. After that, we present our main result which based on a variant of fixed point theorem developed in [14].

2. PRELIMINARIES

In this section we establish the notation used in the paper and we provide a few auxiliary facts which will be needed in our considerations. Moreover, we give here definitions of basic concepts applied in our study and we also indicated some essential properties of the concepts appearing in our reasonings.

Throughout the paper we denote by $\mathbb{R}$ the set of real numbers. The symbol $\mathbb{N}$ stands for the set of natural numbers (positive integers). By the symbol E we will denote a Banach space endowed with the norm $\|.\|_E$ and with zero element θ. In general, we write $\|.\|$ in place of $\|.\|_E$. For $r>0$ the symbol B_r denotes the closed ball centered at θ and with radius r and $\mathcal{D}(A)$ denotes the domain of an operator A. By the symbol $\mathcal{M}_E$ we will denote the collection of all nonempty bounded subsets of E while $\mathcal{W}(E)$ stands for its subfamily consisting of all relatively weakly compact sets. Moreover, for an arbitrarry subset M of the space X the symbol $\overline{M^w}$ will stand for the weak closure of M and the symbol $\text{conv}(M)$ denotes the conve hull of M. Apart from this we use the standard notation M_1+M_2, $\lambda M(\lambda\in\mathbb{R})$ for algebraic operations on sets.

Further, let us recall the concept of the De Blasi measure of weak noncompactness [9] being the function $\omega : \mathcal{M}_E \to \mathbb{R}_+ = [0, \infty)$, defined in the following way

$$\omega(M) = \inf\{r > 0 : \text{ there exists } W \in \mathcal{W}(X) \text{ such that } M \subset W + B_r\}.$$

For our purpose we recall some basic properties of the measure of weak noncompactness [4, 9].

Lemma 2.1. Let M_1, M_2 be two elements of $\mathcal{M}_E$. Then, the following conditions are satisfied:

(1) $M_1 \subseteq M_2$ implies $\omega(M_1) \leq \omega(M_2)$.
(2) $\omega(M_1) = 0$ if, and only if, $\overline{M_1}^w \in \mathcal{W}(E)$.
(3) $\omega(\overline{M_1}^w) = \omega(M_1)$.
(4) $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$.
(5) $\omega(\lambda M_1) = |\lambda|\omega(M_1)$ for all $\lambda \in \mathbb{R}$.
(6) $\omega(conv(M_1)) = \omega(M_1)$.
(7) $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$.
(8) if $(M_n)_{n\geq 1}$ is a decreasing sequence of nonempty bounded and weakly closed subsets of E with $\lim\limits_{n\to\infty} \omega(M_n) = 0$, then $M_\infty := \bigcap\limits_{n=1}^{\infty} M_n$ is nonempty and $\omega(M_\infty) = 0$ i.e., M_∞ is relatively weakly compact. ♢

In L^1 space, the measure $\omega(.)$ possesses the following form (see [2]).

Proposition 2.1. [2] Let Ω be a compact subset of $\mathbb{R}^n$ and let M be a bounded subset of $L^1(\Omega, E)$ where E is a finite dimensional Banach space. Then, $\omega(.)$ possesses the following form

$$\omega(M) = \lim_{\varepsilon\to 0} \sup \left\{ \sup_{\Psi\in M} \left\{ \int_D \|\Psi(t)\| \, dt : meas(D) \leq \varepsilon \right\} \right\},$$

for any nonempty subset D of Ω, where $meas(.)$ denotes the Lebesgue measure. ♢

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ and let E, F be two Banach spaces. A function $f : \Omega \times E \to F$ is said to be a Carathéodory, if

(i) for any $u \in E$, the map $t \longmapsto f(t, u)$ is measurable from Ω to F, and
(ii) for almost all $t \in \Omega$, the map $u \longmapsto f(t, u)$ is continuous from E to F.

Let $m(\Omega, E)$ be the set of all measurable functions $u : \Omega \to E$. If f is a Carathéodory function, then f defines a mapping $N_f : m(\Omega, E) \to m(\Omega, F)$ by $N_f u(t) := f(t, u(t))$, for all $t \in \Omega$. This mapping is called the Nemytskii's operator associated to f. ♢

Let us recall the following Lemma that will be needed in the sequel.

Lemma 2.2. [15] Let $\Omega \subset \mathbb{R}^n$ and let E be a separable Banach space, and $p, q \geq 1$ and let $F : \Omega \times E \to E$ be a Carathéodory function. The Nemytskii operator $\mathcal{N}_f$ associated to f maps continuously the space $L^1(\Omega, E)$ into itself if, and only if,

$$\|f(t, u)\| \leq a(t) + b\|u\|, \forall t \in I, \forall u \in E,$$

where $a \in L^1_+(\Omega, E)$ and b is a nonnegative constant. Here, $L^1_+(\Omega, E)$ stands for the positive cone of the space $L^1(\Omega, E)$. Obviously, we have

$$\|\mathcal{N}_f u\|_{L^1} \leq \|a\|_{L^1} + b\|u\|_{L^1}, \forall u \in L^1(\Omega, E). \quad ♢$$

Definition 2.2. [8,13] The Riemann-Liouville fractional integral of the function u of order $\alpha \geq 0$ is defined by

$$I^{\alpha}u\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\frac{u\left(\tau\right)}{\left(t-\tau\right)^{1-\alpha}}d\tau,$$

where $\Gamma\left(.\right)$ is the Euler gamma function defined by $\Gamma\left(\alpha\right)=\int_{0}^{\infty}e^{-t}t^{\alpha-1}dt.$ ♢

Definition 2.3. [8,13] The Riemann-Liouville fractional derivative of the function u of order $\alpha \in (n-1, n]$, $n \geq 1$ is defined by

$$D^{\alpha}u\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^{n}}{dt^{n}}\int_{0}^{t}\frac{u\left(\tau\right)}{\left(t-\tau\right)^{\alpha-n+1}}d\tau. \quad ♢$$

Let $\alpha > 0$ be a real number, we have following Lemma.

Lemma 2.3. [13] The unique solution of the linear fractional differential equation

$$D^{\alpha}u(t)=0,$$

is given by

$$u(t)=c_{1}t^{\alpha-1}+c_{2}t^{\alpha-2}+c_{3}t^{\alpha-3}+...+c_{n}t^{\alpha-n},\quad c_{i}\in\mathbb{R},\ i=1,2,...,n. \quad ♢$$

3. Existence result

Let E be a Banach space and let $A : \mathcal{D}(A) \subseteq E \to E$ be an operator. Recall the following conditions:

$$(\mathcal{C}1) \qquad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A) \text{ is a weakly convergent sequence in } E, \text{ then} \\ (Ax_n)_{n\in\mathbb{N}} \text{ has a strongly convergent subsequence in } E. \end{cases}$$

$$(\mathcal{C}2) \qquad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A) \text{ is a weakly convergent sequence in } E, \text{ then} \\ (Ax_n)_{n\in\mathbb{N}} \text{ has a weakly convergent subsequence in } E. \end{cases}$$

Note conditions $(\mathcal{C}1)$ and $(\mathcal{C}2)$ were considered in [7,10] and for some applications on maps satisfying these conditions we refer the reader to the monograph [9].

The following variant of fixed point theorem [14] will play a fundamental role in our considerations.

Theorem 3.1. Let M be a nonempty, bounded, closed, and convex subset of a Banach space E, Suppose that $A : M \to E$ and $B : M \to E$ be two operators such that
(i) A is continuous and satisfies $(\mathcal{C}1)$,
(ii) there exists $\gamma \in [0, 1[$ such that $\omega\left(AS + BS\right) \leq \gamma\omega\left(S\right)$ for all $S \subseteq M$,
(iii) B is a contraction and satisfies $(\mathcal{C}2)$, and
(iv) $AM + BM \subseteq M$.
Then, there is $u \in M$ such that $Au + Bu = u$. ♢

Our existence result is based on Theorem 3.1. After showing the existence of solutions for the problem IVP (1.1), we transform it into an equivalent integral equation as follows.

Lemma 3.1. IVP (1.1) is equivalent to the following Volterra type integral equation

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\,f(s,u(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,Hu(s))ds. \tag{3.1}$$

♢

Proof. Using Lemma 2.3, the equation (1.1) can be written as follows

$$\begin{aligned} u(t) &= I^{\alpha}h(t)\,f(t,u(t))+I^{\alpha}g(t,Hu(t))+c_1t^{\alpha-1}+c_2t^{\alpha-2} \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\,f(s,u(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,Hu(s))ds \\ &\quad +c_1t^{\alpha-1}+c_2t^{\alpha-2}. \end{aligned} \tag{3.2}$$

Using condition $\lim_{t\to0^+}t^{2-\alpha}u(t)=0$, we get $c_2=0$, and the condition $\lim_{t\to0^+}t^{2-\alpha}u'(t)=0$ give us $c_1=0$. Substituting in (3.2), we obtain the integral equation

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\,f(s,u(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,Hu(s))ds.$$

□

From Lemma 3.1, it follows that the integral equation (3.1) can be written in the following form

$$u=\mathcal{A}u+\mathcal{B}u$$

where $\mathcal{A}$, and $\mathcal{B}$ are two operators defined from $L^1(I,E)$ into itself by

$$\mathcal{A}:=\mathcal{I}\mathcal{N}_f \text{ and } \mathcal{B}:=\mathcal{J}\mathcal{N}_gH, \tag{3.3}$$

where $\mathcal{N}_f$, and $\mathcal{N}_g$ are the Nemytskii's operators associated to $f(.,.)$ and $g(.,.)$, respectively. The linear operators $\mathcal{I}$ and $\mathcal{J}$ are defined from $L^1(I,E)$ into $L^1(I,E)$ by

$$\mathcal{I}z(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\,z(s)\,ds,$$

and

$$\mathcal{J}z(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}z(s)\,ds.$$

In what follows, we need the following assumption:

($\mathcal{H}1$) The function $h:I\to E$ belongs to $L^{\infty}(I,E)$.

Lemma 3.2. Assume that ($\mathcal{H}1$) is satisfied. The linear operators $\mathcal{I}$ and $\mathcal{J}$ are bounded on $L^1(I,E)$ and we have the following estimates

$$\|\mathcal{I}z\|_{L^1}\le\frac{T^{\alpha}}{\Gamma(\alpha)}\|h\|_{L^\infty}\|z\|_{L^1},$$

and

$$\|\mathcal{J}z\|_{L^1}\le\frac{T^{\alpha}}{\Gamma(\alpha)}\|z\|_{L^1},$$

for all $z\in L^1(I,E)$. ♢

Now, we give the existence of integrable solution for the Volterra type integral equations (3.1). Obviously, every solution of (3.1) is also a solution of (1.1).

So, we consider the following assumptions:

$(\mathcal{H}2)$ The function $g : I \times E \to E$ is a measurable function, $g(.,0) \in L^1(I,E)$ and g is Lipschitzian with respect to the second variable, i.e., there exists a $\lambda \in \mathbb{R}_+$ such that $\|g(t,u) - g(t,v)\| \leq \lambda\|u - v\|$ for all $t \in \Omega$ and $u, v \in E$.

$(\mathcal{H}3)$ $f(.,.)$ is a Carathéodory function and there exists a function $a \in L^1_+(I)$ and a nonnegative constant b such that

$$\|f(t,u)\| \leq a(t) + b\|u\|,$$

for all $(t,u) \in I \times E$.

Lemma 3.3. [14] Let E be a finite dimensional Banach space. Assume that $(\mathcal{H}3)$ holds. Then, the Nemytskii operator $\mathcal{N}_f$ satisfies condition $(\mathcal{C}2)$. ♢

Theorem 3.2. Let E be a finite dimensional Banach space and let $I = [0,T],\ T > 0$ be a compact subset of $\mathbb{R}$. Assume that the conditions $(\mathcal{H}1) - (\mathcal{H}3)$ are satisfied. Then IVP (1.1) has at least one solution in $L^1(I,E)$, provided

$$\frac{T^\alpha}{\Gamma(\alpha)}\left(b\|h\|_{L^\infty} + \lambda\|H\|_{\mathcal{L}}\right) < 1. \tag{3.4}$$

Here, $\|\cdot\|_{\mathcal{L}}$ denotes the standard norm of linear operator spaces. ♢

Proof. To prove Theorem 3.2, it is enough to show that operators $\mathcal{A}$ and $\mathcal{B}$ given by (3.3) satisfy all hypotheses of Theorem 3.1. To this end, we need four claims. Before this, let

$$r \geq \frac{\frac{T^\alpha}{\Gamma(\alpha)}\|h\|_{L^\infty}\|a\|_{L^1} + \frac{T^\alpha}{\Gamma(\alpha)}\|g(.,0)\|_{L^1}}{1 - \left(\frac{bT^\alpha}{\Gamma(\alpha)}\|h\|_{L^\infty} + \frac{\lambda T^\alpha}{\Gamma(\alpha)}\|H\|_{\mathcal{L}}\right)}. \tag{3.5}$$

Clearly, r is positive from (3.4). Let us consider the bounded, closed, convex of $L^1(I,E)$, defined by

$$B_r = \left\{u \in L^1(I,E) : \|u\|_{L^1} \leq r\right\}.$$

Claim 1. Based on Lemma 2.2 and using $(\mathcal{H}3)$, we can see that $\mathcal{A}$ is continuous maps from $L^1(I,E)$ into itself. We show now that $\mathcal{A}$ satisfies $(\mathcal{C}1)$. For this purpose, let $(u_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence of $L^1(I,E)$, then by Lemma 3.3, it follows that $(\mathcal{N}_f u_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence, say, $(\mathcal{N}_f u_{n_k})_{k\in\mathbb{N}}$. From the boundedness of the operator $\mathcal{I}$, it follows that the sequence $(\mathcal{I}\mathcal{N}_f u_{n_k})_{k\in\mathbb{N}}$ converges pointwise for almost all $t \in I$. Now, when applying Vitali convergence theorem ([6], p.150), we deduce that the sequence $(\mathcal{A}u_{n_k})_{k\in\mathbb{N}}$ converges strongly in $L^1(I,E)$. Consequently, $\mathcal{A}$ satisfies $(\mathcal{C}1)$.

Claim 2. We claim that condition (ii) of Theorem 3.1 is fulfilled. Let S be a bounded subset of $L^1(I,E)$, then for all $u \in S$, for all $\varepsilon > 0$ and any nonempty subset J of I, we have

$$\begin{aligned}
\int_J \|\mathcal{N}_f u(t)\|\,dt &\leq \int_J \|f(t,u(t))\|\,dt \\
&\leq \int_J \left(\|a(t)\| + b\|u(t)\|\right)dt \\
&\leq \|a\|_{L^1(J)} + b\int_J \|u(t)\|\,dt.
\end{aligned}$$

Taking into account the fact that the set consisting of one element is weakly compact, by Proposition 2.1 we obtain

$$\limsup_{\varepsilon \to 0} \left\{ \int_J \|a(t)\| \, dt : meas(J) \leq \varepsilon \right\} = 0,$$

then,

$$\omega(\mathcal{N}_f(S)) \leq b\omega(S). \tag{3.6}$$

It follows from (3.6) combined with Lemma 3.2 that

$$\omega(\mathcal{A}S) \leq \frac{bT^\alpha}{\Gamma(\alpha)} \|h\|_{L^\infty} \omega(S). \tag{3.7}$$

Furthermore, if we use the same precedent way for the operator $\mathcal{N}_g H$, and taking into account $(\mathcal{H}2)$, we get

$$\begin{aligned} \int_J \|\mathcal{N}_g Hu(t)\| \, dt &\leq \int_J \|g(t, Hu(t))\| \, dt \\ &\leq \int_J \|g(t,0)\| \, dt + \lambda \int_J \|Hu(t)\| \, dt, \\ &\leq \|g(.,0)\|_{L^1} + \lambda \|H\|_{\mathcal{L}} \int_J \|u(t)\| \, dt. \end{aligned}$$

Then, we obtain

$$\omega(\mathcal{N}_g HS) \leq \lambda \|H\|_{\mathcal{L}} \omega(S).$$

Now, by Lemma 3.2, we get

$$\omega(\mathcal{B}S) \leq \frac{\lambda T^\alpha}{\Gamma(\alpha)} \|H\|_{\mathcal{L}} \omega(S). \tag{3.8}$$

Combining (3.7), (3.8) and Lemma 2.1, it yields

$$\omega(\mathcal{A}S + \mathcal{B}S) \leq \left(\frac{bT^\alpha}{\Gamma(\alpha)} \|h\|_{L^\infty} + \frac{\lambda T^\alpha}{\Gamma(\alpha)} \|H\|_{\mathcal{L}} \right) \omega(S). \tag{3.9}$$

Estimation (3.9) and hypothesis (3.4) prove that condition (ii) of Theorem 3.1 is satisfied.

Claim 3. We will prove that $\mathcal{B}$ is a contraction mapping. For this purpose, let $u, v \in L^1(I, E)$ and by using Lemma 3.2 and assumption $(\mathcal{H}2)$, it follows for all $t \in I$, that

$$\begin{aligned} \|\mathcal{B}u - \mathcal{B}v\|_{L^1} &= \|\mathcal{J}\mathcal{N}_g Hu - \mathcal{J}\mathcal{N}_g Hv\|_{L^1} \\ &\leq \|\mathcal{J}\|_{\mathcal{L}} \|\mathcal{N}_g Hu - \mathcal{N}_g Hv\|_{L^1} \\ &\leq \frac{\lambda T^\alpha}{\Gamma(\alpha)} \|H\|_{\mathcal{L}} \|u - v\|_{L^1}. \end{aligned}$$

Then, $\mathcal{B}$ is a contraction mapping on $L^1(I, E)$ with constant $\dfrac{\lambda T^\alpha}{\Gamma(\alpha)} \|H\|_{\mathcal{L}}$.

Claim 4. It remains to prove that $\mathcal{A}u+\mathcal{B}v \in B_r$. Hence, taking into account assumptions $(\mathcal{H}1)-(\mathcal{H}3)$, and Lemmas 2.2 and Lemma 3.2 we deduce that for all $u, v \in B_r$ we have

$$\begin{aligned}
\|\mathcal{A}u+\mathcal{B}v\|_{L^1} &= \|\mathcal{I}\mathcal{N}_f u+\mathcal{J}\mathcal{N}_g Hv\|_{L^1} \\
&\leq \|\mathcal{I}\|_{\mathcal{L}}\|\mathcal{N}_f u\|_{L^1}+\|\mathcal{J}\|_{\mathcal{L}}\|\mathcal{N}_g Hv\|_{L^1} \\
&\leq \frac{T^\alpha}{\Gamma(\alpha)}\|h\|_{L^\infty}\left(\|a\|_{L^1}+b\|u\|_{L^1}\right)+\frac{T^\alpha}{\Gamma(\alpha)}\left(\|g(.,0)\|_{L^1}+\lambda\|H\|_{\mathcal{L}}\|v\|_{L^1}\right) \\
&\leq \frac{T^\alpha}{\Gamma(\alpha)}\|h\|_{L^\infty}\left(\|a\|_{L^1}+br\right)+\frac{T^\alpha}{\Gamma(\alpha)}\left(\|g(.,0)\|_{L^1}+\lambda r\|H\|_{\mathcal{L}}\right) \\
&\leq \left(\frac{bT^\alpha}{\Gamma(\alpha)}\|h\|_{L^\infty}+\frac{\lambda T^\alpha}{\Gamma(\alpha)}\|H\|_{\mathcal{L}}\right)r+\left(\frac{T^\alpha}{\Gamma(\alpha)}\|h\|_{L^\infty}\|a\|_{L^1}+\frac{T^\alpha}{\Gamma(\alpha)}\|g(.,0)\|_{L^1}\right).
\end{aligned}$$

From (3.5), we get

$$\|\mathcal{A}u+\mathcal{B}v\|_{L^1} \leq r.$$

This achieves Claim 4.

Hence, by applying Theorem 3.1, we conclude that the operator $\mathcal{A}+\mathcal{B}$ has, at least, one fixed point in B_r, which is the solution of IVP (1.1).

□

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