

ON THE NOVEL SUPERPOLYCONVOLUTION OPERATOR FOR SIX INTEGRAL TRANSFORMS FOURIER COSINE, FOURIER SINE, LAPLACE AND APPLICATIONS

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ABSTRACT. The main target of this article is to investigate the novel superpolyconvolution related to the six integral transforms Fourier cosine, Fourier sine and Laplace. The operator generalizes the classical polyconvolution framework by incorporating a six-fold transform decomposition that extends the traditional four-transform structure. Some interesting algebraic properties of this operator are presented. The Watson-type theorem is given. We also establish the Young-type inequalities in mixed L^p spaces, together with a Titchmarsh-type uniqueness theorem ensuring the distinctness of the associated functions. Applications are also discussed, showing that the proposed operator allows explicit solutions to various integral equations of Toeplitz plus Hankel type and to Barbashin-type equations.

1. INTRODUCTION

The research of convolution, generalized convolution and their generalizations, including the notions of polyconvolution for integral transforms, has long played a central role in both theoretical and applied mathematics, mathematical physics problems, boundary value problems, technology and technical fields [3, 7, 9, 10, 12–14, 22, 24–27, 30–33, 36, 40, 42, 43].

These operators constitute a versatile analytical framework for a wide range of disciplines, including engineering, harmonic analysis, heat transfer, image processing, signal processing, data analysis, acoustics, electro-magnetic waves, and so on. The interesting topic about the polyconvolution operator related to the integral transforms was first advanced by V. A. Kakichev in 1997 [11].

For the time being, there are some polyconvolution operators [6, 14–21, 34, 35]. Most of the existing constructions exhibit a factorization property involving four integral transforms. Motivated by this framework, we investigate a superpolyconvolution operator associated with the six integral transforms Fourier cosine, Fourier sine and Laplace. This wide-open idea helps us to solve a lot of practical problems which have many complicated connections.

The proposed operator extends the conventional factorization structure to encompass six integral transforms, enriching the algebraic hierarchy of convolution-type operators. We give a concrete model as follows.

Let $U_i(X_i)$, $i = \overline{1, 6}$, be the space of linear functions that can differ on the same field and $W(Y)$ be an algebra. Examine the integral transforms $T_i : U_i(X_i) \rightarrow W(Y)$, $i = \overline{1, 6}$. Then the superpolyconvolution

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operator of functions $f_1 \in U_1(X_1)$, $f_2 \in U_2(X_2)$, $f_3 \in U_3(X_3)$, $f_4 \in U_4(X_4)$, $f_5 \in U_5(X_5)$ for the integral transforms T_i is a multi-linear operator determined by

$$* : \prod_{i=1}^5 U_i(X_i) \longrightarrow W(Y),$$

determined by the mapping $*(f_1, f_2, f_3, f_4, f_5)$ such that the following factorization identity holds

$$(1.1) \quad T_6 [*(f_1, f_2, f_3, f_4, f_5)](y) = (T_1 f_1)(y) \cdot (T_2 f_2)(y) \cdot (T_3 f_3)(y) \cdot (T_4 f_4)(y) \cdot (T_5 f_5)(y).$$

From (1.1), we receive

$$(1.2) \quad *(f_1, f_2, f_3, f_4, f_5)(x) = T_6^{-1} [(T_1 f_1)(y) \cdot (T_2 f_2)(y) \cdot (T_3 f_3)(y) \cdot (T_4 f_4)(y) \cdot (T_5 f_5)(y)](x),$$

here T_6^{-1} is the inverse integral transform of T_6 .

Within this analytical setting, the superpolyconvolution framework offers an effective means for analyzing and solving various functional equations. It yields closed-form representations for some classes of integral equations of Toeplitz plus Hankel type and of Barbashin-type equations with complex kernels. Beyond the theoretical development, these results provide mathematical tools relevant to problems arising in engineering and physical modeling where convolution-type operators naturally appear.

This superpolyconvolution operator manifests a vivid connection between the three well-known integral transforms Fourier cosine, Fourier sine and Laplace, and builds a bridge linking the applied areas of these integral transforms together.

On the other hand, at first, we enlarge and develop a lot of results related to the classical convolution, convolution with the weight function, generalized convolution, generalized convolution with the weight function, polyconvolution, polyconvolution with the weight function for the superpolyconvolution operator.

2. PRELIMINARIES

In this section, we recall some known convolutions and generalized convolutions. The convolution of two functions f and g in $L_1(\mathbb{R})$ for the Fourier integral transform is well known [28] as

$$(2.1) \quad (f *_F g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-y) \cdot g(y) \cdot dy, \quad x \in \mathbb{R},$$

for which the following factorization property holds

$$(2.2) \quad F(f *_F g)(y) = (Ff)(y) \cdot (Fg)(y), \quad y \in \mathbb{R},$$

where the Fourier integral transform is defined by

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cdot e^{-iyx} \cdot dx, \quad y \in \mathbb{R}.$$

The convolution of two functions f and g for the Laplace transform is of the form [28]

$$(2.3) \quad (f *_L g)(x) = \int_0^x f(x-y) \cdot g(y) \cdot dy, \quad x > 0,$$

which satisfies the factorization identity

$$(2.4) \quad L(f *_L g)(y) = (Lf)(y) \cdot (Lg)(y), \quad y > 0.$$

Here, L denotes the Laplace integral transform [28]

$$(Lf)(y) = \int_0^{+\infty} f(x) \cdot e^{-yx} \cdot dx, \quad y > 0.$$

The convolution of two functions f and g in $L_1(\mathbb{R}_+)$ for the Fourier cosine integral transform is defined by [37]

$$(2.5) \quad (f *_{F_c} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y) \cdot [g(x+y) + g(|x-y|)] \cdot dy, \quad x > 0,$$

with the factorization equality

$$(2.6) \quad F_c(f *_{F_c} g)(y) = (F_c f)(y) \cdot (F_c g)(y), \quad y > 0.$$

Here, the Fourier cosine integral transform is defined by [37]

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cdot \cos(xy) \cdot dx.$$

Here, F_s denotes the Fourier sine integral transform [28]

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cdot \sin(yx) \cdot dx.$$

3. THE SUPERPOLYCONVOLUTION OPERATOR FOR SIX INTEGRAL TRANSFORMS FOURIER COSINE, FOURIER SINE AND LAPLACE

Definition 3.1. The superpolyconvolution operator for six integral transforms Fourier cosine, Fourier sine and Laplace, of the functions f, g, h, k, l , is defined by

$$(3.1) \quad \begin{aligned} * (f, g, h, k, l)(x) = & \frac{\sqrt{2}}{4\pi^2\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \left[\right. \\ & \frac{t}{t^2 + (u+v+w-e+x)^2} + \frac{t}{t^2 + (u+v+w-e-x)^2} \\ & + \frac{t}{t^2 + (u+v-w+e+x)^2} + \frac{t}{t^2 + (u+v-w+e-x)^2} \\ & - \frac{t}{t^2 + (u+v+w+e+x)^2} - \frac{t}{t^2 + (u+v+w+e-x)^2} \\ & - \frac{t}{t^2 + (u+v-w+e+x)^2} - \frac{t}{t^2 + (u+v-w+e-x)^2} \\ & + \frac{t}{t^2 + (u-v+w-e+x)^2} + \frac{t}{t^2 + (u-v+w-e-x)^2} \\ & + \frac{t}{t^2 + (u-v-w+e+x)^2} + \frac{t}{t^2 + (u-v-w+e-x)^2} \\ & - \frac{t}{t^2 + (u-v+w+e+x)^2} - \frac{t}{t^2 + (u-v+w+e-x)^2} \\ & \left. - \frac{t}{t^2 + (u-v-w+e+x)^2} - \frac{t}{t^2 + (u-v-w+e-x)^2} \right] \\ & \cdot f(u) \cdot g(v) \cdot h(w) \cdot k(e) \cdot l(t) \cdot du \cdot dv \cdot dw \cdot de \cdot dt, \quad x > 0. \end{aligned}$$

Theorem 3.2. If the functions f, g, h, k, l are given in the space $L_1(\mathbb{R}_+)$, then the superpolyconvolution operator (3.1) belongs to $L_1(\mathbb{R}_+)$ and satisfies the factorization identity

$$(3.2) \quad F_c [* (f, g, h, k, l)](y) = (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y), \quad \forall y > 0.$$

Furthermore, when $f, g, h, k, l \in L_2(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$, the Parseval type identity holds

$$(3.3) \quad \begin{aligned} & * (f, g, h, k, l)(x) \\ & = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y) \cdot \cos xy \cdot dy. \end{aligned}$$

On the other hand, the superpolyconvolution operator (3.1) belongs to $C_0(\mathbb{R}_+)$, where $C_0(\mathbb{R}_+)$ is the space of bounded continuous functions vanishing at infinity.

Proof. We have the estimate

$$\begin{aligned}
 (3.4) \quad & \left| \int_0^{+\infty} \left[\frac{t}{t^2 + (u+v+w-e+x)^2} + \frac{t}{t^2 + (u+v+w-e-x)^2} \right. \right. \\
 & + \frac{t}{t^2 + (u+v-w+e+x)^2} + \frac{t}{t^2 + (u+v-w+e-x)^2} \\
 & - \frac{t}{t^2 + (u+v+w+e+x)^2} - \frac{t}{t^2 + (u+v+w+e-x)^2} \\
 & - \frac{t}{t^2 + (u+v-w-e+x)^2} - \frac{t}{t^2 + (u+v-w-e-x)^2} \\
 & + \frac{t}{t^2 + (u-v+w-e+x)^2} + \frac{t}{t^2 + (u-v+w-e-x)^2} \\
 & + \frac{t}{t^2 + (u-v-w+e+x)^2} + \frac{t}{t^2 + (u-v-w+e-x)^2} \\
 & - \frac{t}{t^2 + (u-v+w+e+x)^2} - \frac{t}{t^2 + (u-v+w+e-x)^2} \\
 & \left. \left. - \frac{t}{t^2 + (u-v-w-e+x)^2} - \frac{t}{t^2 + (u-v-w-e-x)^2} \right] \cdot dx \right| \\
 & \leq 16 \int_0^{+\infty} \frac{t}{t^2 + s^2} \cdot ds = 8\pi.
 \end{aligned}$$

From (3.1) and (3.4), we receive

$$\begin{aligned}
 |*(f, g, h, k, l)(x)| & \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} |f(u)| \cdot du \cdot \int_0^{+\infty} |g(v)| \cdot dv \cdot \int_0^{+\infty} |h(w)| \cdot dw \\
 & \quad \cdot \int_0^{+\infty} |k(e)| \cdot de \cdot \int_0^{+\infty} |l(t)| \cdot dt \\
 & = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|f\|_{L_1(\mathbb{R}_+)} \cdot \|g\|_{L_1(\mathbb{R}_+)} \cdot \|h\|_{L_1(\mathbb{R}_+)} \cdot \|k\|_{L_1(\mathbb{R}_+)} \cdot \|l\|_{L_1(\mathbb{R}_+)}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \|*(f, g, h, k, l)\|_{L_1(\mathbb{R}_+)} & \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|f\|_{L_1(\mathbb{R}_+)} \cdot \|g\|_{L_1(\mathbb{R}_+)} \cdot \|h\|_{L_1(\mathbb{R}_+)} \cdot \|k\|_{L_1(\mathbb{R}_+)} \cdot \|l\|_{L_1(\mathbb{R}_+)} \\
 & < +\infty.
 \end{aligned}$$

So $*(f, g, h, k, l)(x) \in L_1(\mathbb{R}_+)$.

Now, using (3.1) and the formula in [8]

$$\int_0^{+\infty} e^{-sx} \cdot \cos xy \cdot dx = \frac{s}{s^2 + y^2}, \quad s > 0,$$

we get

$$\begin{aligned}
 *(f, g, h, k, l)(x) & = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \left[\sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(u) \cdot \cos yu \cdot du \cdot \sqrt{\frac{2}{\pi}} \int_0^{+\infty} g(v) \cdot \cos yv \cdot dv \right. \\
 & \quad \cdot \sqrt{\frac{2}{\pi}} \int_0^{+\infty} h(w) \cdot \sin yw \cdot dw \cdot \sqrt{\frac{2}{\pi}} \int_0^{+\infty} k(e) \cdot \sin ye \cdot de \\
 & \quad \left. \cdot \int_0^{+\infty} l(t) \cdot e^{-yt} \cdot dt \right] \cdot \cos xy \cdot dy
 \end{aligned}$$

$$\begin{aligned}
&= \frac{4\sqrt{2}}{\pi^2\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} f(u).g(v).h(w).k(e).l(t) \\
&\quad .\cos yu.\cos yv.\sin yw.\sin ye.\cos xy.e^{-yt} \\
&\quad .du.dv.dw.de.dt.dy \\
&= \frac{\sqrt{2}}{4\pi^2\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \left[\right. \\
&\quad \cos y(u+v+w-e+x) + \cos y(u+v+w-e-x) \\
&\quad + \cos y(u+v-w+e+x) + \cos y(u+v-w+e-x) \\
&\quad - \cos y(u+v+w+e+x) - \cos y(u+v+w+e-x) \\
&\quad - \cos y(u+v-w-e+x) - \cos y(u+v-w-e-x) \\
&\quad + \cos y(u-v+w-e+x) + \cos y(u-v+w-e-x) \\
&\quad + \cos y(u-v-w+e+x) + \cos y(u-v-w+e-x) \\
&\quad - \cos y(u-v+w+e+x) - \cos y(u-v+w+e-x) \\
&\quad \left. - \cos y(u-v-w-e+x) - \cos y(u-v-w-e-x) \right] \\
&\quad .e^{-yt}.f(u).g(v).h(w).k(e).l(t).du.dv.dw.de.dt.dy \\
&= \frac{\sqrt{2}}{4\pi^2\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \left[\right. \\
&\quad \frac{t}{t^2+(u+v+w-e+x)^2} + \frac{t}{t^2+(u+v+w-e-x)^2} \\
&\quad + \frac{t}{t^2+(u+v-w+e+x)^2} + \frac{t}{t^2+(u+v-w+e-x)^2} \\
&\quad - \frac{t}{t^2+(u+v+w+e+x)^2} - \frac{t}{t^2+(u+v+w+e-x)^2} \\
&\quad - \frac{t}{t^2+(u+v-w-e+x)^2} - \frac{t}{t^2+(u+v-w-e-x)^2} \\
&\quad + \frac{t}{t^2+(u-v+w-e+x)^2} + \frac{t}{t^2+(u-v+w-e-x)^2} \\
&\quad + \frac{t}{t^2+(u-v-w+e+x)^2} + \frac{t}{t^2+(u-v-w+e-x)^2} \\
&\quad - \frac{t}{t^2+(u-v+w+e+x)^2} - \frac{t}{t^2+(u-v+w+e-x)^2} \\
&\quad \left. - \frac{t}{t^2+(u-v-w-e+x)^2} - \frac{t}{t^2+(u-v-w-e-x)^2} \right] \\
&\quad .f(u).g(v).h(w).k(e).l(t).du.dv.dw.de.dt.
\end{aligned}$$

Thus, we have (3.3)

$$\begin{aligned}
&*(f, g, h, k, l)(x) \\
&= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (F_c f)(y).(F_c g)(y).(F_s h)(y).(F_s k)(y).(Ll)(y).\cos xy.dy.
\end{aligned}$$

The Riemann–Lebesgue Theorem in [29] states that: If $f \in L_1(\mathbb{R}^n)$, then $(Ff)(y) \rightarrow 0$ as $|y| \rightarrow \infty$, and, hence $(Ff)(y) \in C_0(\mathbb{R}^n)$. This theorem is also true for the F_c integral transform [37]. Thus, $*(f, g, h, k, l)$ belonging to $C_0(\mathbb{R}_+)$ is obvious by virtue of the Riemann–Lebesgue theorem. $\square$

Proposition 3.3. *Let $f, g, h, k, l, m \in L_1(\mathbb{R}_+)$. Then the superpolyconvolution operator (3.1) satisfies the following equalities*

- 1) $*(f, g, h, k, l) = *(g, f, h, k, l);$
- 2) $*(f, g, h, k, l) = *(f, g, k, h, l);$
- 3) $*\left[\left(f *_{{F_c}} g\right), h, k, l, m\right] = *\left[\left(g *_{{F_c}} h\right), f, k, l, m\right];$

$$4) * \left[f, g, h, k, (l *_L m) \right] = * \left[f, g, h, k, (m *_L l) \right].$$

Proof. 1) First of all, from the factorization equality (3.2), we have

$$\begin{aligned} F_c [* (f, g, h, k, l)] (y) &= (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y) \\ &= (F_c g)(y) \cdot (F_c f)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y) \\ &= F_c [* (g, f, h, k, l)] (y), \quad \forall y > 0. \end{aligned}$$

So $*(f, g, h, k, l) = *(g, f, h, k, l)$.

2) Similarly, we can prove 2).

3) Using the factorization identities (2.6) and (3.2), we have

$$\begin{aligned} F_c \left\{ * \left[(f *_F g), h, k, l, m \right] \right\} (y) &= F_c (f *_F g)(y) \cdot (F_c h)(y) \cdot (F_s k)(y) \cdot (F_s l)(y) \cdot (Lm)(y) \\ &= (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (F_s l)(y) \cdot (Lm)(y) \\ &= F_c (g *_F h)(y) \cdot (F_c f)(y) \cdot (F_s k)(y) \cdot (F_s l)(y) \cdot (Lm)(y) \\ &= F_c \left\{ * \left[(g *_F h), f, k, l, m \right] \right\} (y), \quad \forall y > 0. \end{aligned}$$

Thus, we obtain

$$* \left[(f *_F g), h, k, l, m \right] = * \left[(g *_F h), f, k, l, m \right].$$

4) From the factorization identities (2.4) and (3.2), we have

$$\begin{aligned} F_c \left\{ * \left[f, g, h, k, (l *_L m) \right] \right\} (y) &= (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot L(l *_L m)(y) \\ &= (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y) \cdot (Lm)(y) \\ &= (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Lm)(y) \cdot (Ll)(y) \\ &= (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot L(m *_L l)(y) \\ &= F_c \left\{ * \left[f, g, h, k, (m *_L l) \right] \right\} (y), \quad \forall y > 0. \end{aligned}$$

$$\text{Thus } * \left[f, g, h, k, (l *_L m) \right] = * \left[f, g, h, k, (m *_L l) \right]. \quad \square$$

Theorem 3.4 (Titchmarsh type theorem). *Let $f, g, h, k \in L_1(\mathbb{R}_+, e^{\alpha x})$, $\alpha > 0$, $l \in L_1(\mathbb{R}_+)$. If $*(f, g, h, k, l)(x) = 0$, $\forall x > 0$, then either $f(x) = 0$, $\forall x > 0$ or $g(x) = 0$, $\forall x > 0$ or $h(x) = 0$, $\forall x > 0$ or $k(x) = 0$, $\forall x > 0$ or $l(x) = 0$, $\forall x > 0$.*

Proof. Using the factorization (3.2) we have

$$F_c [* (f, g, h, k, l)] (y) = (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y), \quad \forall y > 0.$$

On the other hand, we have

$$\begin{aligned} \left| \frac{d^n}{dy^n} [\cos yx \cdot f(x)] \right| &= \left| f(x) \cdot x^n \cdot \cos \left(yx + \frac{n\pi}{2} \right) \right| \\ &\leq |e^{-\alpha x} \cdot x^n| \cdot |e^{\alpha x} \cdot f(x)| \leq \frac{n!}{\alpha^n} |e^{\alpha x} \cdot f(x)|. \end{aligned}$$

Here, we used the estimation

$$0 \leq e^{-\alpha x} \cdot x^n = e^{-\alpha x} \cdot \frac{(\alpha x)^n}{n!} \cdot \frac{n!}{\alpha^n} \leq e^{-\alpha x} \cdot e^{\alpha x} \cdot \frac{n!}{\alpha^n} = \frac{n!}{\alpha^n}.$$

From $f \in L_1(\mathbb{R}_+, e^{\alpha x})$, we obtain $\frac{d^n}{dy^n} [\cos yx \cdot f(x)] \in L_1(\mathbb{R}_+)$. Since $L_1(\mathbb{R}_+, e^{\alpha x}) \subset L_1(\mathbb{R}_+)$, $(F_c f)(y)$ is analytic in $\mathbb{R}_+$. Similarly, we get that $(F_c g)(y)$, $(F_s h)(y)$, $(F_s k)(y)$ are analytic in $\mathbb{R}_+$. On the other hand, $(Ll)(y)$ is analytic in $\mathbb{R}_+$.

By using the above factorization property, this implies that either $f(x) = 0, \forall x > 0$ or $g(x) = 0, \forall x > 0$ or $h(x) = 0, \forall x > 0$ or $k(x) = 0, \forall x > 0$, or $l(x) = 0, \forall x > 0$. The proof is complete. $\square$

4. INEQUALITIES FOR THE SUPERPOLYCONVOLUTION OPERATOR

In this section, we present the norm inequalities for the superpolyconvolution operator (3.1) in $L_1(\mathbb{R}_+)$ and $L_p(\mathbb{R}_+, \rho)$ with $1 \leq p \leq \infty$ and ρ being a positive weight function. The standard norms are determined as follows

$$\begin{aligned} \|f\|_{L_1(\mathbb{R}_+)} &= \int_0^{+\infty} |f(x)| \cdot dx; & \|f\|_{L_1(\mathbb{R}_+, \rho)} &= \int_0^{+\infty} |f(x)| \cdot \rho(x) \cdot dx; \\ \|f\|_{L_p(\mathbb{R}_+)} &= \left[\int_0^{+\infty} |f(x)|^p \cdot dx \right]^{\frac{1}{p}}; & \|f\|_{L_p(\mathbb{R}_+, \rho)} &= \left[\int_0^{+\infty} |f(x)|^p \cdot \rho(x) \cdot dx \right]^{\frac{1}{p}}. \end{aligned}$$

4.1. The norm inequality for the superpolyconvolution operator in $L_1(\mathbb{R}_+)$.

Theorem 4.1. *If f, g, h, k, l belong to $L_1(\mathbb{R}_+)$, then the following inequality holds*

$$(4.1) \quad \begin{aligned} \|*(f, g, h, k, l)\|_{L_1(\mathbb{R}_+)} &\leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|f\|_{L_1(\mathbb{R}_+)} \cdot \|g\|_{L_1(\mathbb{R}_+)} \cdot \|h\|_{L_1(\mathbb{R}_+)} \\ &\quad \cdot \|k\|_{L_1(\mathbb{R}_+)} \cdot \|l\|_{L_1(\mathbb{R}_+)}. \end{aligned}$$

Proof. From Definition 3.1 and the proof of Theorem 3.2, we receive

$$\begin{aligned} \|*(f, g, h, k, l)\|_{L_1(\mathbb{R}_+)} &= \int_0^{+\infty} |*(f, g, h, k, l)(x)| \cdot dx \\ &\leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|f\|_{L_1(\mathbb{R}_+)} \cdot \|g\|_{L_1(\mathbb{R}_+)} \cdot \|h\|_{L_1(\mathbb{R}_+)} \\ &\quad \cdot \|k\|_{L_1(\mathbb{R}_+)} \cdot \|l\|_{L_1(\mathbb{R}_+)}. \end{aligned}$$

Thus, we get (4.1). $\square$

4.2. The norm inequalities for the superpolyconvolution operator in $L_p(\mathbb{R}_+)$. Now, we research the superpolyconvolution operator on the function space $L_p(\mathbb{R}_+)$ and estimate its norm.

From the convolution for the Fourier integral transform (2.1), in [44] Young established the inequality

$$\left\| (f *_F g)(x) \right\|_{L_r(\mathbb{R})} \leq \|f\|_{L_p(\mathbb{R})} \cdot \|g\|_{L_q(\mathbb{R})},$$

here $p, q, r > 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$. In this section, we first establish a Young-type inequality for this superpolyconvolution operator.

Theorem 4.2 (Young-type inequality). *Suppose $p, q, r, m, n > 1$ and satisfy the condition*

$$(4.2) \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{m} + \frac{1}{n} = 4.$$

Then, the superpolyconvolution operator (3.1) is a bounded function with $x \in \mathbb{R}_+$, and the following inequality holds

$$(4.3) \quad \begin{aligned} \|*(f, g, h, k, l)\|_{L_1(\mathbb{R}_+)} &\leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|f\|_{L_p(\mathbb{R}_+)} \cdot \|g\|_{L_q(\mathbb{R}_+)} \cdot \|h\|_{L_r(\mathbb{R}_+)} \\ &\quad \cdot \|k\|_{L_m(\mathbb{R}_+)} \cdot \|l\|_{L_n(\mathbb{R}_+)}. \end{aligned}$$

Proof. Let p_1, q_1, r_1, m_1, n_1 be the conjugate exponents of p, q, r, m, n respectively, then we have

$$\frac{1}{p} + \frac{1}{p_1} = \frac{1}{q} + \frac{1}{q_1} = \frac{1}{r} + \frac{1}{r_1} = \frac{1}{m} + \frac{1}{m_1} = \frac{1}{n} + \frac{1}{n_1} = 1.$$

From the condition (4.2), we receive

$$(4.4) \quad \left\{ \begin{aligned} &\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} + \frac{1}{n_1} = 1, \\ &p \left(\frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} + \frac{1}{n_1} \right) = q \left(\frac{1}{p_1} + \frac{1}{r_1} + \frac{1}{m_1} + \frac{1}{n_1} \right) \\ &\quad = r \left(\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{m_1} + \frac{1}{n_1} \right) = m \left(\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{n_1} \right) \\ &\quad = n \left(\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} \right) = 1. \end{aligned} \right.$$

As proved in Theorem 3.2, we have

$$\begin{aligned} &\left| \int_0^{+\infty} \left[\frac{t}{t^2 + (u+v+w-e+x)^2} + \frac{t}{t^2 + (u+v+w-e-x)^2} \right. \right. \\ &\quad + \frac{t}{t^2 + (u+v-w+e+x)^2} + \frac{t}{t^2 + (u+v-w+e-x)^2} \\ &\quad - \frac{t}{t^2 + (u+v+w+e+x)^2} - \frac{t}{t^2 + (u+v+w+e-x)^2} \\ &\quad - \frac{t}{t^2 + (u+v-w+e+x)^2} - \frac{t}{t^2 + (u+v-w+e-x)^2} \\ &\quad + \frac{t}{t^2 + (u-v+w-e+x)^2} + \frac{t}{t^2 + (u-v+w-e-x)^2} \\ &\quad + \frac{t}{t^2 + (u-v-w+e+x)^2} + \frac{t}{t^2 + (u-v-w+e-x)^2} \\ &\quad - \frac{t}{t^2 + (u-v+w+e+x)^2} - \frac{t}{t^2 + (u-v+w+e-x)^2} \\ &\quad \left. \left. - \frac{t}{t^2 + (u-v-w+e+x)^2} - \frac{t}{t^2 + (u-v-w+e-x)^2} \right] \cdot dx \right| \leq 8\pi. \end{aligned}$$

Therefore

$$\begin{aligned} \left| \int_0^{+\infty} [*(f, g, h, k, l)(x)] \cdot dx \right| &\leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \\ &\quad |f(u)| \cdot |g(v)| \cdot |h(w)| \cdot |k(e)| \cdot |l(t)| \cdot du \cdot dv \cdot dw \cdot de \cdot dt. \end{aligned}$$

Put

$$\begin{aligned} M(u, v, w, e, t) &= |g(v)|^{\frac{q}{p_1}} \cdot |h(w)|^{\frac{r}{p_1}} \cdot |k(e)|^{\frac{m}{p_1}} \cdot |l(t)|^{\frac{n}{p_1}} \in L_{p_1}(\mathbb{R}_+^5), \\ N(u, v, w, e, t) &= |h(w)|^{\frac{r}{q_1}} \cdot |k(e)|^{\frac{m}{q_1}} \cdot |l(t)|^{\frac{n}{q_1}} \cdot |f(u)|^{\frac{p}{q_1}} \in L_{q_1}(\mathbb{R}_+^5), \\ P(u, v, w, e, t) &= |k(e)|^{\frac{m}{r_1}} \cdot |l(t)|^{\frac{n}{r_1}} \cdot |f(u)|^{\frac{p}{r_1}} \cdot |g(v)|^{\frac{q}{r_1}} \in L_{r_1}(\mathbb{R}_+^5), \\ Q(u, v, w, e, t) &= |l(t)|^{\frac{n}{m_1}} \cdot |f(u)|^{\frac{p}{m_1}} \cdot |g(v)|^{\frac{q}{m_1}} \cdot |h(w)|^{\frac{r}{m_1}} \in L_{m_1}(\mathbb{R}_+^5), \\ R(u, v, w, e, t) &= |f(u)|^{\frac{p}{n_1}} \cdot |g(v)|^{\frac{q}{n_1}} \cdot |h(w)|^{\frac{r}{n_1}} \cdot |k(e)|^{\frac{m}{n_1}} \in L_{n_1}(\mathbb{R}_+^5). \end{aligned}$$

Using (4.4), we receive

$$\begin{aligned} &M(u, v, w, e, t) \cdot N(u, v, w, e, t) \cdot P(u, v, w, e, t) \cdot Q(u, v, w, e, t) \cdot R(u, v, w, e, t) \\ &= |f(u)| \cdot |g(v)| \cdot |h(w)| \cdot |k(e)| \cdot |l(t)|. \end{aligned}$$

Then we get

$$\begin{aligned} (4.5) \quad &\left| \int_0^{+\infty} [*(f, g, h, k, l)(x)] \cdot dx \right| \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \\ &M(u, v, w, e, t) \cdot N(u, v, w, e, t) \cdot P(u, v, w, e, t) \cdot Q(u, v, w, e, t) \cdot R(u, v, w, e, t) \\ &\cdot du \cdot dv \cdot dw \cdot de \cdot dt. \end{aligned}$$

Applying Hölder's inequality for (4.5), we obtain

$$\begin{aligned} (4.6) \quad &\left| \int_0^{+\infty} [*(f, g, h, k, l)(x)] \cdot dx \right| \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|M\|_{L_{p_1}(\mathbb{R}_+^5)} \cdot \|N\|_{L_{q_1}(\mathbb{R}_+^5)} \\ &\cdot \|P\|_{L_{r_1}(\mathbb{R}_+^5)} \cdot \|Q\|_{L_{m_1}(\mathbb{R}_+^5)} \cdot \|R\|_{L_{n_1}(\mathbb{R}_+^5)}. \end{aligned}$$

Using the definition of the norm on the space $L_{p_1}(\mathbb{R}_+^5)$ along with the help of the Fubini Theorem, we have

$$\begin{aligned} \|M\|_{L_{p_1}(\mathbb{R}_+^5)}^{p_1} &= \int_0^{+\infty} |g(v)|^q \cdot \left(\int_0^{+\infty} |h(w)|^r \cdot dw \cdot \int_0^{+\infty} |k(e)|^m \cdot de \right. \\ &\quad \cdot \left. \int_0^{+\infty} |l(t)|^n \cdot dt \right) \cdot dv \\ &\leq \|g\|_{L_q(\mathbb{R}_+)}^q \cdot \|h\|_{L_r(\mathbb{R}_+)}^r \cdot \|k\|_{L_m(\mathbb{R}_+)}^m \cdot \|l\|_{L_n(\mathbb{R}_+)}^n. \end{aligned}$$

Hence

$$(4.7) \quad \|M\|_{L_{p_1}(\mathbb{R}_+^5)} \leq \|g\|_{L_q(\mathbb{R}_+)}^{\frac{q}{p_1}} \cdot \|h\|_{L_r(\mathbb{R}_+)}^{\frac{r}{p_1}} \cdot \|k\|_{L_m(\mathbb{R}_+)}^{\frac{m}{p_1}} \cdot \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{p_1}}.$$

Similarly

$$\begin{aligned} (4.8) \quad &\|N\|_{L_{q_1}(\mathbb{R}_+^5)} \leq \|h\|_{L_r(\mathbb{R}_+)}^{\frac{r}{q_1}} \cdot \|k\|_{L_m(\mathbb{R}_+)}^{\frac{m}{q_1}} \cdot \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{q_1}} \cdot \|f\|_{L_p(\mathbb{R}_+)}^{\frac{p}{q_1}}, \\ &\|P\|_{L_{r_1}(\mathbb{R}_+^5)} \leq \|k\|_{L_m(\mathbb{R}_+)}^{\frac{m}{r_1}} \cdot \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{r_1}} \cdot \|f\|_{L_p(\mathbb{R}_+)}^{\frac{p}{r_1}} \cdot \|g\|_{L_q(\mathbb{R}_+)}^{\frac{q}{r_1}}, \\ &\|Q\|_{L_{m_1}(\mathbb{R}_+^5)} \leq \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{m_1}} \cdot \|f\|_{L_p(\mathbb{R}_+)}^{\frac{p}{m_1}} \cdot \|g\|_{L_q(\mathbb{R}_+)}^{\frac{q}{m_1}} \cdot \|h\|_{L_r(\mathbb{R}_+)}^{\frac{r}{m_1}}, \\ &\|R\|_{L_{n_1}(\mathbb{R}_+^5)} \leq \|f\|_{L_p(\mathbb{R}_+)}^{\frac{p}{n_1}} \cdot \|g\|_{L_q(\mathbb{R}_+)}^{\frac{q}{n_1}} \cdot \|h\|_{L_r(\mathbb{R}_+)}^{\frac{r}{n_1}} \cdot \|k\|_{L_m(\mathbb{R}_+)}^{\frac{m}{n_1}}. \end{aligned}$$

Using (4.7), (4.8) and applying (4.4), we obtain

$$\begin{aligned} \|*(f, g, h, k, l)(x)\|_{L_1(\mathbb{R}_+)} &\leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|f\|_{L_p(\mathbb{R}_+)} \cdot \|g\|_{L_q(\mathbb{R}_+)} \cdot \|h\|_{L_r(\mathbb{R}_+)} \\ &\quad \cdot \|k\|_{L_m(\mathbb{R}_+)} \cdot \|l\|_{L_n(\mathbb{R}_+)}. \end{aligned}$$

We have (4.3). The proof is complete. $\square$

4.3. Young-type theorem for the superpolyconvolution operator. After Young's inequality, Adams and Fournier developed Young's inequality for the convolution related to the Fourier integral transform as follows [1]

$$\left| \int_{\mathbb{R}^n} (f *_F g)(x) \cdot h(x) \cdot dx \right| \leq \|f\|_{L_p(\mathbb{R}^n)} \cdot \|g\|_{L_q(\mathbb{R}^n)} \cdot \|h\|_{L_r(\mathbb{R}^n)},$$

where $p, q, r > 1$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, $f \in L_p(\mathbb{R}^n)$, $g \in L_q(\mathbb{R}^n)$, and $h \in L_r(\mathbb{R}^n)$. Now, we continue to develop this inequality for the superpolyconvolution operator (3.1).

Theorem 4.3 (Young-type theorem). *Suppose $p, q, r, m, n, s \in (1, +\infty)$, satisfying the condition*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{m} + \frac{1}{n} + \frac{1}{s} = 5.$$

If the functions $f \in L_p(\mathbb{R}_+)$, $g \in L_q(\mathbb{R}_+)$, $h \in L_r(\mathbb{R}_+)$, $k \in L_m(\mathbb{R}_+)$, $l \in L_n(\mathbb{R}_+)$, $\theta \in L_s(\mathbb{R}_+)$, then we receive the following estimate

$$(4.9) \quad \left| \int_0^{+\infty} [*(f, g, h, k, l)(x) \cdot \theta(x)] \cdot dx \right| \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|f\|_{L_p(\mathbb{R}_+)} \cdot \|g\|_{L_q(\mathbb{R}_+)} \cdot \|h\|_{L_r(\mathbb{R}_+)} \\ \cdot \|k\|_{L_m(\mathbb{R}_+)} \cdot \|l\|_{L_n(\mathbb{R}_+)} \cdot \|\theta\|_{L_s(\mathbb{R}_+)}.$$

Proof. From (3.1) and the proof of Theorem 3.2, we have the estimate

$$(4.10) \quad \left| \int_0^{+\infty} [*(f, g, h, k, l)(x) \cdot \theta(x)] \cdot dx \right| \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \\ |f(u)| \cdot |g(v)| \cdot |h(w)| \cdot |k(e)| \cdot |l(t)| \cdot |\theta(x)| \cdot du \cdot dv \cdot dw \cdot de \cdot dt \cdot dx.$$

Let $p_1, q_1, r_1, m_1, n_1, s_1$ be the conjugate exponents of p, q, r, m, n, s

$$\frac{1}{p} + \frac{1}{p_1} = \frac{1}{q} + \frac{1}{q_1} = \frac{1}{r} + \frac{1}{r_1} = \frac{1}{m} + \frac{1}{m_1} = \frac{1}{n} + \frac{1}{n_1} = \frac{1}{s} + \frac{1}{s_1} = 1.$$

From the hypothesis $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{m} + \frac{1}{n} + \frac{1}{s} = 5$, we receive

$$(4.11) \quad \begin{cases} \frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} + \frac{1}{n_1} + \frac{1}{s_1} = 1, \\ p \left(\frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} + \frac{1}{n_1} + \frac{1}{s_1} \right) = q \left(\frac{1}{p_1} + \frac{1}{r_1} + \frac{1}{m_1} + \frac{1}{n_1} + \frac{1}{s_1} \right) \\ = r \left(\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{m_1} + \frac{1}{n_1} + \frac{1}{s_1} \right) = m \left(\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{n_1} + \frac{1}{s_1} \right) \\ = n \left(\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} + \frac{1}{s_1} \right) = s \left(\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} + \frac{1}{n_1} \right) = 1. \end{cases}$$

Putting

$$\begin{aligned} H_1(x, u, v, w, e, t) &= |g(v)|^{\frac{q}{p_1}} \cdot |h(w)|^{\frac{r}{p_1}} \cdot |k(e)|^{\frac{m}{p_1}} \cdot |l(t)|^{\frac{n}{p_1}} \cdot |\theta(x)|^{\frac{s}{p_1}} \in L_{p_1}(\mathbb{R}_+^6), \\ H_2(x, u, v, w, e, t) &= |h(w)|^{\frac{r}{q_1}} \cdot |k(e)|^{\frac{m}{q_1}} \cdot |l(t)|^{\frac{n}{q_1}} \cdot |\theta(x)|^{\frac{s}{q_1}} \cdot |f(u)|^{\frac{p}{q_1}} \in L_{q_1}(\mathbb{R}_+^6), \\ H_3(x, u, v, w, e, t) &= |k(e)|^{\frac{m}{r_1}} \cdot |l(t)|^{\frac{n}{r_1}} \cdot |\theta(x)|^{\frac{s}{r_1}} \cdot |f(u)|^{\frac{p}{r_1}} \cdot |g(v)|^{\frac{q}{r_1}} \in L_{r_1}(\mathbb{R}_+^6), \\ H_4(x, u, v, w, e, t) &= |l(t)|^{\frac{n}{m_1}} \cdot |\theta(x)|^{\frac{s}{m_1}} \cdot |f(u)|^{\frac{p}{m_1}} \cdot |g(v)|^{\frac{q}{m_1}} \cdot |h(w)|^{\frac{r}{m_1}} \in L_{m_1}(\mathbb{R}_+^6), \\ H_5(x, u, v, w, e, t) &= |\theta(x)|^{\frac{s}{n_1}} \cdot |f(u)|^{\frac{p}{n_1}} \cdot |g(v)|^{\frac{q}{n_1}} \cdot |h(w)|^{\frac{r}{n_1}} \cdot |k(e)|^{\frac{m}{n_1}} \in L_{n_1}(\mathbb{R}_+^6), \\ H_6(x, u, v, w, e, t) &= |f(u)|^{\frac{p}{s_1}} \cdot |g(v)|^{\frac{q}{s_1}} \cdot |h(w)|^{\frac{r}{s_1}} \cdot |k(e)|^{\frac{m}{s_1}} \cdot |l(t)|^{\frac{n}{s_1}} \in L_{s_1}(\mathbb{R}_+^6). \end{aligned}$$

It follows that

$$\begin{aligned} & \left| \int_0^{+\infty} [*(f, g, h, k, l)(x) \cdot \theta(x)] \cdot dx \right| \\ & \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} H_1(x, u, v, w, e, t) \\ & \quad \cdot H_2(x, u, v, w, e, t) \cdot H_3(x, u, v, w, e, t) \cdot H_4(x, u, v, w, e, t) \\ & \quad \cdot H_5(x, u, v, w, e, t) \cdot H_6(x, u, v, w, e, t) \cdot dx \cdot du \cdot dv \cdot dw \cdot de \cdot dt. \end{aligned}$$

From $\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} + \frac{1}{n_1} + \frac{1}{s_1} = 1$, and using Hölder inequality, we have the following estimate

$$\begin{aligned} & \left| \int_0^{+\infty} [*(f, g, h, k, l)(x) \cdot \theta(x)] \cdot dx \right| \\ & \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \left(\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |H_1(x, u, v, w, e, t)|^{p_1} \right. \\ & \quad \cdot dx \cdot du \cdot dv \cdot dw \cdot de \cdot dt \Big)^{\frac{1}{p_1}} \\ & \quad \cdot \left(\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |H_2(x, u, v, w, e, t)|^{q_1} \right. \\ & \quad \cdot dx \cdot du \cdot dv \cdot dw \cdot de \cdot dt \Big)^{\frac{1}{q_1}} \\ & \quad \cdot \left(\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |H_3(x, u, v, w, e, t)|^{r_1} \right. \\ & \quad \cdot dx \cdot du \cdot dv \cdot dw \cdot de \cdot dt \Big)^{\frac{1}{r_1}} \\ & \quad \cdot \left(\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |H_4(x, u, v, w, e, t)|^{m_1} \right. \\ & \quad \cdot dx \cdot du \cdot dv \cdot dw \cdot de \cdot dt \Big)^{\frac{1}{m_1}} \\ & \quad \cdot \left(\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |H_5(x, u, v, w, e, t)|^{n_1} \right. \\ & \quad \cdot dx \cdot du \cdot dv \cdot dw \cdot de \cdot dt \Big)^{\frac{1}{n_1}} \\ & \quad \cdot \left(\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |H_6(x, u, v, w, e, t)|^{s_1} \right. \\ & \quad \cdot dx \cdot du \cdot dv \cdot dw \cdot de \cdot dt \Big)^{\frac{1}{s_1}} \\ & = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|H_1\|_{L_{p_1}(\mathbb{R}_+^6)} \cdot \|H_2\|_{L_{q_1}(\mathbb{R}_+^6)} \cdot \|H_3\|_{L_{r_1}(\mathbb{R}_+^6)} \\ & \quad \cdot \|H_4\|_{L_{m_1}(\mathbb{R}_+^6)} \cdot \|H_5\|_{L_{n_1}(\mathbb{R}_+^6)} \cdot \|H_6\|_{L_{s_1}(\mathbb{R}_+^6)}. \end{aligned}$$

Furthermore, for $g \in L_q(\mathbb{R}_+)$, $h \in L_r(\mathbb{R}_+)$, $k \in L_m(\mathbb{R}_+)$, $l \in L_n(\mathbb{R}_+)$, $\theta \in L_s(\mathbb{R}_+)$, along with the help of the Fubini Theorem, we have

$$\begin{aligned} \|H_1\|_{L_{p_1}(\mathbb{R}_+^6)}^{p_1} &= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |g(v)|^q \cdot |h(w)|^r \cdot |k(e)|^m \cdot |l(t)|^n \cdot |\theta(x)|^s \\ & \quad \cdot dx \cdot du \cdot dv \cdot dw \cdot de \cdot dt \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{+\infty} |g(v)|^q \cdot dv \cdot \int_0^{+\infty} |h(w)|^r \cdot dw \cdot \int_0^{+\infty} |k(e)|^m \cdot de \\
&\quad \cdot \int_0^{+\infty} |l(t)|^n \cdot dt \cdot \int_0^{+\infty} |\theta(x)|^s \cdot dx \\
&= \|g\|_{L_q(\mathbb{R}_+)}^q \cdot \|h\|_{L_r(\mathbb{R}_+)}^r \cdot \|k\|_{L_m(\mathbb{R}_+)}^m \cdot \|l\|_{L_n(\mathbb{R}_+)}^n \cdot \|\theta\|_{L_s(\mathbb{R}_+)}^s.
\end{aligned}$$

So, we have the norm estimate of $H_1(x, u, v, w, e, t)$ in $L_{p_1}(\mathbb{R}_+^6)$ as follows

$$\begin{aligned}
(4.12) \quad \|H_1\|_{L_{p_1}(\mathbb{R}_+^6)} &\leq \|g\|_{L_q(\mathbb{R}_+)}^{\frac{q}{p_1}} \cdot \|h\|_{L_r(\mathbb{R}_+)}^{\frac{r}{p_1}} \cdot \|k\|_{L_m(\mathbb{R}_+)}^{\frac{m}{p_1}} \cdot \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{p_1}} \cdot \|\theta\|_{L_s(\mathbb{R}_+)}^{\frac{s}{p_1}} \\
&\quad \cdot \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{p_1}} \cdot \|\theta\|_{L_s(\mathbb{R}_+)}^{\frac{s}{p_1}}.
\end{aligned}$$

Similarly, we have the norm estimates for H_2, H_3, H_4, H_5, H_6 as follows

$$\begin{aligned}
(4.13) \quad \|H_2\|_{L_{q_1}(\mathbb{R}_+^6)} &\leq \|h\|_{L_r(\mathbb{R}_+)}^{\frac{r}{q_1}} \cdot \|k\|_{L_m(\mathbb{R}_+)}^{\frac{m}{q_1}} \cdot \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{q_1}} \\
&\quad \cdot \|\theta\|_{L_s(\mathbb{R}_+)}^{\frac{s}{q_1}} \cdot \|f\|_{L_p(\mathbb{R}_+)}^{\frac{p}{q_1}},
\end{aligned}$$

$$\begin{aligned}
(4.14) \quad \|H_3\|_{L_{r_1}(\mathbb{R}_+^6)} &\leq \|k\|_{L_m(\mathbb{R}_+)}^{\frac{m}{r_1}} \cdot \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{r_1}} \cdot \|\theta\|_{L_s(\mathbb{R}_+)}^{\frac{s}{r_1}} \\
&\quad \cdot \|f\|_{L_p(\mathbb{R}_+)}^{\frac{p}{r_1}} \cdot \|g\|_{L_q(\mathbb{R}_+)}^{\frac{q}{r_1}},
\end{aligned}$$

$$\begin{aligned}
(4.15) \quad \|H_4\|_{L_{m_1}(\mathbb{R}_+^6)} &\leq \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{m_1}} \cdot \|\theta\|_{L_s(\mathbb{R}_+)}^{\frac{s}{m_1}} \cdot \|f\|_{L_p(\mathbb{R}_+)}^{\frac{p}{m_1}} \\
&\quad \cdot \|g\|_{L_q(\mathbb{R}_+)}^{\frac{q}{m_1}} \cdot \|h\|_{L_r(\mathbb{R}_+)}^{\frac{r}{m_1}},
\end{aligned}$$

$$\begin{aligned}
(4.16) \quad \|H_5\|_{L_{n_1}(\mathbb{R}_+^6)} &\leq \|\theta\|_{L_s(\mathbb{R}_+)}^{\frac{s}{n_1}} \cdot \|f\|_{L_p(\mathbb{R}_+)}^{\frac{p}{n_1}} \cdot \|g\|_{L_q(\mathbb{R}_+)}^{\frac{q}{n_1}} \\
&\quad \cdot \|h\|_{L_r(\mathbb{R}_+)}^{\frac{r}{n_1}} \cdot \|k\|_{L_m(\mathbb{R}_+)}^{\frac{m}{n_1}},
\end{aligned}$$

$$\begin{aligned}
(4.17) \quad \|H_6\|_{L_{s_1}(\mathbb{R}_+^6)} &\leq \|f\|_{L_p(\mathbb{R}_+)}^{\frac{p}{s_1}} \cdot \|g\|_{L_q(\mathbb{R}_+)}^{\frac{q}{s_1}} \cdot \|h\|_{L_r(\mathbb{R}_+)}^{\frac{r}{s_1}} \\
&\quad \cdot \|k\|_{L_m(\mathbb{R}_+)}^{\frac{m}{s_1}} \cdot \|l\|_{L_n(\mathbb{R}_+)}^{\frac{n}{s_1}}.
\end{aligned}$$

From (4.12), (4.13), (4.14), (4.15), (4.16), (4.17) we obtain

$$\begin{aligned}
&\left| \int_0^{+\infty} [*(f, g, h, k, l)(x) \cdot \theta(x)] \cdot dx \right| \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|f\|_{L_p(\mathbb{R}_+)} \cdot \|g\|_{L_q(\mathbb{R}_+)} \cdot \|h\|_{L_r(\mathbb{R}_+)} \\
&\quad \cdot \|k\|_{L_m(\mathbb{R}_+)} \cdot \|l\|_{L_n(\mathbb{R}_+)} \cdot \|\theta\|_{L_s(\mathbb{R}_+)}.
\end{aligned}$$

We have proved the estimate (4.9). The proof is complete. $\square$

4.4. Saitoh-type inequality for the superpolyconvolution operator. We know that in the case $f, g \in L_2(\mathbb{R})$, Young's inequality is not true. In [26], Saitoh established the weighted $L_p(\mathbb{R}, |\rho_i|)$ norm inequality for the Fourier convolution

$$\left\| \left((F_1 \rho_1) *_F (F_2 \rho_2) \right) \cdot (\rho_1 *_F \rho_2)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R})} \leq \|F_1\|_{L_p(\mathbb{R}, |\rho_1|)} \cdot \|F_2\|_{L_p(\mathbb{R}, |\rho_2|)}, \quad p > 1,$$

here ρ_i are non-vanishing functions, $F_i \in L_p(\mathbb{R}, |\rho_i|)$, $i = 1, 2$. The norm of F_i in the weighted space $L_p(\mathbb{R}, \rho_i)$ is defined as

$$\|F_i\|_{L_p(\mathbb{R}, \rho_i)} = \left\{ \int_{-\infty}^{+\infty} |F_i(x)|^p \cdot \rho_i(x) \cdot dx \right\}^{\frac{1}{p}}.$$

In this section, we establish the Saitoh-type inequality for the superpolyconvolution operator. Here, we use the weighted space $L_p(\mathbb{R}_+, \rho_i)$ which is determined in $\mathbb{R}_+$ such that

$$L_p(\mathbb{R}_+, \rho_i) := \left\{ f(x) : \int_0^{+\infty} |f(u)|^p \cdot \rho_i(u) \cdot du < \infty \right\},$$

with respect to a positive measure $\rho_i(u)du$, equipped with the norm

$$\|f\|_{L_p(\mathbb{R}_+, \rho_i)} := \left(\int_0^{+\infty} |f(u)|^p \cdot \rho_i(u) \cdot du \right)^{\frac{1}{p}},$$

here $1 \leq p < \infty$.

Theorem 4.4. Suppose that ρ_i , $i = \overline{1, 5}$, are non-vanishing positive functions for all $x \in \mathbb{R}_+$, then $*(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)(x)$ given by the formula (3.1) is well-defined. Moreover, for any functions $K_i \in L_p(\mathbb{R}_+, \rho_i)$, $i = \overline{1, 5}$, with $p > 1$, the following inequality holds

$$\begin{aligned} & \left\| * (K_1 \rho_1, K_2 \rho_2, K_3 \rho_3, K_4 \rho_4, K_5 \rho_5) \cdot * (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\ (4.18) \quad & \leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|K_1\|_{L_p(\mathbb{R}_+, \rho_1)}^p \cdot \|K_2\|_{L_p(\mathbb{R}_+, \rho_2)}^p \cdot \|K_3\|_{L_p(\mathbb{R}_+, \rho_3)}^p \\ & \quad \cdot \|K_4\|_{L_p(\mathbb{R}_+, \rho_4)}^p \cdot \|K_5\|_{L_p(\mathbb{R}_+, \rho_5)}^p. \end{aligned}$$

Proof. From (3.1), we get

$$\begin{aligned} & \left\| * (K_1 \rho_1, K_2 \rho_2, K_3 \rho_3, K_4 \rho_4, K_5 \rho_5) \cdot * (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\ & = \int_0^{+\infty} |* (K_1 \rho_1, K_2 \rho_2, K_3 \rho_3, K_4 \rho_4, K_5 \rho_5)|^p \cdot |* (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)|^{1-p} \cdot dx \\ & = \frac{\sqrt{2}}{4\pi^2\sqrt{\pi}} \int_0^{+\infty} \left| \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \theta(x, u, v, w, e, t) \right. \\ (4.19) \quad & \quad \cdot (K_1 \rho_1)(u) \cdot (K_2 \rho_2)(v) \cdot (K_3 \rho_3)(w) \cdot (K_4 \rho_4)(e) \cdot (K_5 \rho_5)(t) \\ & \quad \cdot du \cdot dv \cdot dw \cdot de \cdot dt \Big|^p \\ & \quad \times \left| \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \theta(x, u, v, w, e, t) \right. \\ & \quad \cdot \rho_1(u) \cdot \rho_2(v) \cdot \rho_3(w) \cdot \rho_4(e) \cdot \rho_5(t) \cdot du \cdot dv \cdot dw \cdot de \cdot dt \Big|^{1-p} \cdot dx, \end{aligned}$$

where

$$\begin{aligned} \theta(x, u, v, w, e, t) = & \left[\frac{t}{t^2 + (u+v+w-e+x)^2} + \frac{t}{t^2 + (u+v+w-e-x)^2} \right. \\ & + \frac{t}{t^2 + (u+v-w+e+x)^2} + \frac{t}{t^2 + (u+v-w+e-x)^2} \\ & - \frac{t}{t^2 + (u+v+w+e+x)^2} - \frac{t}{t^2 + (u+v+w+e-x)^2} \\ & - \frac{t}{t^2 + (u+v-w-e+x)^2} - \frac{t}{t^2 + (u+v-w-e-x)^2} \\ & + \frac{t}{t^2 + (u-v+w-e+x)^2} + \frac{t}{t^2 + (u-v+w-e-x)^2} \\ & + \frac{t}{t^2 + (u-v-w+e+x)^2} + \frac{t}{t^2 + (u-v-w+e-x)^2} \\ & - \frac{t}{t^2 + (u-v+w+e+x)^2} - \frac{t}{t^2 + (u-v+w+e-x)^2} \\ & \left. - \frac{t}{t^2 + (u-v-w-e+x)^2} - \frac{t}{t^2 + (u-v-w-e-x)^2} \right]. \end{aligned}$$

Using Hölder's inequality for q as the conjugate exponent of p , we receive

$$\begin{aligned}
 & \left| \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \theta(x, u, v, w, e, t) \cdot (K_1 \rho_1)(u) \cdot (K_2 \rho_2)(v) \cdot (K_3 \rho_3)(w) \right. \\
 & \quad \left. \cdot (K_4 \rho_4)(e) \cdot (K_5 \rho_5)(t) \cdot du \cdot dv \cdot dw \cdot de \cdot dt \right| \\
 & \leq \left\{ \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |\theta(x, u, v, w, e, t)| \right. \\
 & \quad \cdot |(K_1 \rho_1)(u)|^p \cdot \rho_1(u) \cdot |(K_2 \rho_2)(v)|^p \cdot \rho_2(v) \\
 & \quad \cdot |(K_3 \rho_3)(w)|^p \cdot \rho_3(w) \cdot |(K_4 \rho_4)(e)|^p \cdot \rho_4(e) \cdot |(K_5 \rho_5)(t)|^p \cdot \rho_5(t) \\
 & \quad \left. \cdot du \cdot dv \cdot dw \cdot de \cdot dt \right\}^{\frac{1}{p}} \\
 & \quad \cdot \left\{ \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |\theta(x, u, v, w, e, t)| \cdot \rho_1(u) \cdot \rho_2(v) \cdot \rho_3(w) \cdot \rho_4(e) \cdot \rho_5(t) \right. \\
 & \quad \left. \cdot du \cdot dv \cdot dw \cdot de \cdot dt \right\}^{\frac{1}{q}}.
 \end{aligned} \tag{4.20}$$

From (4.19) and (4.20), we receive

$$\begin{aligned}
 & \left\| * (K_1 \rho_1, K_2 \rho_2, K_3 \rho_3, K_4 \rho_4, K_5 \rho_5) \cdot * (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\
 & \leq \frac{\sqrt{2}}{4\pi^2 \sqrt{\pi}} \int_0^{+\infty} \left\{ \left[\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |\theta(x, u, v, w, e, t)| \right. \right. \\
 & \quad \cdot |K_1(u)|^p \cdot \rho_1(u) \cdot |K_2(v)|^p \cdot \rho_2(v) \cdot |K_3(w)|^p \cdot \rho_3(w) \\
 & \quad \cdot |K_4(e)|^p \cdot \rho_4(e) \cdot |K_5(t)|^p \cdot \rho_5(t) \cdot du \cdot dv \cdot dw \cdot de \cdot dt \Big] \\
 & \quad \times \left[\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |\theta(x, u, v, w, e, t)| \cdot \rho_1(u) \cdot \rho_2(v) \cdot \rho_3(w) \cdot \rho_4(e) \cdot \rho_5(t) \right. \\
 & \quad \left. \cdot du \cdot dv \cdot dw \cdot de \cdot dt \right]^{\frac{p}{q}} \\
 & \quad \times \left[\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |\theta(x, u, v, w, e, t)| \cdot \rho_1(u) \cdot \rho_2(v) \cdot \rho_3(w) \cdot \rho_4(e) \cdot \rho_5(t) \right. \\
 & \quad \left. \cdot du \cdot dv \cdot dw \cdot de \cdot dt \right]^{1-p} \Big\} \cdot dx.
 \end{aligned}$$

Because $\frac{1}{p} + \frac{1}{q} = 1$, then $\frac{p}{q} + 1 - p = 0$. Therefore, we receive that

$$\begin{aligned}
 & \left\| * (K_1 \rho_1, K_2 \rho_2, K_3 \rho_3, K_4 \rho_4, K_5 \rho_5) \cdot * (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\
 & \leq \frac{\sqrt{2}}{4\pi^2 \sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} |\theta(x, u, v, w, e, t)| \cdot |K_1(u)|^p \cdot \rho_1(u) \\
 & \quad \cdot |K_2(v)|^p \cdot \rho_2(v) \cdot |K_3(w)|^p \cdot \rho_3(w) \cdot |K_4(e)|^p \cdot \rho_4(e) \cdot |K_5(t)|^p \cdot \rho_5(t) \\
 & \quad \cdot du \cdot dv \cdot dw \cdot de \cdot dt \cdot dx.
 \end{aligned}$$

From the supposition $K_i \in L_p(\mathbb{R}_+, \rho_i)$, using Fubini's theorem for this estimate, we get

$$\left\| * (K_1 \rho_1, K_2 \rho_2, K_3 \rho_3, K_4 \rho_4, K_5 \rho_5) \cdot * (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p$$

$$\begin{aligned}
&\leq \frac{\sqrt{2}}{4\pi^2\sqrt{\pi}} \left[\int_0^{+\infty} |\theta(x, u, v, w, e, t)| \cdot dx \right] \cdot \left[\int_0^{+\infty} |K_1(u)|^p \cdot \rho_1(u) \cdot du \right] \\
&\quad \cdot \left[\int_0^{+\infty} |K_2(v)|^p \cdot \rho_2(v) \cdot dv \right] \cdot \left[\int_0^{+\infty} |K_3(w)|^p \cdot \rho_3(w) \cdot dw \right] \\
&\quad \cdot \left[\int_0^{+\infty} |K_4(e)|^p \cdot \rho_4(e) \cdot de \right] \cdot \left[\int_0^{+\infty} |K_5(t)|^p \cdot \rho_5(t) \cdot dt \right] \\
&\leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \cdot \left[\int_0^{+\infty} |K_1(u)|^p \cdot \rho_1(u) \cdot du \right] \cdot \left[\int_0^{+\infty} |K_2(v)|^p \cdot \rho_2(v) \cdot dv \right] \\
&\quad \cdot \left[\int_0^{+\infty} |K_3(w)|^p \cdot \rho_3(w) \cdot dw \right] \cdot \left[\int_0^{+\infty} |K_4(e)|^p \cdot \rho_4(e) \cdot de \right] \\
&\quad \cdot \left[\int_0^{+\infty} |K_5(t)|^p \cdot \rho_5(t) \cdot dt \right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\| * (K_1\rho_1, K_2\rho_2, K_3\rho_3, K_4\rho_4, K_5\rho_5) \cdot * (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\
&\leq \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \|K_1\|_{L_p(\mathbb{R}_+, \rho_1)}^p \cdot \|K_2\|_{L_p(\mathbb{R}_+, \rho_2)}^p \cdot \|K_3\|_{L_p(\mathbb{R}_+, \rho_3)}^p \\
&\quad \cdot \|K_4\|_{L_p(\mathbb{R}_+, \rho_4)}^p \cdot \|K_5\|_{L_p(\mathbb{R}_+, \rho_5)}^p.
\end{aligned}$$

We have (4.18). The theorem is proved. $\square$

5. THE WATSON INTEGRAL TRANSFORM RELATED TO THE SUPERPOLYCONVOLUTION OPERATOR

In [41], Watson affirmed that the Mellin convolution type integral transform

$$h(x) = (Tf)(x) = \int_0^{+\infty} k(xy) \cdot f(y) \cdot dy,$$

such that its inverse has the following form

$$f(x) = (T^*h)(x) = \int_0^{+\infty} \bar{k}(xy) \cdot h(y) \cdot dy,$$

here $\bar{k}(u)$ is defined as the conjugate kernel of $k(u)$. Now, we can research the Watson-type integral transform for the superpolyconvolution operator (3.1) as follows

$$\begin{aligned}
&T_{ghkl} : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+), \\
&f \longmapsto \eta(x) = (T_{ghkl}f)(x) = \left(1 - \frac{d^2}{dx^2} \right) [* (f, g, h, k, l)](x),
\end{aligned}$$

here T_{ghkl} is a differential operator.

There are several previously published articles related to the Watson-type theorem for the convolutions, polyconvolutions involving to the Fourier cosine integral transform [39], Hartley, Fourier cosine and Fourier sine integral transforms [34, 35], Hartley integral transforms [16–18], Fourier cosine and Laplace integral transforms [15].

In this section, we research the Watson-type integral transform related to the superpolyconvolution operator (3.1) by fixing one function, and letting the remaining functions vary in certain functional spaces. We also set up necessary and sufficient conditions for this operator to be unitary on $L_2(\mathbb{R}_+)$ and receive its inverse represented in the conjugate symmetric form.

Now we investigate the following operator

$$\begin{aligned}
&T_{ghkl} : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+), \\
&f \longmapsto \eta(x) = (T_{ghkl}f)(x) = \left(1 - \frac{d^2}{dx^2} \right) [* (f, g, h, k, l)(x)],
\end{aligned}$$

where

$$\begin{aligned}
 \eta(x) = (T_{ghkl}f)(x) = & \left(1 - \frac{d^2}{dx^2}\right) \left[\frac{\sqrt{2}}{4\pi^2\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \left(\right. \right. \\
 & \frac{t}{t^2 + (u+v+w-e+x)^2} + \frac{t}{t^2 + (u+v+w-e-x)^2} \\
 & + \frac{t}{t^2 + (u+v-w+e+x)^2} + \frac{t}{t^2 + (u+v-w+e-x)^2} \\
 & - \frac{t}{t^2 + (u+v+w+e+x)^2} - \frac{t}{t^2 + (u+v+w+e-x)^2} \\
 & - \frac{t}{t^2 + (u+v-w-e+x)^2} - \frac{t}{t^2 + (u+v-w-e-x)^2} \\
 & + \frac{t}{t^2 + (u-v+w-e+x)^2} + \frac{t}{t^2 + (u-v+w-e-x)^2} \\
 & + \frac{t}{t^2 + (u-v-w+e+x)^2} + \frac{t}{t^2 + (u-v-w+e-x)^2} \\
 & - \frac{t}{t^2 + (u-v+w+e+x)^2} - \frac{t}{t^2 + (u-v+w+e-x)^2} \\
 & \left. \left. - \frac{t}{t^2 + (u-v-w-e+x)^2} - \frac{t}{t^2 + (u-v-w-e-x)^2} \right) \right. \\
 & \left. \cdot f(u) \cdot g(v) \cdot h(w) \cdot k(e) \cdot l(t) \cdot du \cdot dv \cdot dw \cdot de \cdot dt \right], \quad x \in \mathbb{R}_+.
 \end{aligned}
 \tag{5.1}$$

Definition 5.1. Symbol $\mathcal{M}(\mathbb{R}_+)$ is the set of all functions on $\mathbb{R}_+$, such that their images for the Laplace integral transform belong to $L_\infty(\mathbb{R}_+)$

$$\mathcal{M}(\mathbb{R}_+) := \{f(x), x \in \mathbb{R}_+ : (Lf) \in L_\infty(\mathbb{R}_+)\}.$$

It is obvious that with this definition, for any function f in $L_1(\mathbb{R}_+)$, then f evidently belongs to $\mathcal{M}(\mathbb{R}_+)$.

Lemma 5.2. If l is a function belonging to $\mathcal{M}(\mathbb{R}_+)$ and satisfies

$$(1+y^2)|(Ll)(y)| \text{ bounded for any } y > 0.$$

Then, for any functions f, g, h, k belonging to $L_2(\mathbb{R}_+)$, we receive

$$\begin{aligned}
 (T_{ghkl}f)(x) &= \left(1 - \frac{d^2}{dx^2}\right) [*(f, g, h, k, l)](x) \\
 &= F_c [(1+y^2) \cdot (F_c f)(y) \cdot (F_c g)(y) \cdot (F_c h)(y) \cdot (F_c k)(y) \cdot (Ll)(y)](x) \in L_2(\mathbb{R}_+).
 \end{aligned}
 \tag{5.3}$$

Proof. We know that, the functions $\eta(y)$, $y\eta(y)$, $y^2\eta(y)$, $\dots$, $y^n\eta(y)$ belong to the space $L_2(\mathbb{R})$ if and only if $(F\eta)(x)$, $\frac{d}{dx}[F\eta](x)$, $\frac{d^2}{dx^2}(F\eta)(x)$, $\dots$, $\frac{d^n}{dx^n}(F\eta)(x) \in L_2(\mathbb{R})$ [28,37]. This affirmation is also true for the Fourier cosine integral transform F_c on $L_2(\mathbb{R}_+)$. On the other hand, we have

$$\begin{aligned}
 \frac{d}{dx}[F_c\eta](x) &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \eta(y) \cdot \left(\frac{d}{dx} \cos xy\right) \cdot dy = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (-y) \cdot \eta(y) \cdot \sin xy \cdot dy; \\
 \frac{d^2}{dx^2}(F_c\eta)(x) &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (-y) \cdot \eta(y) \cdot \left(\frac{d}{dx} \sin xy\right) \cdot dy \\
 &= F_c(-y^2 \cdot \eta(y))(x) \in L_2(\mathbb{R}_+).
 \end{aligned}$$

This means, if $\eta(y)$, $y^2\eta(y)$ belong to $L_2(\mathbb{R}_+)$ then we get

$$\left(1 - \frac{d^2}{dx^2}\right) [(F_c\eta)(y)](x) = F_c[(1+y^2)\eta(y)](x).$$

From the condition (5.2), we infer that $(1 + y^2)(Ll)(y)$ is a bounded function $\forall y > 0$, and $f, g, h, k \in L_2(\mathbb{R}_+)$ imply that $F_c f, F_c g, F_s h, F_s k$ belong to $L_2(\mathbb{R}_+)$. So $(1 + y^2)(F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y)$ belongs to the $L_2(\mathbb{R}_+)$.

Therefore, from the Parseval's identity (3.3) and (5.4) we have

$$\begin{aligned} (T_{ghkl}f)(x) &= \left(1 - \frac{d^2}{dx^2}\right) [* (f, g, h, k, l)](x) \\ &= \left(1 - \frac{d^2}{dx^2}\right) \{F_c [(F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y)](x)\} \\ &= F_c [(1 + y^2) \cdot (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y)](x) \in L_2(\mathbb{R}_+). \end{aligned}$$

We have (5.3). □

Theorem 5.3. Suppose that $g, h, k \in L_1(\mathbb{R}_+)$, $l \in \mathcal{M}(\mathbb{R}_+)$ are given functions and satisfy

$$(5.5) \quad |(F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y)| = \frac{1}{1 + y^2}, \quad \forall y > 0.$$

Then the condition (5.5) is the necessary and sufficient condition for the operator T_{ghkl} to become an isometric isomorphism mapping on $L_2(\mathbb{R}_+)$. Moreover, the inverse operator of T_{ghkl} has a symmetric form and is represented by

$$\begin{aligned} f(x) &= \left(T_{\bar{g}\bar{h}\bar{k}\bar{l}}^{-1}\eta\right)(x) = \left(1 - \frac{d^2}{dx^2}\right) [* (\eta, \bar{g}, \bar{h}, \bar{k}, \bar{l})](x) \\ &= \left(1 - \frac{d^2}{dx^2}\right) \cdot \frac{\sqrt{2}}{4\pi^2\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \left[\right. \\ &\quad \frac{t}{t^2 + (u + v + w - e + x)^2} + \frac{t}{t^2 + (u + v + w - e - x)^2} \\ &\quad + \frac{t}{t^2 + (u + v - w + e + x)^2} + \frac{t}{t^2 + (u + v - w + e - x)^2} \\ &\quad - \frac{t}{t^2 + (u + v + w + e + x)^2} - \frac{t}{t^2 + (u + v + w + e - x)^2} \\ &\quad - \frac{t}{t^2 + (u + v - w - e + x)^2} - \frac{t}{t^2 + (u + v - w - e - x)^2} \\ &\quad + \frac{t}{t^2 + (u - v + w - e + x)^2} + \frac{t}{t^2 + (u - v + w - e - x)^2} \\ &\quad + \frac{t}{t^2 + (u - v - w + e + x)^2} + \frac{t}{t^2 + (u - v - w + e - x)^2} \\ &\quad - \frac{t}{t^2 + (u - v + w + e + x)^2} - \frac{t}{t^2 + (u - v + w + e - x)^2} \\ &\quad \left. - \frac{t}{t^2 + (u - v - w - e + x)^2} - \frac{t}{t^2 + (u - v - w - e - x)^2} \right] \\ &\quad \cdot \eta(u) \cdot \bar{g}(v) \cdot \bar{h}(w) \cdot \bar{k}(e) \cdot \bar{l}(t) \cdot du \cdot dv \cdot dw \cdot de \cdot dt, \quad \forall x > 0, \end{aligned}$$

where $\bar{g}(x), \bar{h}(x), \bar{k}(x), \bar{l}(x)$ are complex conjugate functions of g, h, k, l respectively.

Proof. Necessary condition. By the condition (5.5), we have that $(1 + y^2)(F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y)$ is bounded, together with $(F_c f)(y)$ belonging to $L_2(\mathbb{R}_+)$, we infer that

$$(1 + y^2)(F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y) \in L_2(\mathbb{R}_+).$$

Applying (3.3) to (5.1), we get

$$(5.6) \quad \begin{aligned} \eta(x) &= (T_{ghkl}f)(x) = \left(1 - \frac{d^2}{dx^2}\right) [* (f, g, h, k, l)] (x) \\ &= F_c [(1 + y^2) \cdot (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y)] (x) \in L_2(\mathbb{R}_+). \end{aligned}$$

According to Theorem 9.13 in [23], for any function f belonging to $L_2(\mathbb{R})$, we have $\|Ff\|_{L_2(\mathbb{R})} = \|f\|_{L_2(\mathbb{R})}$. This is also true for the Fourier cosine integral transform F_c , which implies that

$$\|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}, \quad \forall f \in L_2(\mathbb{R}_+).$$

Applying this to (5.6) and using the condition (5.5), we receive

$$\begin{aligned} \|\eta\|_{L_2(\mathbb{R}_+)} &= \left\| \left(1 - \frac{d^2}{dx^2}\right) [* (f, g, h, k, l)] \right\|_{L_2(\mathbb{R}_+)} \\ &= (1 + y^2) \cdot |(F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y)| \cdot \|F_c f\|_{L_2(\mathbb{R}_+)} \\ &= \|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}. \end{aligned}$$

This means that T_{ghkl} is an isometric isomorphism (unitary) transformation in $L_2(\mathbb{R}_+)$.

We continue to prove that the inverse operator of T_{ghkl} has a symmetric form. In $L_2(\mathbb{R}_+)$, applying the unitary property of mapping Fourier cosine $F_c : L_2(\mathbb{R}_+) \rightarrow L_2(\mathbb{R}_+)$ [37] for the (5.6), we obtain

$$(5.7) \quad (F_c \eta)(y) = (1 + y^2) \cdot (F_c f)(y) \cdot (F_c g)(y) \cdot (F_s h)(y) \cdot (F_s k)(y) \cdot (Ll)(y) \in L_2(\mathbb{R}_+).$$

From $(\overline{Ll})(y) = (L\bar{l})(y)$, $(\overline{F_c g})(y) = (F_c \bar{g})(y)$, $(\overline{F_s h})(y) = (F_s \bar{h})(y)$, $(\overline{F_s k})(y) = (F_s \bar{k})(y)$, we get

$$\begin{aligned} (1 + y^2) &\left| (\overline{F_c g})(y) \cdot (\overline{F_s h})(y) \cdot (\overline{F_s k})(y) \cdot (\overline{Ll})(y) \right| \\ &= (1 + y^2) \left| (F_c \bar{g})(y) \cdot (F_s \bar{h})(y) \cdot (F_s \bar{k})(y) \cdot (L\bar{l})(y) \right|. \end{aligned}$$

Due to (5.5), it follows that

$$(1 + y^2) \left| (F_c \bar{g})(y) \cdot (F_s \bar{h})(y) \cdot (F_s \bar{k})(y) \cdot (L\bar{l})(y) \right| = 1.$$

This leads to

$$(F_c \bar{g})(y) \cdot (F_s \bar{h})(y) \cdot (F_s \bar{k})(y) \cdot (L\bar{l})(y) \cdot (F_c \eta)(y) = \frac{1}{1 + y^2} \cdot (F_c f)(y),$$

based on (5.7). We get

$$(5.8) \quad (1 + y^2) \cdot (F_c \bar{g})(y) \cdot (F_s \bar{h})(y) \cdot (F_s \bar{k})(y) \cdot (L\bar{l})(y) \cdot (F_c \eta)(y) = (F_c f)(y).$$

By the same reasoning as above, we get $F_c \eta \in L_2(\mathbb{R}_+)$ and $(1 + y^2)(F_c \bar{g})(y) \cdot (F_s \bar{h})(y) \cdot (F_s \bar{k})(y) \cdot (L\bar{l})(y)$ is a bounded function, implying that

$$(1 + y^2)(F_c \bar{g})(y) \cdot (F_s \bar{h})(y) \cdot (F_s \bar{k})(y) \cdot (L\bar{l})(y) \cdot (F_c \eta)(y) \in L_2(\mathbb{R}_+).$$

From (5.3) and (5.8), we receive

$$\begin{aligned} f(x) &= F_c [(1 + y^2) \cdot (F_c \bar{g})(y) \cdot (F_s \bar{h})(y) \cdot (F_s \bar{k})(y) \cdot (L\bar{l})(y) \cdot (F_c \eta)(y)] (x) \\ &= \left(1 - \frac{d^2}{dx^2}\right) [* (\eta, \bar{g}, \bar{h}, \bar{k}, \bar{l})] (x). \end{aligned}$$

Sufficient condition. Suppose that T_{ghkl} is unitary on $L_2(\mathbb{R}_+)$ and has an inverse operator. We need to prove that the functions g, h, k, l must satisfy the condition (5.5). Since T_{ghkl} has the unitary property on

$L_2(\mathbb{R}_+)$, for any function η belonging to $L_2(\mathbb{R}_+)$, we receive

$$\begin{aligned}\|(F_c\eta)\|_{L_2(\mathbb{R}_+)} &= \|\eta\|_{L_2(\mathbb{R}_+)} \\ &= \|F_c [(1+y^2)(F_cf)(y).(F_cg)(y).(F_sh)(y).(F_sk)(y).(Ll)(y)]\|_{L_2(\mathbb{R}_+)} \\ &= (1+y^2) \cdot |(F_cg)(y).(F_sh)(y).(F_sk)(y).(Ll)(y)| \cdot \|F_cf\|_{L_2(\mathbb{R}_+)} \\ &= \|F_cf\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}, \quad \forall f \in L_2(\mathbb{R}_+).\end{aligned}$$

This shows that there exists a multiplication operator of the form $H_\xi[\cdot]$ defined by

$$H_\xi[f](y) := \xi(y) \cdot f(y),$$

where the function

$$\xi(y) := (1+y^2) |(F_cg)(y).(F_sh)(y).(F_sk)(y).(Ll)(y)|, \quad \forall y > 0.$$

The above expression can be represented as

$$\|F_cf\|_{L_2(\mathbb{R}_+)} = \|H_\xi(F_cf)\|_{L_2(\mathbb{R}_+)},$$

for any $f \in L_2(\mathbb{R}_+)$. This means that $H_\xi[\cdot]$ is an isometric isomorphism (unitary) on $L_2(\mathbb{R}_+)$, and this happens if and only if

$$|(F_cg)(y).(F_sh)(y).(F_sk)(y).(Ll)(y)| = \frac{1}{1+y^2}.$$

In conclusion, g, h, k, l must satisfy the condition (5.5). The proof is complete. $\square$

6. SOME APPLICATIONS OF THIS SUPERPOLYCONVOLUTION OPERATOR

6.1. Integral equation. The Fredholm integral equation [38]

$$f(x) + \int_0^{+\infty} K(x, y) \cdot f(y) \cdot dy = g(x), \quad \forall x \geq 0,$$

when the kernel $K(x, y)$ is the sum of a Toeplitz and a Hankel kernel, i.e.

$$K(x, y) = K_1(x - y) + K_2(x + y), \quad x, y \geq 0.$$

Now, the solution of this integral equation is an open problem. A lot of solutions of integral equations of this type have many useful applications in such diverse fields as scattering theory, fluid dynamics and linear filtering theory, ...

In this section, we develop and apply the new results of the superpolyconvolution operator (3.1) in solving an integral equation with the complex kernel of Toeplitz plus Hankel type and receive the solution

in closed form. We examine the following integral equation

$$\begin{aligned}
 (6.1) \quad & f(x) + \frac{\sqrt{2}}{4\pi^2\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \left[\right. \\
 & \frac{t}{t^2 + (u+v+w-e+x)^2} + \frac{t}{t^2 + (u+v+w-e-x)^2} \\
 & + \frac{t}{t^2 + (u+v-w+e+x)^2} + \frac{t}{t^2 + (u+v-w+e-x)^2} \\
 & - \frac{t}{t^2 + (u+v+w+e+x)^2} - \frac{t}{t^2 + (u+v+w+e-x)^2} \\
 & - \frac{t}{t^2 + (u+v-w-e+x)^2} - \frac{t}{t^2 + (u+v-w-e-x)^2} \\
 & + \frac{t}{t^2 + (u-v+w-e+x)^2} + \frac{t}{t^2 + (u-v+w-e-x)^2} \\
 & + \frac{t}{t^2 + (u-v-w+e+x)^2} + \frac{t}{t^2 + (u-v-w+e-x)^2} \\
 & - \frac{t}{t^2 + (u-v+w+e+x)^2} - \frac{t}{t^2 + (u-v+w+e-x)^2} \\
 & \left. - \frac{t}{t^2 + (u-v-w-e+x)^2} - \frac{t}{t^2 + (u-v-w-e-x)^2} \right] \\
 & \cdot (f *_{F_c} q)(u) \cdot g(v) \cdot h(w) \cdot k(e) \cdot l(t) \cdot du \cdot dv \cdot dw \cdot de \cdot dt = p(x), \quad x \in \mathbb{R}_+.
 \end{aligned}$$

Theorem 6.1. Let $g, h, k, l, p, q \in L_1(\mathbb{R}_+)$ be given. Then the integral equation (6.1) has a unique solution $f \in L_1(\mathbb{R}_+)$ if $1 + F_c[*](q, g, h, k, l)(y) \neq 0, \forall y > 0$. The solution has the following form

$$f(x) = p(x) - (p *_{F_c} m)(x),$$

where

$$(F_c m)(y) = \frac{F_c[*](q, g, h, k, l)(y)}{1 + F_c[*](q, g, h, k, l)(y)}.$$

Proof. The integral equation (6.1) can be represented in the following form

$$f(x) + \left\{ * \left[(f *_{F_c} q), g, h, k, l \right] \right\} (x) = p(x), \quad x \in \mathbb{R}_+.$$

Due to Theorem 3.2 and (2.6), (3.2), we receive

$$\begin{aligned}
 & (F_c f)(y) + (F_c f)(y) \cdot (F_c q)(y) \cdot (F_c g)(y) \cdot (F_c h)(y) \cdot (F_c k)(y) \cdot (F_c l)(y) \\
 & = (F_c p)(y), \quad \forall y > 0, \\
 & \Leftrightarrow (F_c f)(y) \{1 + F_c[*](q, g, h, k, l)(y)\} = (F_c p)(y), \quad \forall y > 0.
 \end{aligned}$$

With the condition $1 + F_c[*](q, g, h, k, l)(y) \neq 0, \forall y > 0$, we get

$$(F_c f)(y) = (F_c p)(y) \left[1 - \frac{F_c[*](q, g, h, k, l)(y)}{1 + F_c[*](q, g, h, k, l)(y)} \right].$$

Thus, by Wiener–Lévy’s theorem [2], there exists a function $m \in L_1(\mathbb{R}_+)$ such that

$$(F_c m)(y) = \frac{F_c[*](q, g, h, k, l)(y)}{1 + F_c[*](q, g, h, k, l)(y)}.$$

Hence,

$$\begin{aligned}
 & (F_c f)(y) = (F_c p)(y) [1 - (F_c m)(y)], \quad \forall y > 0, \\
 & \Leftrightarrow (F_c f)(y) = (F_c p)(y) - F_c(p *_{F_c} m)(y), \quad \forall y > 0.
 \end{aligned}$$

Thus, $f(x) = p(x) - (p *_{F_c} m)(x) \in L_1(\mathbb{R}_+)$. The theorem is proved. $\square$

6.2. Integro-differential equation of Barbashin type related to the superpolyconvolution operator. Barbashin's integro-differential equation has been applied to a lot of fields of mathematics and technology such as mathematical physics, radiation propagation, mathematical biology, transport problem ... [4,5].

In this section, we investigate the following integro-differential equation

$$(6.2) \quad \left(1 - \frac{d^2}{dx^2}\right) (f *_{F_c} m)(x) + \left\{ * \left[(q *_{F_c} f), g, h, k, l \right] \right\} (x) = p(x), \quad x > 0.$$

Theorem 6.2. Suppose that g, h, k, l, m, p, q belong to $L_2(\mathbb{R}_+)$ and are the given functions satisfying the following condition

$$(6.3) \quad \frac{(F_c p)(y)}{(1 + y^2)(F_c m)(y) + F_c [* (q, g, h, k, l)](y)} \in L_2(\mathbb{R}_+), \quad y \in \mathbb{R}_+.$$

Then, the integro-differential equation (6.2) has a unique solution in $L_2(\mathbb{R}_+)$ which can be presented as follows

$$(6.4) \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{(F_c p)(y) \cdot \cos(xy) \cdot dy}{(1 + y^2)(F_c m)(y) + F_c [* (q, g, h, k, l)](y)}.$$

Proof. Using the results in (2.6), (3.2)

$$\begin{aligned} (1 + y^2)(F_c f)(y) \cdot (F_c m)(y) + (F_c f)(y) \cdot (F_c q)(y) \cdot (F_c g)(y) \cdot (F_c h)(y) \cdot (F_c k)(y) \cdot (Ll)(y) \\ = (F_c p)(y), \quad \forall y > 0, \\ \Leftrightarrow (F_c f)(y) \{ (1 + y^2)(F_c m)(y) + F_c [* (q, g, h, k, l)](y) \} = (F_c p)(y). \end{aligned}$$

With the condition (6.3), we have

$$(F_c f)(y) = \frac{(F_c p)(y)}{(1 + y^2)(F_c m)(y) + F_c [* (q, g, h, k, l)](y)}.$$

Applying the inverse transform of operator F_c in $L_2(\mathbb{R}_+)$, we receive the solution of the integro-differential equation (6.2) in $L_2(\mathbb{R}_+)$ expressed in the following form

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{(F_c p)(y) \cdot \cos(xy) \cdot dy}{(1 + y^2)(F_c m)(y) + F_c [* (q, g, h, k, l)](y)}, \quad x \in \mathbb{R}_+.$$

We have (6.4). The proof is complete. $\square$

6.3. System of integral equations of superpolyconvolution type. Now, we apply the new superpolyconvolution operator (3.1) in solving the system of integral equations of Toeplitz plus Hankel type and receive the solution in closed form. We consider the following system

$$(6.5) \quad \begin{cases} f(x) + \left\{ * \left[(p *_{F_c} s), g, h, k, l \right] \right\} (x) = \alpha(x), \\ (q *_{F_c} f)(x) + s(x) = \beta(x), \quad \forall x \in \mathbb{R}_+. \end{cases}$$

Here, $g, h, k, l, p, q, \alpha, \beta$ are given functions of $L_1(\mathbb{R}_+)$, f and s are unknown functions.

Theorem 6.3. Let $g, h, k, l, p, q, \alpha, \beta \in L_1(\mathbb{R}_+)$ be given. Then the system (6.5) has a unique solution f, s in $L_1(\mathbb{R}_+)$ if

$$1 - F_c \left\{ q *_{F_c} [* (p, g, h, k, l)] \right\} (y) \neq 0, \quad \forall y > 0.$$

The solution has the following form

$$\begin{aligned} f(x) &= \alpha(x) + (\alpha *_{F_c} m)(x) - \left\{ \beta *_{F_c} [* (p, g, h, k, l)] \right\} (x) \\ &\quad - \left\{ \left[\beta *_{F_c} (* (p, g, h, k, l)) \right] *_{F_c} m \right\} (x), \\ s(x) &= \beta(x) + (\beta *_{F_c} m)(x) - (q *_{F_c} \alpha)(x) - \left[(q *_{F_c} \alpha) *_{F_c} m \right] (x), \end{aligned}$$

where

$$(F_c m)(y) = \frac{F_c \left\{ q *_{F_c} [* (p, g, h, k, l)] \right\} (y)}{1 - F_c \left\{ q *_{F_c} [* (p, g, h, k, l)] \right\} (y)}.$$

Proof. Due to Theorem 3.2, (2.6), (3.2), we get the system of linear algebraic equations

$$\begin{cases} (F_c f)(y) + (F_c s)(y) \cdot F_c [* (p, g, h, k, l)] (y) = (F_c \alpha)(y), \\ (F_c f)(y) \cdot (F_c q)(y) + (F_c s)(y) = (F_c \beta)(y), \quad \forall y > 0. \end{cases}$$

We have

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & F_c [* (p, g, h, k, l)] (y) \\ (F_c q)(y) & 1 \end{vmatrix} = 1 - F_c \left\{ q *_{F_c} [* (p, g, h, k, l)] \right\} (y), \\ \Delta_1 &= \begin{vmatrix} (F_c \alpha)(y) & F_c [* (p, g, h, k, l)] (y) \\ (F_c \beta)(y) & 1 \end{vmatrix} = (F_c \alpha)(y) - F_c \left\{ \beta *_{F_c} [* (p, g, h, k, l)] \right\} (y), \\ \Delta_2 &= \begin{vmatrix} 1 & (F_c \alpha)(y) \\ (F_c q)(y) & (F_c \beta)(y) \end{vmatrix} = (F_c \beta)(y) - F_c (q *_{F_c} \alpha)(y). \end{aligned}$$

With the condition $1 - F_c \left\{ q *_{F_c} [* (p, g, h, k, l)] \right\} (y) \neq 0, \forall y > 0$, we have

$$\begin{aligned} (F_c f)(y) &= \frac{\Delta_1}{\Delta} = \left\{ (F_c \alpha)(y) - F_c \left\{ \beta *_{F_c} [* (p, g, h, k, l)] \right\} (y) \right\} \\ &\quad \cdot \left\{ 1 + \frac{F_c \left\{ q *_{F_c} [* (p, g, h, k, l)] \right\} (y)}{1 - F_c \left\{ q *_{F_c} [* (p, g, h, k, l)] \right\} (y)} \right\}. \end{aligned}$$

Furthermore, according to Wiener–Lévy’s theorem [2], there exists a function $m \in L_1(\mathbb{R}_+)$ such that

$$(F_c m)(y) = \frac{F_c \left\{ q *_{F_c} [* (p, g, h, k, l)] \right\} (y)}{1 - F_c \left\{ q *_{F_c} [* (p, g, h, k, l)] \right\} (y)}.$$

It follows that

$$\begin{aligned} (F_c f)(y) &= (F_c \alpha)(y) + F_c (\alpha *_{F_c} m)(y) - F_c \left\{ \beta *_{F_c} [* (p, g, h, k, l)] \right\} (y) \\ &\quad - F_c \left\{ \left[\beta *_{F_c} (* (p, g, h, k, l)) \right] *_{F_c} m \right\} (y), \quad \forall y > 0. \end{aligned}$$

So,

$$f(x) = \alpha(x) + (\alpha *_{F_c} m)(x) - \left\{ \beta *_{F_c} [* (p, g, h, k, l)] \right\} (x) \\ - \left\{ \left[\beta *_{F_c} (* (p, g, h, k, l)) \right] *_{F_c} m \right\} (x) \in L_1(\mathbb{R}_+).$$

Similarly, we get

$$(F_c s)(y) = \frac{\Delta_2}{\Delta} = \left[(F_c \beta)(y) - F_c(q *_{F_c} \alpha)(y) \right] \cdot \{1 + (F_c m)(y)\} \\ = (F_c \beta)(y) + F_c(\beta *_{F_c} m)(y) - F_c(q *_{F_c} \alpha)(y) \\ - F_c \left[(q *_{F_c} \alpha) *_{F_c} m \right] (y), \quad \forall y > 0.$$

Thus,

$$s(x) = \beta(x) + (\beta *_{F_c} m)(x) - (q *_{F_c} \alpha)(x) - \left[(q *_{F_c} \alpha) *_{F_c} m \right] (x) \in L_1(\mathbb{R}_+).$$

The proof is complete. $\square$

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