

ON RESTRAINED HOP ROMAN DOMINATION IN GRAPHS

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ABSTRACT. Let $G = (V(G), E(G))$ be a graph of order $n \in \mathbb{N}$ and let $f : V(G) \rightarrow \{0, 1, 2\}$ be a *hop Roman dominating function* (HRDF) on G . For each $j \in \{0, 1, 2\}$, let $V_j = \{y \in V(G) : f(y) = j\}$. Then f can be written as $f = (V_0, V_1, V_2)$. A function f is a *restrained hop Roman dominating function* (RHRDF) on G provided that for every $v \in V_0$, there exists $u \in V_0 \setminus \{v\}$, and $w \in V_2$ such that $d_G(u, v) = 1$ and $d_G(v, w) = 2$. The *weight* of RHRDF f denoted by $\omega_G^{rhR}(f)$ is the sum $\omega_G^{rhR}(f) = \sum_{x \in V(G)} f(x)$, that is, $\omega_G^{rhR}(f) = |V_1| + 2|V_2|$. The *restrained hop Roman domination number* of a graph G denoted by $\gamma_{rhR}(G)$ is the minimum weight of an RHRDF on G , that is, $\gamma_{rhR}(G) = \min\{\omega_G^{rhR}(f) : f \text{ is an RHRDF on } G\}$. Every RHRDF f on G with the restriction $\omega_G^{rhR}(f) = \gamma_{rhR}(G)$ is called a γ_{rhR} -function on G . This paper introduces the concept of restrained hop Roman domination in graphs as a new restricted parameter and discusses some important theoretical properties.

1. INTRODUCTION

Domination in graphs seeks the minimum number of dominating vertices that guard all other vertices (the non-dominating set) in a graph [11]. Some interesting studies on domination in graphs can be found in the following references: [2–6, 8, 9, 16]. In 2004, Roman domination was introduced by Cockayne et al. [10], becoming a fascinating restricted parameter in graph domination theory. The concept of Roman domination involves a labeling technique that is based on military defense strategies to protect a region using legions of soldiers during the reign of Roman Emperor Constantine the Great. Currently, hop Roman domination is extensively studied by several graph theorists and combinatorics enthusiasts, and some interesting papers can be found in [1, 7, 15, 17]. Meanwhile, another motivating domination parameter is the restrained dominating set problem in graphs, a variation that requires connectivity among some non-dominating vertices; that is, no isolated non-dominating vertex exists [12]. In this paper, the author is interested in combining the two parameters, namely hop Roman domination and restrained domination, to come up with a new restricted and stronger parameter called the restrained hop Roman dominating function in graphs. The new parameter can be used as a baseline for some graph theorists for their further studies.

2. PRELIMINARIES

Let $G = (V(G), E(G))$ be a graph where $V(G)$ is a set of vertices in G and $E(G)$ is a set of edges in G . Let $v \in V(G)$. The set $N_G^2(v) = \{u \in V(G) : \deg_G(u, v) = 2\}$ is so-called the *open hop-neighborhood* of v and each element of the set $N_G^2(v)$ is so-called *hop-neighbor* of v in G . In addition, the *close hop-neighborhood* of v denoted by $N_G^2[v]$ is defined as $N_G^2[v] = N_G^2(v) \cup \{v\}$. Let $D \subseteq V(G)$. Then D is called a *dominating set* of G if

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every vertex in $V(G) \setminus D$ is adjacent to some vertex in D [14]. The smallest cardinality of a dominating set D is the so-called *domination number* on G denoted by $\gamma(G)$. Every dominating set D satisfying the condition $|D| = \gamma(G)$ is called a γ -set on G . A dominating set $R \subseteq V(G)$ is called a *restrained dominating set* of G if every vertex in $V(G) \setminus R$ is adjacent to a vertex $v \in R$ and another vertex $u \in V(G) \setminus R$ [12]. The smallest cardinality of a restrained dominating set of G is the so-called *restrained domination number*, denoted by $\gamma_r(G)$. A restrained dominating set of cardinality $\gamma_r(G)$ is called γ_r -set in G . Let $D_h \subseteq V(G)$. Then D_h is so-called a *hop dominating set* of G if for every $v \in V(G) \setminus D_h$ there exists $u \in D_h$ such that $v \in N_G^2(u)$. The smallest cardinality of a hop dominating set D_h is the so-called *hop domination number* of G , denoted by $\gamma_h(G)$ [15]. A hop dominating set D_h satisfying the condition $|D_h| = \gamma_h(G)$ is so-called a γ_h -set on G . Let R_h be a subset of $V(G)$. Then R_h is so-called a *restrained hop dominating set* of G if for every $v \in V(G) \setminus R_h$, there exists $u \in R_h$ and $w \in V(G) \setminus (R_h \cup \{v\})$ such that $d_G(u, v) = 2$ and $d_G(v, w) = 1$. The *restrained hop domination number* of G is the smallest cardinality of R_h on G and is denoted by $\gamma_{rh}(G)$. A restrained hop dominating set R_h is called γ_{rh} -set on G provided that $|R_h| = \gamma_{rh}(G)$.

Let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function on G and let $V_0 = \{x \in V(G) : f(x) = 0\}$, $V_1 = \{x \in V(G) : f(x) = 1\}$, and $V_2 = \{x \in V(G) : f(x) = 2\}$. Then the function f can be written as $f = (V_0, V_1, V_2)$. A function $f = (V_0, V_1, V_2)$ is so-called a *hop Roman dominating function* (HRDF) on G if for every $v \in V_0$, there exists $y \in V_2$ such that $v \in N_G^2(u)$ [17]. The *weight* of an HRDF f is denoted by $\omega_G^{hR}(f)$ and is defined as $\omega_G^{hR}(f) = \sum_{z \in V(G)} f(z) = |V_1| + 2|V_2|$. The *hop Roman domination number* of graph G is the minimum weight of an HRDF f on G , denoted by $\gamma_{hR}(G)$. In that case, we have $\gamma_{hR}(G) = \min\{\omega_G^{hR}(f) : f \text{ is an HRDF on } G\}$. Every HRDF f on G with $\omega_G^{hR}(f) = \gamma_{hR}(G)$ is so-called a γ_{hR} -function on G . A function $f = (V_0, V_1, V_2)$ is a *restrained hop Roman dominating function* (RHRDF) on G if for every $v \in V_0$, there exists $u \in V_2$ and $w \in V_0 \setminus \{v\}$ such that $d_G(u, v) = 2$ and $d_G(w, v) = 1$. The *weight* of RHRDF f denoted by $\omega_G^{rhR}(f)$ is the sum $\omega_G^{rhR}(f) = \sum_{u \in V(G)} f(u)$, that is, $\omega_G^{rhR}(f) = |V_1| + 2|V_2|$. The *restrained hop Roman domination number* of a graph G denoted by $\gamma_{rhR}(G)$ is the minimum weight of an RHRDF on G , that is, $\gamma_{rhR}(G) = \min\{\omega_G^{rhR}(f) : f \text{ is an RHRDF on } G\}$. In that case, any RHRDF f on graph G with $\omega_G^{rhR}(f) = \gamma_{rhR}(G)$ is called a γ_{rhR} -function on G . Consider the graph G with $|V(G)| = 8$ in the Figure 1. Let $f = (V_0, V_1, V_2)$ be a function on G such that $V_0 = \{v_1, v_2, v_3, v_7, v_8\}$, $V_1 = \{v_5\}$, and $V_2 = \{v_4, v_6\}$. Observe that f is a γ_{rhR} -function on G by definition of RHRDF. Hence, $\gamma_{rhR}(G) = 5$.

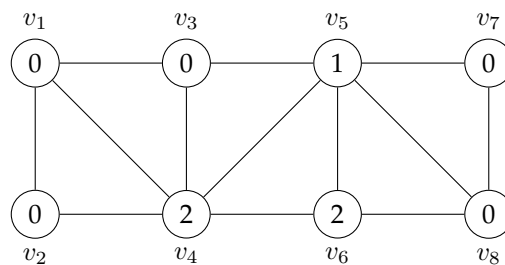


FIGURE 1. A graph G with $\gamma_{rhR}(G) = 5$

A function f is a *Roman point-wise nondominating function* (RPNDF) on G provided that for each $v \in V_0$, there exists $u \in V_2$ such that $v \notin N_G(u)$, that is, $u \in V_2 \setminus N_G(v)$. A function f is a *restrained Roman point-wise nondominating function* (RRPNDF) on G if for each $v \in V_0$, there exists $u \in V_2 \setminus N_G(v)$ and there exists $w \in V_0 \setminus \{v\}$ such that $d_G(v, w) = 1$. The *weight* of RRPNDF f , denoted by $\omega_G^{rRpnD}(f)$, is defined by $\omega_G^{rRpnD}(f) = \sum_{x \in V(G)} f(x) = |V_1| + 2|V_2|$. The *restrained Roman point-wise nondomination number* of G , denoted by $\gamma_{rRpnD}(G)$, is defined by $\gamma_{rRpnD}(G) = \min\{\omega_G^{rRpnD}(f) : f \text{ is an RRPNDF on } G\}$. Every RRPNDF

f on G with the restriction $\omega_G^{rRpd}(f) = \gamma_{rRpd}(G)$ is called a γ_{rRpd} -function on G . Consider the same graph G in Figure 1. Let $f' = (V'_0, V'_1, V'_2)$ be a function on G such that $V'_0 = \{v_3, v_5, v_6, v_7, v_8\}$, $V'_1 = \{v_1, v_4\}$, and $V'_2 = \{v_2\}$ (See Figure 2). By construction, it follows that f' is a γ_{rRpd} -function on G . Therefore, we obtain $\gamma_{rRpd}(G) = 4$.

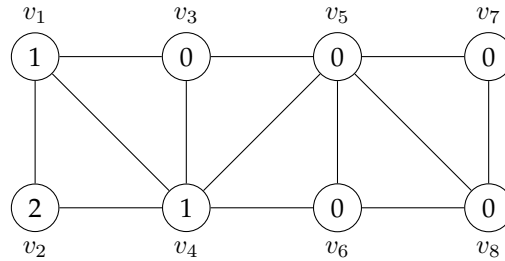


FIGURE 2. A graph G with $\gamma_{rRpd}(G) = 4$

There are several concepts and terminologies in graph theory that are not introduced in this study; hence, readers are advised to read the following papers [5–7, 13, 14]. This research study introduces a new parameter of a hop Roman dominating function on any graph called the restrained hop Roman domination. Moreover, important graph-theoretic properties of the restrained hop Roman dominating function were investigated, and realization problem and some characterizations were discussed.

3. RESULTS

In this section, important results involving the theoretical properties of the restrained hop Roman dominating function on a graph were presented.

Remark 1. Let G be a graph of order $n \geq 1$ and let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . If $\gamma_{rhR}(G) < n$, then $|V_0| \geq 2$ and for every $x \in V_0$, there exists $y \in V_0 \setminus \{x\}$ such that $xy \in E(G)$.

Theorem 2. Let G be a graph of order $n \geq 1$ and $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Then $|V_0| = 0$ if and only if $|V_2| = 0$. In that case, $\gamma_{rhR}(G) = n$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Assume that $|V_0| = 0$. Using a proof by contradiction, we suppose for a moment that $|V_2| \neq 0$. Then, let $u \in V_2$ and let $|W_0| = |V_0| = 0$, $W_1 = V_1 \cup \{u\}$, and $W_2 = V_2 \setminus \{u\}$. Since $W_0 = \emptyset$, it follows that $f' = (W_0, W_1, W_2)$ is an RHRDF on G . Thus, we obtain $\omega_G^{rhR}(f') = |W_1| + 2|W_2| = (|V_1| + 1) + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 < \omega_G^{rhR}(f) = \gamma_{rhR}(G)$. This contradicts to our assumption. Hence, we conclude that $|V_2| = 0$. Conversely, we assume that $|V_2| = 0$. Since f is a γ_{rhR} -function on G , it is forced that $|V_0| = 0$. Therefore, we end up with $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. This proves the assertion. $\square$

Corollary 3. Let G be a graph of order n and let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . If $\gamma_{rhR}(G) = n$ and $|V_0| = 0$, then $|V_1| = n$.

Theorem 4. Let G be a graph and let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Then $V_1 \cup V_2$ is a restrained hop dominating set on G .

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Then it means that f is an RHRDF on G . Hence, for each $v \in V_0$, there exists $u \in V_0 \setminus \{v\}$ and $w \in V_2$ such that $uv \in E(G)$ and $d_G(v, w) = 2$. Since $V_2 \subseteq V_1 \cup V_2$, it follows that $w \in V_1 \cup V_2$. Therefore, it suffices to conclude that $V_1 \cup V_2$ is a restrained hop dominating set on G . $\square$

Theorem 5. Let G be graph and let $f = (V_0, V_1, V_2)$ be an RHRDF on G such that $|V_1| = 0$. Then V_2 is a γ_{rh} -set on G if and only if f is a γ_{rhR} -function on G . Moreover, $\gamma_{rhR}(G) = 2\gamma_{rh}(G)$.

Proof. Let $f = (V_0, V_1 = \emptyset, V_2)$ be an RHRDF on G . Assume that V_2 is a γ_{rh} -set on G . Since $V_1 = \emptyset$, it follows that $V_0 = V(G) \setminus V_2$ and for every $a \in V_0$, there exists $b \in V_0 \setminus \{a\}$ and $c \in V_2$ such that $ab \in E(G)$ and $d_G(a, c) = 2$. Hence, it follows that f is an RHRDF on G . Seeking a contradiction. Suppose for a moment that f is not a γ_{rhR} -function on G . Then there exists an RHRDF $g = (U_0, U_1 = \emptyset, U_2)$ such that g is a γ_{rhR} -function on G . In that case, we have $\gamma_{rhR}(G) = \omega_G^{rhR}(g) = |U_1| + 2|U_2| = 2|U_2| < 2|V_2| = |V_1| + 2|V_2| = \omega_G^{rhR}(f)$. Consequently, we get $|U_2| < |V_2|$. This is a contradiction to the fact that V_2 is a γ_{rh} -set on G . Therefore, it suffices to conclude that f is a γ_{rhR} -function on G . For the converse, we assume that f is a γ_{rhR} -function on G . Then, by Theorem 4, it implies that V_2 is a restrained hop dominating set on G . Seeking a contradiction. Suppose for a moment that V_2 is not a γ_{rh} -set on G . Then there exists a restrained hop dominating set V'_2 such that V'_2 is a γ_{rh} -set on G . This means that $|V'_2| < |V_2|$. Now, let $f' = (W_0, W_1, W_2)$ be an HRDF function on G where $W_0 = V(G) \setminus V'_2$, $|W_1| = 0$ and $W_2 = V'_2$. Since V'_2 is a γ_{rh} -set on G , it follows that for each $x \in W_0$, there exists $y \in W_0 \setminus \{x\}$ and $z \in W_2$ such that $xy \in E(G)$ and $d_G(x, z) = 2$. This implies that f' is an RHRDF on G . Hence, we have $\omega_G^{rhR}(f') = |W_1| + 2|W_2| = 2|W_2| = 2|V'_2| < 2|V_2| = \omega_G^{rhR}(f) = \gamma_{rhR}(G)$. This is a contradiction to the assumption that f is a γ_{rhR} -function on G . Thus, V_2 is a γ_{rh} -set on G , that is, $\gamma_{rh}(G) = |V_2|$. Moreover, we conclude that $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = 2|V_2| = 2\gamma_{rh}(G)$. This proves the assertion. $\square$

Theorem 6. Let G be a graph of order n and $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Then $|V_0| = |V_2|$ if and only if $\gamma_{rhR}(G) = n$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . First, we assume that $|V_0| = |V_2|$. Then if $|V_0| = |V_2| = 0$, then by Theorem 2, it follows that $\gamma_{rhR}(G) = n$. Suppose that $|V_0| = |V_2| \geq 1$. Then we get $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = |V_0| + |V_1| + |V_2| = |V(G)| = n$. As for the converse, we assume that $\gamma_{rhR}(G) = n$. Seeking a contradiction. Suppose for a moment that $|V_0| \neq |V_2|$. Then we get either $|V_0| > |V_2|$ or $|V_0| < |V_2|$. If $|V_0| > |V_2|$, then $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| < |V_0| + |V_1| + |V_2| = |V(G)| = n$. If $|V_0| < |V_2|$, then $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V(G)| = n$. Both contradict our assumption. Therefore, we end up with $|V_0| = |V_2|$. $\square$

Theorem 7. Let G be a graph of order n and $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Then $|V_0| > |V_2|$ if and only if $\gamma_{rhR}(G) < n$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Then f is an RHRDF on G . Assume that $|V_0| > |V_2|$. Then it implies that $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| < |V_1| + |V_2| + |V_0| = |V(G)| = n$. On the other hand, assume that $\gamma_{rhR}(G) < n$. By Remark 1, it follows that $V_0 \neq \emptyset$. Since f is an RHRDF on G , we have that $|V_2| \geq 1$. Since $\gamma_{rhR}(G) < n$, by Theorem 5, we get $|V_0| \neq |V_2|$. Thus, either $|V_0| < |V_2|$ or $|V_0| > |V_2|$. Suppose for a moment that $|V_2| > |V_0|$, then $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| > |V_1| + |V_2| + |V_0| = |V(G)| = n$. A contradiction to the fact that $\gamma_{rhR}(G) < n$. Therefore, we have $|V_2| < |V_0|$. And the result follows. $\square$

The following corollary is a consequence of Theorem 7.

Corollary 8. Let G be a graph of order n . If $\gamma_{rhR}(G) < n$ and $|V_2| = 1$, then $|V_1| \neq 0$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Assume that $\gamma_{rhR}(G) < n$ and $|V_2| = 1$. By Theorem 7, we have that $|V_2| < |V_0|$. Seeking a contradiction. Suppose for a moment that $|V_1| = 0$. Let $V_2 = \{x\}$ and $y \in V_0$. Then $d_G(y, x) = 2$ and there exists $w \in V_0 \setminus \{y\}$ such that $yw \in E(G)$. This follows that there exists $w \in N_G(x) \cap N_G(y)$. Now, since $|V_2| = 1$ and $|V_1| = 0$, it means that $w \in V_0 \setminus \{y\}$. However, since

$w \in N_G(x)$, it implies that $N_G^2(w) \cap V_2 = \emptyset$. This indicates that $V_2 = \{u\}$ is not a restrained hop dominating set on G , a contradiction. Therefore, $|V_1| \neq 0$. This completes the proof. $\square$

Theorem 9. Let G be a graph of order n . If $\gamma_{rhR}(G) < n$, then $|V_2| \leq |V_0|$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Assume that $\gamma_{rhR}(G) < n$. Suppose, for the contrary, that $\emptyset \neq |V_0| < |V_2|$. Let $x \in V_0$. Since f is a RHRDF on G , it implies that there exists $y_x \in V_2$ and $w \in V_0 \setminus \{x\}$ such that $x \in N_G^2(y_x)$ and $x \in N_G(w)$. Let $R = \{y_x : x \in V_0\}$. Then $|R| \leq |V_0| < |V_2|$. Let $f^* = (V_0, V_1 \cup (V_2 \setminus R), R)$. Then f^* is an RHRDF on G . In that case, we get $\omega_G^{rhR}(f^*) = |V_1 \cup (V_2 \setminus R)| + 2|R| = |V_1| + |V_2| - |R| + 2|R| = |V_1| + |V_2| + |R| < |V_1| + 2|V_2| = \omega_G^{rhR}(f) = \gamma_{rhR}(G)$. This is a contradiction. Therefore, it suffices to conclude that $|V_2| \leq |V_0|$. $\square$

Proposition 10. If $G = K_n$ where n is a positive integer, then $\gamma_{rhR}(G) = n$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on $G = K_n$ where n is positive integer. Seeking a contradiction. Assume for a moment that $\gamma_{rhR}(G) < n$. Then by Remark 1, $|V_0| \geq 2$. In view Theorem 2, $V_2 \neq \emptyset$. Let $x \in V_0$. Then there exists $y \in V_2$ and $w \in V_0 \setminus \{x\}$ such that $d_G(x, y) = 2$ and $xw \in E(G)$. This is a contradiction since for every $u, v \in V(G)$, $d_G(u, v) \leq 1$. Therefore, we concluded that $\gamma_{rhR}(G) = n$. This completes the proof. $\square$

Proposition 11. Let $G = P_n$ where $n \geq 1$. Then

$$\gamma_{rhR}(G) = \begin{cases} n, & \text{if } n \leq 5, \\ \frac{2n}{3}, & \text{if } n \equiv 0 \pmod{6}, \\ \frac{2n+1}{3}, & \text{if } n \equiv 1 \pmod{6}, \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \text{ or } 5 \pmod{6}, \\ \frac{2n+3}{3}, & \text{if } n \equiv 3 \pmod{6}, \\ \frac{2n+4}{3}, & \text{if } n \equiv 4 \pmod{6}. \end{cases}$$

Proof. Assume that $G = P_n$ where $n \geq 1$ and let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . On one hand, let $n \leq 5$. Then it is easy to see that $|V_0| = |V_2| \geq 0$. By Theorem 6, we have that $\gamma_{rhR}(G) = n$. On the other hand, let $G = P_n = [x_1, x_2, \dots, x_n]$ where $n \geq 6$. Then consider the following 6 cases below:

Case 1: $n \equiv 0 \pmod{6}$

Let k be a positive integer. Then $n = 6k$ and so, $k = \frac{n}{6}$. Consider the set $A_i = \{x_{6i-3}, x_{6i-2}\}$. Then we let $U_1 = \emptyset$ and $U_2 = \bigcup_{i=1}^k A_i = \bigcup_{i=1}^{\frac{n}{6}} \{x_{6i-3}, x_{6i-2}\}$. Hence, we have that $U_0 = V(G) \setminus (U_1 \cup U_2)$. In that case, for every $v = x_p \in U_0$ there exists $u = x_q \in U_2$ and $w = x_r \in U_0 \setminus \{v\}$ such that $d_G(u, v) = 2$ and $d_G(v, w) = 1$. This follows that $\xi = (U_0, U_1, U_2)$ is an RHRDF on G . By construction, we have that ξ is a γ_{rhR} -function on G . Thus, we get $\gamma_{rhR}(G) = \omega_G^{rhR}(\xi) = |U_1| + 2|U_2| = 2|\bigcup_{i=1}^{\frac{n}{6}} \{x_{6i-3}, x_{6i-2}\}| = \frac{2n}{3}$.

Case 2: $n \equiv 1 \pmod{6}$

Let k' be a positive integer. Then $n = 6k' + 1$ and so, $k' = \frac{n-1}{6}$. Then we consider the set $A'_i = \{x_{6i-3}, x_{6i-2}\}$. Then let $U'_1 = \{x_n\}$ and $U'_2 = \bigcup_{i=1}^{k'} A'_i = \bigcup_{i=1}^{\frac{n-1}{6}} \{x_{6i-3}, x_{6i-2}\}$. Hence, we let $U'_0 = V(G) \setminus (U'_1 \cup U'_2)$. So, for every $v' = x_p \in U'_0$ there exists $u' = x_q \in U'_2$ and $w' = x_r \in U'_0 \setminus \{v'\}$ such that $d_G(u', v') = 2$ and $d_G(v', w') = 1$. This means that $\xi' = (U'_0, U'_1, U'_2)$ is an RHRDF on G . By construction, it follows that ξ' is a γ_{rhR} -function on G . So, we get $\gamma_{rhR}(G) = \omega_G^{rhR}(\xi') = |U'_1| + 2|U'_2| = |\{x_n\}| + 2|\bigcup_{i=1}^{\frac{n-1}{6}} \{x_{6i-3}, x_{6i-2}\}| = \frac{2n+1}{3}$.

Case 3: $n \equiv 2 \pmod{6}$

Let k'' be a positive integer. Then $n = 6k'' + 2$ and so, $k'' = \frac{n-2}{6}$. Set $A''_i = \{x_{6i-3}, x_{6i-2}\}$. Then we let $U''_1 = \{x_{n-1}, x_n\}$ and $U''_2 = \bigcup_{i=1}^{k''} A''_i = \bigcup_{i=1}^{\frac{n-2}{6}} \{x_{6i-3}, x_{6i-2}\}$. Hence, $U''_0 = V(G) \setminus (U''_1 \cup U''_2)$. Then

for each $v'' = x_p \in U_0''$ there exists $u'' = x_q \in U_2''$ and $w'' = x_r \in U_0'' \setminus \{v''\}$ such that $d_G(u'', v'') = 2$ and $d_G(v'', w'') = 1$. This means that $\xi'' = (U_0'', U_1'', U_2'')$ is an RHRDF on G . By construction, it implies that ξ'' is a γ_{rhR} -function on G . Therefore, we obtain $\gamma_{rhR}(G) = \omega_G^{rhR}(\xi'') = |U_1''| + 2|U_2''| = |\{x_{n-1}, x_n\}| + 2|\bigcup_{i=1}^{\frac{n-2}{6}} \{x_{6i-3}, x_{6i-2}\}| = \frac{2n+2}{3}$.

Case 4: $n \equiv 3 \pmod{6}$

Let k''' be a positive integer. Then $n = 6k''' + 3$ and we have, $k''' = \frac{n-3}{6}$. Again, we set $A_i''' = \{x_{6i-3}, x_{6i-2}\}$. Then let $U_1''' = \{x_{n-2}, x_{n-1}, x_n\}$ and $U_2''' = \bigcup_{i=1}^{k'''} A_i''' = \bigcup_{i=1}^{\frac{n-3}{6}} \{x_{6i-3}, x_{6i-2}\}$. In that case, we have that $U_0''' = V(G) \setminus (U_1''' \cup U_2''')$. Hence, for each $v''' = x_p \in U_0'''$ there exists $u''' = x_q \in U_2'''$ and $w''' = x_r \in U_0''' \setminus \{v'''\}$ such that $d_G(u''', v''') = 2$ and $d_G(v''', w''') = 1$. This implies that $\xi''' = (U_0''', U_1''', U_2''')$ is an RHRDF on G . By construction, it follows that ξ''' is a γ_{rhR} -function on G . Therefore, we have $\gamma_{rhR}(G) = \omega_G^{rhR}(\xi''') = |U_1'''| + 2|U_2'''| = |\{x_{n-2}, x_{n-1}, x_n\}| + 2|\bigcup_{i=1}^{\frac{n-3}{6}} \{x_{6i-3}, x_{6i-2}\}| = \frac{2n+3}{3}$.

Case 5: $n \equiv 4 \pmod{6}$

Let $k^{(4)}$ be a positive integer. Then $n = 6k^{(4)} + 4$ and so, $k^{(4)} = \frac{n-4}{6}$. Set $A_i^{(4)} = \{x_{6i-3}, x_{6i-2}\}$. Then we let $U_1^{(4)} = \emptyset$ and $U_2^{(4)} = \bigcup_{i=1}^{k^{(4)}+1} A_i = \bigcup_{i=1}^{\frac{n+2}{6}} \{x_{6i-3}, x_{6i-2}\}$. Hence, we have $U_0^{(4)} = V(G) \setminus (U_1^{(4)} \cup U_2^{(4)})$. Hence, for every $v^{(4)} = x_p \in U_0^{(4)}$ there exists $u^{(4)} = x_q \in U_2^{(4)}$ and $w^{(4)} = x_r \in U_0^{(4)} \setminus \{v^{(4)}\}$ such that $d_G(u^{(4)}, v^{(4)}) = 2$ and $d_G(v^{(4)}, w^{(4)}) = 1$. This follows that $\xi^{(4)} = (U_0^{(4)}, U_1^{(4)}, U_2^{(4)})$ is an RHRDF on G . By construction, $\xi^{(4)}$ is a γ_{rhR} -function on G . Hence, we obtain $\gamma_{rhR}(G) = \omega_G^{rhR}(\xi^{(4)}) = |U_1^{(4)}| + 2|U_2^{(4)}| = 2|\bigcup_{i=1}^{\frac{n+2}{6}} \{x_{6i-3}, x_{6i-2}\}| = \frac{2n+4}{3}$.

Case 6: $n \equiv 5 \pmod{6}$

Let $k^{(5)}$ be a positive integer. Then $n = 6k^{(5)} + 5$ and so, $k^{(5)} = \frac{n-5}{6}$. Set $A_i^{(5)} = \{x_{6i-3}, x_{6i-2}\}$. Then we let $U_1^{(5)} = \emptyset$ and $U_2^{(5)} = \bigcup_{i=1}^{k^{(5)}} A_i = \bigcup_{i=1}^{\frac{n-5}{6}} \{x_{6i-3}, x_{6i-2}\}$. So, we have $U_0^{(5)} = V(G) \setminus (U_1^{(5)} \cup U_2^{(5)})$. Thus, for every $v^{(5)} = x_p \in U_0^{(5)}$ there exists $u^{(5)} = x_q \in U_2^{(5)}$ and $w^{(5)} = x_r \in U_0^{(5)} \setminus \{v^{(5)}\}$ such that $d_G(u^{(5)}, v^{(5)}) = 2$ and $d_G(v^{(5)}, w^{(5)}) = 1$. This follows that $\xi^{(5)} = (U_0^{(5)}, U_1^{(5)}, U_2^{(5)})$ is an RHRDF on G . By construction, it implies that $\xi^{(5)}$ is a γ_{rhR} -function on G . Therefore, we end up with $\gamma_{rhR}(G) = \omega_G^{rhR}(\xi^{(5)}) = |U_1^{(5)}| + 2|U_2^{(5)}| = 2|\bigcup_{i=1}^{\frac{n-5}{6}} \{x_{6i-3}, x_{6i-2}\}| = \frac{2n+2}{3}$.

This completes the proof. $\square$

Proposition 12. Let $G = C_n$ where $n \geq 3$. Then

$$\gamma_{rhR}(G) = \begin{cases} n, & \text{if } n \leq 4, \\ \frac{2n}{3}, & \text{if } n \equiv 0 \pmod{6}, \\ \frac{2n+1}{3}, & \text{if } n \equiv 1 \pmod{6}, \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \text{ or } 5 \pmod{6}, \\ \frac{2n+3}{3}, & \text{if } n \equiv 3 \pmod{6}, \\ \frac{2n+4}{3}, & \text{if } n \equiv 4 \pmod{6}. \end{cases}$$

Proof. Similar to Proposition 11. $\square$

Let G be a graph. The *shadow graph* of G is denoted by $D_2(G)$ and is defined by taking two copies of the graph G , say G_1 and G_2 , and connecting every vertex $z \in V(G_1)$ to the adjacent vertex of the corresponding vertex $z' \in V(G_2)$.

Theorem 13. Let G_1 and G_2 be the two copies of a graph G . If $f = (V_0, V_1, V_2)$ is an RHRDF on G_1 , then $f' = (V_0, V_1 \cup V(G_2), V_2)$ is an RHRDF on $D_2(G)$.

Proof. Assume that $f = (V_0, V_1, V_2)$ is an RHRDF on G_1 . Let $A_0 = V_0$, $A_1 = V_1 \cup V(G_2)$ and $A_2 = V_2$. Let $f' = (A_0, A_1, A_2)$ and let $v \in A_0$. Since f is an RHRDF on G_1 , there exists $u \in V_2$ and $w \in V_0 \setminus \{v\}$ such that

$d_{G_1}(u, v) = 2$ and $vw \in E(G_1)$. Since $A_0 = V_0$ and $A_2 = V_2$, it follows that $u \in A_2$, $d_{D_2(G)}(u, v) = 2$ and $w \in A_0 \setminus \{v\}$, $vw \in E(D_2(G))$. Therefore, f' is an RHRDF on $D_2(G)$. This proves the assertion. $\square$

Corollary 14. Let H be a graph with $|V(H)| = m \geq 1$. Then $\gamma_{rhR}(D_2(H)) \leq m + \gamma_{rhR}(H)$.

Proof. Let H_1 and H_2 be the two copies of graph H of order $m \in \mathbb{N}$, and let $f = (V_0, V_1, V_2)$ be an RHRDF on H . Then f is an RHRDF on H_1 . Let $X_0 = V_0$, $X_1 = V_1 \cup V(H_2)$ and $X_2 = V_2$. Then by Theorem 13, it follows that $f' = (X_0, X_1, X_2)$ is an RHRDF on $D_2(H)$. Hence, we end up with

$$\begin{aligned} \gamma_{rhR}(D_2(H)) &\leq \omega_H^{rhR}(f) = |X_1| + 2|X_2| \\ &= (|V_1 \cup V(H_2)|) + 2|V_2| \\ &= |V_1| + |V(H_2)| + 2|V_2| \\ &= m + |V_1| + 2|V_2| \\ &= m + \gamma_{rhR}(H). \end{aligned}$$

This completes the proof. $\square$

The following result characterizes the RHRDF on any graph with several components.

Theorem 15. Let $H = H_1 \cup H_2 \cup \dots \cup H_m$ be a graph. Then $f = (V_0, V_1, V_2)$ is an RHRDF on H if and only if $f|_{H_i} = (V_0^i, V_1^i, V_2^i)$ is an RHRDF on H_i for each $i \in \{1, 2, \dots, m\}$.

Proof. Suppose that $f = (V_0, V_1, V_2)$ is an RHRDF on $H = H_1 \cup H_2 \cup \dots \cup H_m$. For all $i \in \{1, 2, \dots, m\}$, we let $V_0^i = V_0 \cap V(H_i)$, $V_1^i = V_1 \cap V(H_i)$ and $V_2^i = V_2 \cap V(H_i)$. Then $f|_{H_i} = (V_0^i, V_1^i, V_2^i)$ for each $i \in \{1, 2, \dots, m\}$. Let $x \in V_0^r$ where $r \in \{1, 2, \dots, m\}$. Then since $V_0^i \subseteq V_0$ for each $i \in \{1, 2, \dots, m\}$, it follows that $x \in V_0$. Note that f is an interior hop Roman dominating function on G . Then it implies that there exists $y \in V_2$ such that $x \in N_H^2(y)$ and there exists $w \in V_0 \setminus \{x\}$ such that $xw \in E(H)$. Since $d_H(a, b) = \infty$ for all $a \in H_p$ and for all $b \in H_q$ where $p \neq q$ and for any $p, q \in \{1, 2, \dots, m\}$. Then it implies that $y \in V_2^r$ for which $x \in N_{H_r}^2(y)$, and $w \in V_0^r \setminus \{x\}$ for which $xw \in E(H_r)$ where $r \in \{1, 2, \dots, m\}$. Hence, we have $f|_{H_i} = (V_0^i, V_1^i, V_2^i)$ is an RHRDF on H_i for each $i \in \{1, 2, \dots, m\}$. Conversely, we suppose that $f|_{H_i} = (V_0^i, V_1^i, V_2^i)$ is an RHRDF on H_i for each $i \in \{1, 2, \dots, m\}$. Set $V_0 = \bigcup_{i=1}^m V_0^i$, $V_1 = \bigcup_{i=1}^m V_1^i$ and $V_2 = \bigcup_{i=1}^m V_2^i$. Then $f = (V_0, V_1, V_2)$ is a function on $H = H_1 \cup H_2 \cup \dots \cup H_m$. Let $x' \in V_0$. Then $x' \in V_0^i$ for some $i \in \{1, 2, \dots, m\}$. Since $f|_{H_i}$ is an RHRDF on H_i , it follows that there exists $y' \in V_2^i$ such that $x' \in N_{H_i}^2(y')$ and there exists $w' \in V_0^i \setminus \{x'\}$ such that $x'w' \in E(H_i)$. Note that $V_0 = \bigcup_{i=1}^m V_0^i$ and $V_2 = \bigcup_{i=1}^m V_2^i$, it implies that $y' \in V_2$ and $w' \in V_0 \setminus \{x'\}$ for which $x' \in N_H^2(y')$ and $x'w' \in E(H)$. Therefore, f is an RHRDF on H . $\square$

The corollary below is a consequence of Theorem 15 above.

Corollary 16. Let $H = H_1 \cup H_2 \cup \dots \cup H_m$ be a graph. Then, $\gamma_{rhR}(H) = \sum_{i=1}^m \gamma_{rhR}(H_i)$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on $H = H_1 \cup H_2 \cup \dots \cup H_k$. For each $i \in \{1, 2, \dots, m\}$, we let $V_0^i = V_0 \cap V(H_i)$, $V_1^i = V_1 \cap V(H_i)$ and $V_2^i = V_2 \cap V(H_i)$. In that case, $f|_{H_i} = (V_0^i, V_1^i, V_2^i)$ is an HRDF on H_i for each $i \in \{1, 2, \dots, m\}$. By Theorem 15, it follows that $f|_{H_i}$ is an RHRDF on H_i for each $i \in \{1, 2, \dots, m\}$. Thus, we obtain

$$\begin{aligned} \gamma_{rhR}(H) &= \omega_H^{rhR}(f) = |V_1| + 2|V_2| \\ &= \sum_{i=1}^m |V_1^i| + 2 \sum_{i=1}^m |V_2^i| \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^m (|V_1^i| + 2|V_2^i|) \\
&= \sum_{i=1}^m \omega_{H_i}^{rhR}(f|_{H_i}) \\
&\geq \sum_{i=1}^m \gamma_{rhR}(H_i).
\end{aligned}$$

Now, let $f|_{H_i} = (W_0^i, W_1^i, W_2^i)$ be an RHRDF on H_i for each $i \in \{1, 2, \dots, m\}$. Set $W_0 = \bigcup_{i=1}^m W_0^i$, $W_1 = \bigcup_{i=1}^m W_1^i$ and $W_2 = \bigcup_{i=1}^m W_2^i$. Then, $f = (W_0, W_1, W_2)$ is an HRDF on H . By Theorem 15, it implies that f is an RHRDF on H . So, we obtain

$$\begin{aligned}
\sum_{i=1}^m \gamma_{rhR}(H_i) &= \sum_{i=1}^m \omega_H^{rhR}(f|_{H_i}) = \sum_{i=1}^m (|W_1^i| + 2|W_2^i|) \\
&= \sum_{i=1}^m |W_1^i| + 2 \sum_{i=1}^m |W_2^i| \\
&= |W_1| + 2|W_2| \\
&= \omega_G^{rhR}(f) \\
&\geq \gamma_{rhR}(H).
\end{aligned}$$

Therefore, we end up with $\gamma_{rhR}(H) = \sum_{i=1}^m \gamma_{rhR}(H_i)$. □

Remark 17. Let $H = H_1 \cup H_2 \cup \dots \cup H_m$ be a graph with $|V(H)| = n$. If $H_i = K_{n_i}$ where $n_i \geq 1$ for each $i \in \{1, 2, \dots, m\}$, then $\gamma_{rhR}(G) = n$. In particular, $\gamma_{rhR}(\overline{K}_n) = n$.

Theorem 18. Let G be any graph. Then the following statements hold true:

- (i) $\gamma_{rhR}(G) = 1$ if and only if $G = K_1$; and
- (ii) $\gamma_{rhR}(G) = 2$ if and only if $G \in \{\overline{K}_2, K_2\}$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . First, we assume that $\gamma_{rhR}(G) = 1$. Then we have $|V_1| + 2|V_2| = 1$. Thus, we obtain $|V_2| = 0$ and so, $|V_1| = 1$. In view of Theorem 2, we get $|V_0| = 0$. Hence, $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 1$. Accordingly, we obtain $G = K_1$. Conversely, we assume that $G = K_1$. Then it is easy to see that $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| = |V(G)| = 1$. Thus, (i) holds. Secondly, we assume that $\gamma_{rhR}(G) = 2$. Then we get $|V_1| + 2|V_2| = 2$. thus, we have that $|V_2| \leq 1$. Suppose that $|V_2| = 1$. Then $|V_1| = 0$ and $|V_0| \geq 2$. Let $V_2 = \{u\}$ and let $v \in V_0$. Then it follows that $d_G(u, v) = 2$ and there exists $w \in V_0 \setminus \{v\}$ such that $vw \in E(G)$. Now, since $d_G(u, v) = 2$, it follows that there exists $z \in N_G(u) \cap N_G(v)$. Since $V_1 = \emptyset$, it means that $z \in V_0$. However, we have that $N_G^2(z) \cap V_2 = \emptyset$. This is a contradiction since V_2 is a hop dominating set. Hence, it implies that $|V_2| = 0$. In view of Theorem 2, we get $|V_0| = 0$. Hence, $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + |V_2| = |V_1| = |V(G)| = 2$. Therefore, it suffices to say that $G \in \{\overline{K}_2, K_2\}$. Conversely, we assume that $G \in \{\overline{K}_2, K_2\}$. Then it follows that $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| = |V(G)| = 2$. Hence, (ii) is satisfied. This completes the proof. □

Theorem 19. Let G be a graph. Then $\gamma_{rhR}(G) = 3$ if and only if $G \in \{\overline{K}_3, K_1 \cup K_2, P_3, K_1 + H\}$ for which $H \in \{K_2, P_1 \cup J\}$ where J is any graph with $|V(J)| \geq 2$ and for every $v \in V(J)$ there exists $z \in V(J) \setminus \{v\}$ such that $d_G(v, z) = 1$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on a graph G . Then f is an RHRDF on G . Assume that $\gamma_{rhR}(G) = 3$. Then it follows that $|V_1| + 2|V_2| = 3$. Thus, we have that $|V_2| \leq 1$. Suppose $|V_2| = 0$. In that

case, we have $|V_1| = 3$. So, we obtain $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 3$. Since G is any graph, it follows that $G \in \{\overline{K_3}, K_1 \cup K_2, K_3, P_3\}$. Suppose $|V_2| = 1$. Then it implies that $|V_1| = 1$. Since f is an RHRDF on G , then for every $v \in V_0$ there exists $z \in V_0 \setminus \{v\}$ such that $d_G(v, z) = 1$. Let $V_2 = \{u\}$ and $V_1 = \{w\}$. Then $d_G(u, v) = 2$ and $d_G(w, v) = 1$ for every $v \in V_0$. Hence, it follows that $G = \langle V_1 \rangle + \langle V_0 \cup V_2 \rangle$. Equivalently, we have that $G = K_1 + H$ for which $H = P_1 \cup J$ where J is any graph with $|V(J)| \geq 2$ and for every $v \in V(J)$ there exists $z \in V(J) \setminus \{v\}$ such that $d_G(v, z) = 1$. Therefore, we conclude that $G \in \{\overline{K_3}, K_1 \cup K_2, P_3, K_1 + H\}$ where $H \in \{K_2, P_1 \cup J\}$ where J is any graph with $|V(J)| \geq 2$ and for every $v \in V(J)$ there exists $z \in V(J) \setminus \{v\}$ such that $d_G(v, z) = 1$. Conversely, if $G \in \{\overline{K_3}, K_1 \cup K_2, P_3, K_3\}$, then it is easy to see that $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 3$. Now, let $G = K_1 + H$ for which $H = P_1 \cup J$ where J is any graph with $|V(J)| \geq 2$ and for every $v \in V(J)$ there exists $z \in V(J) \setminus \{v\}$ such that $d_G(v, z) = 1$. Let $V_1 = V(K_1) = \{w\}$ and $V_2 = V(P_1) = \{u\}$. Then it follows that $V_0 = V(G) \setminus (V_1 \cup V_2) = V(J)$. Since for every $v \in V_0$ there exists $z \in V_0 \setminus \{v\}$ such that $d_G(v, z) = 1$ and $d_G(u, v) = 2$, it follows that $f = (V_0, V_1, V_2)$ is an RHRDF on G . Therefore, by construction, we have $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = |\{w\}| + 2|\{u\}| = 3$. This completes the proof. $\square$

The following corollary is a consequence of Theorem 19.

Corollary 20. Let $G \notin \{\overline{K_3}, K_1 \cup K_2, P_3, K_1 + H\}$ be a graph for which $H \in \{K_2, P_1 \cup J\}$ where J is any graph with $|V(J)| \geq 2$ and for every $v \in V(J)$ there exists $z \in V(J) \setminus \{v\}$ such that $d_G(v, z) = 1$ and let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Then $|V_1| = 0$ and $\gamma_{rhR}(G) = 4$ if and only if there exists a hop dominating set $\{x, y\}$ such that for every $a \in V(G) \setminus \{x, y\}$, there exists $b \in V(G) \setminus \{a, x, y\}$ such that $ab \in E(G)$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on a connected graph G . Assume that $|V_1| = 0$ and $\gamma_{rhR}(G) = 4$. Then it follows that $\omega_G^{rhR}(f) = 2|V_2| = 4$ and so, $|V_2| = 2$. Since $V_1 = \emptyset$, by Theorem 5, V_2 is a γ_{rhR} -set on G . Let $V_2 = \{x, y\}$. Then $\{x, y\}$ is a restrained hop dominating set on G , and hence, a hop dominating set on G . Let $V_0 = V(G) \setminus V_2 = \{x, y\}$. In that case, for every $a \in V(G) \setminus \{x, y\}$, there exists $b \in V(G) \setminus \{a, x, y\}$ such that $ab \in E(G)$. Conversely, assume that there exists a hop dominating set $\{x, y\}$ such that for every $a \in V(G) \setminus \{x, y\}$, there exists $b \in V(G) \setminus \{a, x, y\}$ such that $ab \in E(G)$. Then it implies that $\{x, y\}$ is a restrained hop dominating set on G . This follows that $V_0 = V(G) \setminus \{x, y\}$ and $|V_1| = 0$. Invoking Theorem 5, we have that V_2 is a γ_{rh} -set on G . In that case, we obtain $2 = |\{x, y\}| \geq |V_2|$. Suppose that $|V_2| = 0$. Then we get $\gamma_{rhR}(G) = 0$. A contradiction. Now, suppose that $|V_2| = 1$. Then by Theorem 19, $|V_1| = 1$ and $G \in \{\overline{K_3}, K_1 \cup K_2, P_3, K_1 + H\}$ for which $H \in \{K_2, P_1 \cup J\}$ where J is any graph with $|V(J)| \geq 2$ and for every $v \in V(J)$ there exists $z \in V(J) \setminus \{v\}$ such that $d_G(v, z) = 1$. This is a contradiction to our assumption. Accordingly, we obtain $|V_2| \geq 2$. Thus, we get $|V_2| = 2$ and $|V_1| = 0$. Therefore, we end up with $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = 2|V_2| = 4$. This completes the proof. $\square$

Proposition 21. If $G = K_{m,n}$ where $m, n \geq 2$, then $\gamma_{rhR}(G) = 4$.

Proof. Let $G = (V, V', E) = K_{m,n}$ be a bipartite graph where V and V' are the partition parts of the vertex sets and E denotes the edges of G . Then we have that $|V| = m \geq 2$ and $|V'| = n \geq 2$. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . And let $V = \{u_1, u_2, \dots, u_m\}$ and $V' = \{v_1, v_2, \dots, v_n\}$. Pick $u_j \in V$ for some $j \in \{1, 2, 3, \dots, m\}$. Then we get $d_G(u_j, u_i) = 2$ for every $i \neq j \in \{1, 2, 3, \dots, m\}$. This follows that $V \subseteq N_G^2[u_j]$ for each $j \in \{1, 2, 3, \dots, m\}$. In a similar manner, $V' \subseteq N_G^2[v_k]$ for each $k \in \{1, 2, 3, \dots, n\}$. In that case, we let $V_2 = \{u_j, v_k\}$ for some $j \in \{1, 2, 3, \dots, m\}$ and for some $k \in \{1, 2, 3, \dots, n\}$. This implies that $V(G) = N_G^2[u_j, v_k]$, indicating that $\{u_j, v_k\}$ is a hop dominating set on G . In addition to that, since $m, n \geq 2$, it means that for every $x \in V(G) \setminus \{u_j, v_k\}$, there exists $y \in V(G) \setminus \{x, u_j, v_k\}$ such that $xy \in E$. Hence, we have that $V_1 = \emptyset$ and $V_0 = V(G) \setminus V_2$. Invoking Corollary 20, it is concluded that $\gamma_{rhR}(G) = 4$. This completes the proof. $\square$

The next result presents the lower and upper bounds of restrained hop domination number of a graph.

Theorem 22. *Let G be a graph of order n . Then the following holds:*

$$\max\{\gamma_R(G), \gamma_{rh}(G)\} \leq \gamma_{rhR}(G) \leq \min\{2\gamma_{rh}(G), n\}.$$

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rhR}(G)$ -function on G . Then by Theorem 4, it follows that $V_1 \cup V_2$ is a restrained hop dominating set on G . Thus, we get $\gamma_{rh}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \omega_G^{rhR}(f) = \gamma_{rhR}(G)$. Since all restrained hop Roman dominating functions are also Roman dominating functions on G , it follows that $\gamma_R(G) \leq \gamma_{rhR}(G)$. Therefore, it suffices to say that $\max\{\gamma_R(G), \gamma_{rh}(G)\} \leq \gamma_{rhR}(G)$. On the other hand, we let $V_0 = \emptyset$. Since f is a γ_{rhR} -function on G , by Theorem 2, we have that $V_2 = \emptyset$. Let $f = (\emptyset, V_1 = V(G), \emptyset)$. Then f is an RHRDF on G . Hence, we get $\gamma_{rhR}(G) \leq \omega_G^{rhR}(f) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Moreover, we suppose that f is a γ_{rhR} -function on G for which $V_1 = \emptyset$. By Theorem 4, we have that V_2 is a restrained hop dominating set on G . Since f is a γ_{rhR} -function on G , by Theorem 5, we have $\gamma_{rhR}(G) = |V_2|$. Thus, we obtain $\gamma_{rhR}(G) \leq \omega_G^{rhR}(f) = |V_1| + 2|V_2| = 2|V_2| = 2\gamma_{rh}(G)$. Therefore, we end up with $\gamma_{rhR}(G) \leq \min\{n, 2\gamma_{rh}(G)\}$. And the conclusion follows. $\square$

Theorem 23. *Let G be a graph and let $f = (V_0, V_1, V_2)$ be an RHRDF on G . Then, $V_1 \cup V_2$ is a minimal restrained hop dominating set on G if and only if for every $u \in V_2$, there exists $v \in V_0$ such that $N_G^2(v) \cap V_2 = \{u\}$ or $d_G(u, w) \neq 2$ for all $w \in (V_1 \cup V_2) \setminus \{u\}$.*

Proof. Let $f = (V_0, V_1, V_2)$ be an RHRDF on G . By Theorem 4, it follows that $V_1 \cup V_2$ is a restrained hop dominating set on G . Assume that $V_1 \cup V_2$ is a minimal restrained hop dominating set on G . This means that for each $u \in V_1 \cup V_2$, $(V_1 \cup V_2) \setminus \{u\}$ is not a restrained hop dominating set on G . Thus, it follows that there exists $v \in V(G) \setminus ((V_1 \cup V_2) \setminus \{u\})$ such that $d_G(v, w) \neq 2$ for all $w \in (V_1 \cup V_2) \setminus \{u\}$. Suppose that $v \neq u$. Since $V_1 \cup V_2$ is a restrained hop dominating set, v is hop dominated by $V_1 \cup V_2$. Hence, we have that $d_G(u, v) = 2$, indicating that $N_G^2(v) \cap V_2 = \{u\}$. In addition, we suppose that $v = u$. Then it means that $d_G(u, w) \neq 2$ for every $w \in (V_1 \cup V_2) \setminus \{u\}$.

Conversely, we assume that for every $u \in V_2$, there exists $v \in V_0$ such that $N_G^2(v) \cap V_2 = \{u\}$. Then $v \in V_0$ is not hop dominated by the set $(V_1 \cup V_2) \setminus \{u\}$. This implies that $V_1 \cup V_2$ is a minimal restrained hop dominating set on G . Additionally, we assume that for every $u \in V_2$, $d_G(u, w) \neq 2$ for every $w \in (V_1 \cup V_2) \setminus \{u\}$. This follows that u can not be hop dominated for every vertex $w \in (V_1 \cup V_2) \setminus \{u\}$. Hence, $(V_1 \cup V_2) \setminus \{u\}$ is not a restrained hop dominating set on G . Therefore, we conclude that $V_1 \cup V_2$ is a minimal restrained hop dominating set on G . $\square$

Theorem 24. *Let G be a graph of order n for which $\gamma_{rh}(G) < \gamma_{rhR}(G)$. Then there exist a non-empty set $R \subseteq V(G)$ and a vertex $u \in V(G)$ such that $R \cup \{u\}$ is a γ_{rh} -set on G and $V(G) \setminus (R \cup \{u\}) \subseteq N_G^2(u)$ if and only if $\gamma_{rhR}(G) = \gamma_{rh}(G) + 1$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G for which $\gamma_{rh}(G) < \gamma_{rhR}(G)$. Assume that there exist a set R such that $\emptyset \neq R \subseteq V(G)$ and $u \in V(G)$ for which $R \cup \{u\}$ is a γ_{rh} -set on G and $V(G) \setminus (R \cup \{u\}) \subseteq N_G^2(u)$. Let $V_0 = V(G) \setminus (R \cup \{u\})$, $V_1 = R$, and $V_2 = \{u\}$. Set $f' = (V_0, V_1, V_2)$. Since $R \cup \{u\}$ is a γ_{rh} -set on G , it follows that f' is an RHRDF on G . Hence, we have that $\gamma_{rhR}(G) \leq \omega_G^{rhR}(f') = |V_1| + 2|V_2| = |R| + 2(1) = (\gamma_{rh}(G) - 1) + 2 = \gamma_{rh}(G) + 1$. Since $\gamma_{rh}(G) < \gamma_{rhR}(G)$, it follows that $\gamma_{rh}(G) = \gamma_{rhR}(G) + 1$.

On the other hand, we assume that $\gamma_{rhR}(G) = \gamma_{rh}(G) + 1$. Note that $\gamma_{rh}(G) \leq |V_1| + |V_2|$. First, we suppose that $\gamma_{rh}(G) = |V_1| + |V_2|$. Then it implies that $V_1 \cup V_2$ is a γ_{rh} -set on G . Hence, it follows that $|V_1| + 2|V_2| = |V_1| + |V_2| + 1$. This means that $|V_2| = 1$ and $|V_1| = \gamma_{rh}(G) - 1$. Let $V_2 = \{u\}$ and $R = V_1$. By Corollary 8, the set $V_1 = R$ is a non-empty set. Hence, we have that $V_0 = V(G) \setminus (V_1 \cup V_2) = V(G) \setminus (R \cup \{u\})$. Let $v \in V_0$. Since β is an RHRDF on G , we have that $d_G(u, v) = 2$ and there exists $w \in V_0 \setminus \{v\}$ such that

$vv \in E(G)$. This follows that $V(G) \setminus (R \cup \{u\}) \subseteq N_G^2(u)$. Secondly, we suppose that $\gamma_{rh}(G) < |V_1| + |V_2|$. Then we have $\gamma_{rh}(G) + 1 \leq |V_1| + |V_2|$. Since $\gamma_{rhR}(G) = \gamma_{rh}(G) + 1$, it follows that $|V_1| + 2|V_2| \leq |V_1| + |V_2|$. Hence, we have $|V_2| = 0$. In view of Theorem 2, it implies that $|V_0| = 0$. Hence, $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| = |V(G)| = n$. Thus, we have that $\gamma_{rh}(G) = n - 1$. Now, let $S = V(G) \setminus \{v\}$ be a γ_{rh} -set on G . This implies that there exists $u \in S$ such that $d_G(u, v) = 2$. In that case, we let $R = S \setminus \{u\}$. Therefore, we have $R \cup \{u\} = S$ is γ_{rh} -set on G and so, $V(G) \setminus (R \cup \{u\}) = \{v\} \subseteq N_G^2(u)$. This completes the proof. $\square$

Theorem 25. Let H be any non-complete graph. Then $f = (V_0, V_1, V_2)$ is a RHRDF on $G = K_n + H$ where n is a positive integer if and only if the following holds: (i) $V(K_n) \subseteq V_1 \cup V_2$; and (ii) $f|_H = (V_0 = V_0^H, V_1^H, V_2^H)$ is an RRPNDF on H where $V_i^H = V_i \cap V(H)$ for each $i \in \{0, 1, 2\}$.

Proof. Assume that $f = (V_0, V_1, V_2)$ is an RHRDF on G . Then f is an HRDF on G . Let $x \in V(K_n)$. Since $xy \in E(G)$ for all $y \in V(G) \setminus \{x\}$ and f is an HRDF on G , it suffices to say that x is not in V_0 . Thus, we have $x \in V_1 \cup V_2$. Hence, (i) holds. Additionally, if we let $V_i^H = V_i \cap V(H)$ for each $i \in \{0, 1, 2\}$, then we have that $f|_H = (V_0 = V_0^H, V_1^H, V_2^H)$. Let $v \in V_0$. Then $v \in V_0^H$. Since f is an HRDF on G , there exists $u \in V_2 \cap N_G^2(v)$. Since for all $z \in V(K_n)$, $zv \in E(G)$, it follows that $u \in V_2^H \cap N_G^2(v)$. This indicates that there exists $u \in V_2^H \setminus N_H(v)$. This implies that $f|_H$ is an RPNDF on H . Since f is an RHRDF on G , there exists $w \in V_0 \setminus \{v\}$ such that $d_G(v, w) = 1$. And since $V_0 = V_0^H$, it means that there exists $w \in V_0^H \setminus \{v\}$ such that $d_H(v, w) = 1$. So, we therefore conclude that $f|_H$ is an RRPNDF on H . Conversely, we assume that (i) and (ii) both hold. Let $v \in V_0$. Then by (ii), $v \in V_0^H$. Since $f|_H$ is an RRPNDF on H by (ii), there exists $u \in V_2^H \setminus N_H(v)$ and there exists $w \in V_0^H \setminus \{v\}$ such that $d_H(v, w) = 1$. Since $V_i^H \subseteq V_i$ for all $i \in \{0, 1, 2\}$ and by (i), it follows that $u \in V_2 \cap N_G^2(v)$. Since $V_0 = V_0^H$, it also indicates that $w \in V_0 \setminus \{v\}$ for which $d_G(v, w) = 1$. Therefore, we can conclude that $f = (V_0, V_1, V_2)$ is an RHRDF on G . $\square$

Corollary 26. Let $G = K_n + H$ be a graph where H is a non-complete graph and n is a positive integer. Then

$$\gamma_{rhR}(G) = n + \gamma_{rRpnd}(H).$$

Proof. Let $g = (W_0, W_1, W_2)$ be a γ_{rRpnd} -function on H . Then it follows that $f = (W_0, V(K_n) \cup W_1, W_2)$ is an SchRDF on G by Theorem 25. Hence, we get

$$\begin{aligned} \gamma_{rhR}(G) &\leq \omega_G^{rhR}(f) = |V(K_n) \cup W_1| + 2|W_2| \\ &= n + |W_1| + 2|W_2| \\ &= n + \omega_H^{rRpnd}(g) \\ &= n + \gamma_{rRpnd}(H). \end{aligned}$$

On the other hand, we suppose that $h = (V_0, V_1, V_2)$ is a γ_{rhR} -function on G . By Theorem 25 and for the reason that h is a γ_{rhR} -function on G , we are forced that $V(K_n) \subseteq V_1$. Hence, it follows that $h = (V_0, V(K_n) \cup V_1^H, V_2)$, where $V_1^H = V_1 \cap V(H)$. Invoking Theorem 25, we have that $h|_H = (V_0, V_1^H, V_2)$ is an RRPNDF on H . Thus, we obtain

$$\begin{aligned} \gamma_{rhR}(G) &= \omega_G^{rhR}(h) = |V(K_n) \cup V_1^H| + 2|V_2| \\ &= n + |V_1^H| + 2|V_2| \\ &= n + \omega_H^{rRpnd}(h|_H) \\ &\geq n + \gamma_{rRpnd}(H). \end{aligned}$$

Therefore, the desired equality holds. $\square$

Theorem 27. Let G and H be any non-complete graphs. If $f|_G = (V_0^G, V_1^G, V_2^G)$ and $f|_H = (V_0^H, V_1^H, V_2^H)$ are RRPNDF on G and H , respectively, where $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$ for any $i \in \{0, 1, 2\}$, then $f = (V_0, V_1, V_2)$ is an RHRDF on $G + H$.

Proof. Let $f = (V_0, V_1, V_2)$ be a function on $G + H$. For each $i \in \{0, 1, 2\}$, let $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$. Then $f|_G = (V_0^G, V_1^G, V_2^G)$ and $f|_H = (V_0^H, V_1^H, V_2^H)$ are functions on G and H , respectively. Assume that $f|_G = (V_0^G, V_1^G, V_2^G)$ and $f|_H = (V_0^H, V_1^H, V_2^H)$ are RRPNDF on G and H , respectively. Then it implies that $f|_G$ and $f|_H$ are RPNDF on G and H , respectively. Let $v \in V_0$. Then we have that either $v \in V_0^G$ or $v \in V_0^H$. Without loss of generality, we consider $v \in V_0^G$. Since $f|_G$ are RPNDF on G , it indicates that there exists $u \in V_2^G \setminus N_G(v)$. This follows that there exists $u \in V_2 \cap N_{G+H}^2(v)$. Hence, f is an HRDF on G . Since $f|_G$ is an RRPNDF on G , there exists $w \in V_0^G \setminus \{v\}$ such that $d_G(v, w) = 1$. This yields that there exists $w \in V_0 \setminus \{v\}$ such that $d_{G+H}(v, w) = 1$. Therefore, it suffices to conclude that f is an RHRDF on G . This proves the assertion. $\square$

We have to note that the converse of Theorem 27 is not always true since f is still RHRDF even if V_0^G can be an independent set and V_0^H is not empty. However, in particular, if V_0^G is an independent set, then $f|_G$ is not an RRPNDF on G . The next result is an immediate consequence of Theorem 27.

Corollary 28. Let G and H be any non-complete graphs. Then $\gamma_{rhR}(G + H) \leq \gamma_{rRpd}(G) + \gamma_{rRpd}(H)$.

Proof. Let $f|_G = (V_0^G, V_1^G, V_2^G)$ and $f|_H = (V_0^H, V_1^H, V_2^H)$ be γ_{rRpd} -functions on G and H , respectively. In addition, we let $V_i = V_i^G \cup V_i^H$ for each $i \in \{0, 1, 2\}$. Applying the idea of Theorem 27, it follows that $f = (V_0, V_1, V_2)$ is an RHRDF on G . Hence, we obtain

$$\begin{aligned} \gamma_{rhR}(G + H) &\leq \omega_{G+H}^{rhR}(f) = |V_1| + 2|V_2| \\ &= |V_1^G \cup V_1^H| + 2|V_2^G \cup V_2^H| \\ &= (|V_1^G| + 2|V_2^G|) + (|V_1^H| + 2|V_2^H|) \\ &= \omega_G^{rRpd}(f|_G) + \omega_H^{rRpd}(f|_H) \\ &= \gamma_{rRpd}(G) + \gamma_{rRpd}(H). \end{aligned}$$

This establishes the desired inequality. $\square$

Remark 29. Let G and H be complete graphs of order n and m , respectively. Then $\gamma_{rhR}(G + H) = n + m$.

The next result is a realization problem for hop Roman domination and restrained hop Roman domination numbers.

Theorem 30. Let a, b and n be positive integers with $1 \leq a \leq b \leq n$. Then there exists a connected graph G of order n such that $\gamma_{hR}(G) = a$ and $\gamma_{rhR}(G) = b$.

Proof. Suppose that G is a graph of order $n \geq 1$, and let a and b be positive integers satisfying the condition $1 \leq a \leq b \leq m$. Let $f' = (V'_0, V'_1, V'_2)$ be a γ_{hR} -function on G and $f = (V_0, V_1, V_2)$ be a γ_{rhR} -function on G . Additionally, we suppose that $\gamma_{hR}(G) = a$ and $\gamma_{rhR}(G) = b$. Then the following cases are considered:

Case 1. $1 \leq a \leq b = n$

Here, we let $G = K_1$. Then it is easy to check that $\gamma_{hR}(G) = 1 = \gamma_{rhR}(G)$. Let $G = K_1 + \overline{K_m}$ where $m \geq 3$. Then $|V(G)| \geq 4$. Let $x \in V(G) \setminus V(K_1) = V(\overline{K_m})$. Then we let $V'_2 = \{x\}$. In that case, we have that $V(\overline{K_m}) \setminus \{x\} \subseteq N_G^2(V'_2)$. This follows that $V'_1 = V(K_1)$ and $V'_0 = V(\overline{K_m}) \setminus \{x\}$. Hence, by construction, it implies that $\gamma_{hR}(G) = \omega_G^{hR}(f') = |V'_1| + 2|V'_2| = |V(K_1)| + 2|\{x\}| = 3$. Notice that for every $a \in V(\overline{K_m})$, there does not exists a vertex in $V(\overline{K_m}) \setminus \{a\}$ such that the distance is 1 from a . Hence, it is forced that $V_0 = \emptyset$, $V_1 = V(G)$ and $V_2 = \emptyset$. In view of Theorem 2, we obtain $\gamma_{rhR}(G) = n$. Consequently, we conclude

that $1 < \gamma_{hR}(H) < \gamma_{rhR}(G) = n$. So, Case 1 holds.

Case 2. $1 < a < b < n$

Let $G = K_1 + H$ where $H = P_m \cup \overline{K_2}$ and $m \geq 2$. Then $|V(G)| = m + 3 = n \geq 5$. Let $x \in V(\overline{K_2})$. Then $x \in V(G)$ and it follows that $V(G) \setminus V(K_1) \subseteq N_G^2[x]$. Hence, we set $V'_0 = V(G) \setminus (\{x\} \cup V(K_1))$, $V'_1 = V(K_1)$ and $V'_2 = \{x\}$. By construction, we have that $\gamma_{hR}(G) = \omega_G^{hR}(f') = |V'_1| + 2|V'_2| = |V(K_1)| + 2|\{x\}| = 3$. Moreover, we let $V_2 = \{x\}$. Note that there does not exist a vertex in $V(P_m)$ that is adjacent to $y \in V(\overline{K_2}) \setminus \{x\}$. In that case, it is forced that $V_1 = V(K_1) \cup \{x\}$ and so, $V_0 = V(P_m)$. By construction, we get $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = |V(K_1) \cup \{x\}| + 2|\{x\}| = 4$. Since $n \geq 5$, we have that $1 < \gamma_{hR}(G) < \gamma_{rhR}(G) < n$. Therefore, Case 2 holds.

Case 3. $1 < a = b < n$

Let $G = F_m$ where $m \geq 4$. Then $|V(G)| = n = m + 1 \geq 5$ and $G = K_1 + P_m$. Let $P_m = [x_1, x_2, \dots, x_m]$. Let $V'_2 = \{x_1\}$. Then it follows that $V(P_m) \setminus \{x_2\} \subseteq N_G^2[x_1]$. This implies that $V'_1 = V(K_1) \cup \{x_2\}$ and thus, $V'_0 = V(P_m) \setminus \{x_1, x_2\}$. By construction, it yields that $\gamma_{hR}(H) = \omega_H^{hR}(f') = |V'_1| + 2|V'_2| = |V(K_1) \cup \{x_2\}| + 2|\{x_1\}| = 4$. It is worthy to note that for every $x \in V(P_m) \setminus \{x_1, x_2\}$, there exists $y \in V(P_m) \setminus \{x, x_1, x_2\}$ such that $d_G(x, y) = 1$. Consequently, we also have that $V_2 = V'_2 = \{x_1\}$, $V_1 = V'_1 = V(K_1) \cup \{x_2\}$ and $V_0 = V'_0 = V(P_m) \setminus \{x_1, x_2\}$. Therefore, by construction, we end up with $\gamma_{rhR}(G) = \omega_G^{rhR}(f) = |V_1| + 2|V_2| = |V(K_1) \cup \{x_2\}| + 2|\{x_1\}| = 4$. Since $n \geq 5$, it indicates that $1 < \gamma_{hR}(G) = \gamma_{rhR}(G) < n$. Accordingly, Case 3 holds.

And the conclusion follows. $\square$

The corollaries below are direct consequences of Theorem 30.

Corollary 31. *Let G be a graph. Then $\gamma_{rhR}(G) - \gamma_{hR}(G)$ can be made arbitrarily large.*

Corollary 32. *Let G be a graph of order $n \geq 1$. Then $n - \gamma_{rhR}(G)$ can be made arbitrarily large.*

4. CONCLUSION

This research paper introduced and investigated a new parameter of hop Roman domination called the restrained hop Roman dominating function on any graph. The study constructed some characterizations of some small values of the restrained hop Roman domination number and obtained some bounds for any graph class. In addition, some combinatorial results for a restrained hop Roman dominating function in the join of two graphs were constructed. Furthermore, a realization problem for the restrained hop Roman domination number and hop Roman domination number was constructed. As for further studies, the said new parameter can be investigated for some binary operations in graphs that include corona, Cartesian, and lexicographic products.

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