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STABILITY CHAOS AND PERIODIC SOLUTION OF DELAYED RATIONAL RECURSIVE SEQUENCE

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ABSTRACT. In this paper, we will investigate a non-linear rational difference equation of higher order. Our concentration is on invariant intervals, periodic character, the character of semi cycles and global asymptotic stability of all positive solutions of

$$s_{n+1} = As_n + Bs_{n-l} + \frac{\alpha + \beta s_{n-k}}{A_0 + B_0 s_{n-k}}, \quad n = 0, 1, ...,$$

where the parameters A, B, α, β and A_0, B_0 and the initial conditions $s_{-r},\ s_{-r+1},\ s_{-r+2}, ...,\ s_0$ are arbitrary positive real numbers, $r = max\{l, k\}$. Finally, we study the global stability of this equation through numerically solved examples and confirm our theoretical discussion through it.

1. INTRODUCTION

Our aim is to investigate the global stability character and the periodicity of the solutions of the following rational higher order difference equation

$$s_{n+1} = As_n + Bs_{n-l} + \frac{\alpha + \beta s_{n-k}}{A_0 + B_0 s_{n-k}}, \quad n = 0, 1, ..., \tag{1}$$

where the parameters $A_0,\ B_0$, α, β and A, B and the initial conditions $s_{-k},\ s_{-k+1}, ..., s_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$ is a positive integer and the initial conditions $s_{-k},\ s_{-k+1}, ..., s_0$ are non-negative real numbers.

Discrete dynamical systems or difference equations are varied field because various biological systems naturally leads to their study by means of a discrete variable. Every dynamical system $u_{n+1} = f(u_n)$ determines a difference equation and vice versa. The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have appeared recently in many areas. The theory of difference equations occupies a central position in applied sciences and applicable areas. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations, which model various diverse phenomena in biology, ecology, physiology,

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physics, engineering, economics, probability theory, genetics, psychology and resource management. It is very interesting to investigate the behaviour of solutions of a higher-order rational difference equation and to discuss the local asymptotic stability of its equilibrium points. Rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such equations. For more results for the rational difference equations, we refer the interested reader to [6-10].
"Saleh and Baha et al. [27] investigated the global attractivity of the following rational recursive sequence

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Ax_n + Bx_{n-k}}.$$

Several other researchers have studied the behavior of the solution of difference equations, for example, in [15] Elsayed et al. investigated the solution of the following non-linear difference equation.

$$z_{n+1} = az_n + \frac{bz_n^2}{cz_n + dz_{n-1}^2}.$$

Elabbasy et al. [16] studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation.

$$x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{Ax_{n-l} + Bx_{n-k}}.$$

Keratas et al.[20] gave the solution of the following difference equation

$$y_{n+1} = \frac{y_{n-5}}{1 + y_{n-2}y_{n-5}}.$$

Elabbasy et al. [17] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} + cx_{n-q}}.$$

Yalçınkaya et al. [18] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Wang et. al. [19] existence and uniqueness of the positive solutions and the asymptotic behaviour of the equilibrium points of the fuzzy difference equation

$$y_{n+1} = \frac{Ax_{n-1}x_{n-2}}{D + Bx_{n-3} + Cx_{n-4}}.$$

where x_n is a sequence of positive fuzzy numbers, the parameters $A,\ B,\ C,\ D$ and the initial conditions $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are positive fuzzy numbers.

Elsayed et al. [22] studied the global behaviour of rational recursive sequence

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-k} + cx_{n-s}}{d + ex_{n-t}}.$$

where the initial conditions $x_{-r},\ x_{-r+1},\ x_{-r+2},\ ...,\ x_0$ are arbitrary positive real numbers, $r = max\{l, k, s, t\}$ is non-negative integer and a, b, c, d, e are positive constants:

L.alseda et al. [4] studied the following rational difference equation

$$x_{n+1}=\frac{1}{A_n+x_n}.$$

Q. Wang et al. [31] investigated the local stability, asymptotic behaviour, periodicity and oscillation of solutions for the difference equation

$$x_{n+1}=\frac{Ax_{n-k}}{B+C\Pi_{i=0}^{k}x_{n-i}}.$$

with the initial conditions $x_{-i}=b_{-i},\ i=0,1,2,...,k$, where k is a non-negative integer, $b_{-k},\ b_{-k+1},...,b_0$ are given $k+1$ constants, $A,\ B,\ C$ are positive constants.
As a matter of fact, numerous papers negotiate with the problem of solving non-linear difference equations in any way possible, see, for instance [7]-[15]. The long-term behaviour and solutions of rational difference equations of order greater than one has been extensively studied during the last decade. For example, various results about periodicity, boundedness, stability, and closed form solution of the second-order rational difference equations, see [5-9, 21-29].

Other related work on rational difference equations see in refs. [30,31,32].

2. Preliminaries and Definitions

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F:I^{k+1}\to I,$$

be a continuously differentiable function. Then for every set of initial conditions $s_{-k},s_{-k+1},...,s_0\in I$, the difference equation

$$s_{n+1}=F(s_n,s_{n-1},...,s_{n-k}),\ \ n=0,1,..., \tag{2}$$

has a unique solution $\{s_n\}_{n=-k}^{\infty}$.

Definition 1. *(Equilibrium Point)*

A point $\overline{s}\in I$ is called an equilibrium point of Eq.(2.1) if

$$\overline{s}=F(\overline{s},\overline{s},...,\overline{s}).$$

That is, $s_n=\overline{s}$ for $n\geq 0$, is a solution of Eq.(2.1), or equivalently, $\overline{s}$ is a fixed point of F.

Definition 2. *(Periodicity)*

A Sequence $\{s_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $s_{n+p}=s_n$ for all $n\geq -k$.

Definition 3. *(Stability)*

(i) The equilibrium point $\overline{s}$ of Eq.(2.1) is locally stable if for every $\epsilon>0$, there exists $\delta>0$ such that for all $s_{-k},s_{-k+1},...,s_{-1},\ s_0\in I$ with

$$|s_{-k}-\overline{s}|+|s_{-k+1}-\overline{s}|+...+|s_0-\overline{s}|<\delta,$$

we have

$$|x_n-\overline{x}|<\epsilon \quad \textit{for all} \quad n\geq -k.$$

(ii) The equilibrium point $\overline{s}$ *of Eq.(2.1) is locally asymptotically stable if* $\overline{s}$ *is locally stable solution of Eq.(2.1) and there exists* $\gamma > 0$*, such that for all* $s_{-k},\ s_{-k+1}, ..., s_{-1}, s_0 \in I$ *with*

$$|s_{-k} - \overline{s}| + |s_{-k+1} - \overline{s}| + ... + |s_0 - \overline{s}| < \gamma,$$

we have

$$\lim_{n\to\infty} s_n = \overline{s}.$$

(iii) The equilibrium point $\overline{s}$ *of Eq.(2.1) is global attractor if for all* $s_{-k},\ s_{-k+1}, ..., s_{-1}, s_0 \in I$*, we have*

$$\lim_{n\to\infty} s_n = \overline{s}.$$

(iv) The equilibrium point $\overline{s}$ *of Eq.(2.1) is globally asymptotically stable if* $\overline{s}$ *is locally stable and* $\overline{s}$ *is also a global attractor of Eq.(2.1).*
(v) The equilibrium point $\overline{s}$ *of Eq.(2.1) is unstable if* $\overline{s}$ *is not locally stable.*
(vi) The linearized equation of Eq.(2.1) about the equilibrium $\overline{s}$ *is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{k} \tfrac{\partial F(\overline{s},\overline{s},...,\overline{s})}{\partial s_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [27]: Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1, \tag{4}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$s_{n+1} + ps_n + qs_{n-k} = 0, \quad n = 0, 1, ...$$

The following theorem will be useful for the proof of our results in this paper.
Theorem B [28]: Let $f\ : [a, b]^{k+1} \to [a, b]$ be a continuous function, where k is a positive integer and $[a, b]$ is an interval of real numbers. Consider the difference equation

$$s_{n+1} = f(s_n, s_{n-1}, ..., s_{n-k}),\ \ n = 0, 1, ... \tag{5}$$

Suppose that f satisfies the following conditions:

(i) For every integer i with $1 \le i \le k+1$, the function $f(s_1, s_2, ..., s_{k+1})$ is weakly monotonic in z_i , for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(ii) If $m,\ M$ is a solution of the system

$$m = f(m_1, m_2, ..., m_{k+1}) \text{ and } M = f(M_1, M_2, ..., M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, ..., k+1$, we set

$$\begin{aligned} m_i &= \{m \text{ if } f \text{ is non-decreasing in } z_i \\ &= \{M \text{ if } f \text{ is non-increasing in } z_i \end{aligned}$$

and

$$\begin{aligned} M_i &= \{M \text{ if } f \text{ is non-decreasing in } z_i \\ &= \{m \text{ if } f \text{ is non-increasing in } z_i \end{aligned}$$

Then Eq. (2.4) has a unique equilibrium $\overline{s} \in [a, b]$ and every solution of Eq.(2.4) converges to $\overline{s}$.

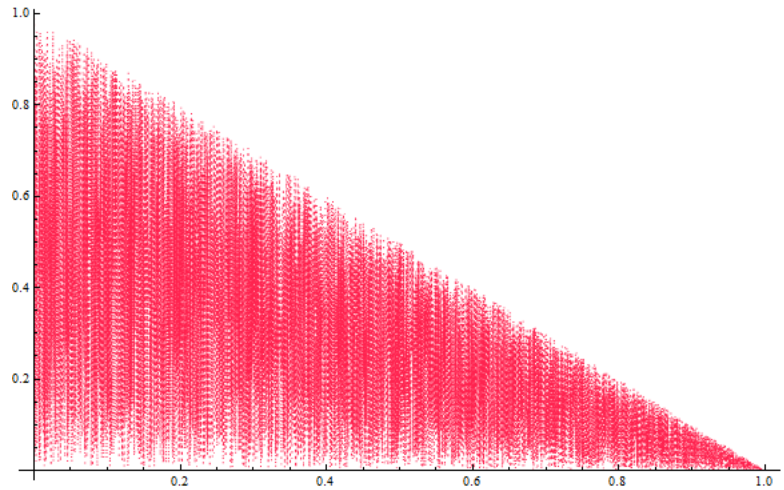

FIGURE 1. Parameters (A, B) plot for which the positive equilibrium exists

3. EQUILIBRIUM POINTS OF EQ.(1.1)

In this section we will study the equilibrium points of Eq.(1.1). The equilibrium points of Eq.(1.1) are the positive solutions of the equation

$$\overline{s} = A\overline{s} + B\overline{s} + \frac{\alpha + \beta\overline{s}}{A_0 + B_0\overline{s}}$$

or,

$$B_0(1 - A - B)\overline{s}^2 + (A_0 - AA_0 - A_0B - \beta)\overline{s} - \alpha = 0$$

if $0 < A < 1 \& 0 < B < 1 - A$ then the only positive equilibrium point of Eq.(1.1) is given by

$$\overline{s} = \frac{(A_0A + A_0B + \beta - A_0) + \sqrt{(A_0A + A_0B + \beta - A_0)^2 + 4\alpha B_0(1 - A - B)}}{2B_0(1 - A - B)}$$

A set of parameters (A and B) for which the positive equilibrium exists are obtained and figured out in the Fig. 1.

To find the linearization for our problem, consider

$$f(u, v, w) = Au + Bv + \frac{\alpha + \beta w}{A_0 + B_0 w}$$

Now,

$$\begin{aligned}
\frac{\partial f(u, v, w)}{\partial u} &= A\ , \\
\frac{\partial f(u, v, w)}{\partial v} &= B, \\
\frac{\partial f(u, v, w)}{\partial w} &= \frac{(\beta A_0 - B_0\alpha)}{(A_0 + B_0 w)^2}\ .
\end{aligned}$$

Hence, for $\overline{s} = \frac{(A_0A + A_0B + \beta - A_0) + \sqrt{(A_0A + A_0B + \beta - A_0)^2 + 4\alpha B_0(1 - A - B)}}{2B_0(1 - A - B)}$,

$$f_u(\overline{s},\ \overline{s},\ \overline{s}) = A = p_1,$$

$$f_v(\overline{s},\ \overline{s},\ \overline{s}) = B = p_2.$$

and

$$f_w(\overline{s},\ \overline{s},\ \overline{s}) = \frac{(\beta A_0 - B_0\alpha)}{(A_0 + B_0\overline{s})^2} = p_3.$$

So, the linearized equation about the $\overline{s} = \frac{(A_0A+A_0B+\beta-A_0)+\sqrt{(A_0A+A_0B+\beta-A_0)^2+4\alpha B_0(1-A-B)}}{2B_0(1-A-B)}$

(6) $$y_{n+1} - Ay_{n-1} - By_n - \left(\frac{(\beta A_0 - B_0\alpha)}{(A_0+B_0\overline{x})^2}\right) y_{n-2} = 0$$

4. Local Asymptotic Stability of the Equilibrium of the Eq.(1.1)

Local stability about $\overline{s} = \frac{(A_0A+A_0B+\beta-A_0)+\sqrt{(A_0A+A_0B+\beta-A_0)^2+4\alpha B_0(1-A-B)}}{2B_0(1-A-B)}$

Theorem 1. *Eq.(1.1) is locally asypmtotically stable if and only if.*

(7) $$A + B < 1$$

Proof. Let $f : (0,\infty)^3 \to (0,\infty)$ be a continuous function defined by

(8) $$f(u,v,w) = Au + Bv + \frac{\alpha + \beta w}{A_0 + B_0 w}.$$

It follows from **Theorem A** that, the equilibrium $\bar{s}$ of the Eq.(1.1) is locally asymptotically stable $\Leftrightarrow$

$$|p_1| + |p_2| + |p_3| < 1.$$

Here at the positive equilibrium $\bar{s}$,
$p_1 = A$, $p_2 = B$ and $p_3 = -\frac{4(A+B-1)^2(\alpha\text{B0}-\text{A0}\beta)}{\left(\sqrt{(\text{A0}(A+B-1)+\beta)^2-4\alpha\text{B0}(A+B-1)}-\text{A0}(A+B-1)+\beta\right)^2}$

Thus

$$|A| + |B| + \left| -\frac{4(A+B-1)^2(\alpha B_0 - A_0\beta)}{\left(\sqrt{(A_0(A+B-1)+\beta)^2 - 4\alpha B_0(A+B-1)} - A_0(A+B-1) + \beta\right)^2} \right| < 1,$$

On simple algebraic simplification, we get,

$$A + B < 1$$

∎

Here we present an example with different pairs of (l,k) with fixed $A = 0.771941$ and $B = 0.115124$ of local stability of the positive equilibrium.

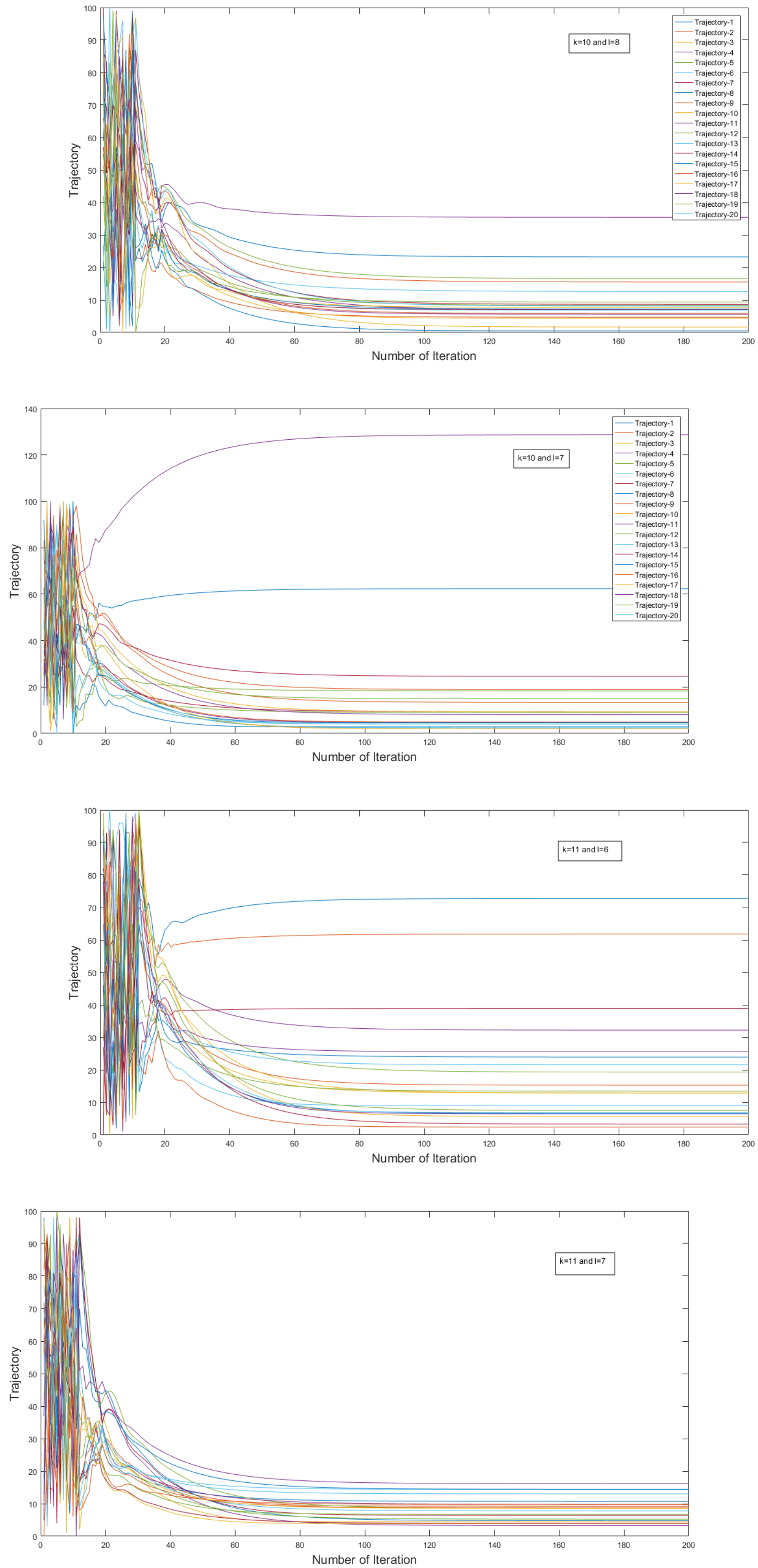

FIGURE 2. Locally Asymptotically stable trajectories for different initial values with four possible pairs of l and k.

In all the above four cases of even and odd pair of (l, k), the trajectories for different choice of parameters except (A, B) are locally asymptotically stable as shown in the Fig. 2.

5. Existence of Periodic solution

In this section, we will investigate positive prime period two solution of Eq.(1.1). The following theorems states the necessary and sufficient conditions that this equation has periodic solution of prime period two.

Theorem 2. *Eq.(1.1) has positive prime period two solution if and only if*

$$(i)\quad [A_0(B-1)-(A_0A+\beta)]^2(B-A-1)+4A\{A_0(1-B)[A_0(B-1)-(A_0A+\beta)]+\alpha AB_0\}>0,$$
$$(l-odd,\ k-even)$$

$$(ii)\ (1+A+B)[(\beta-A_0(1+A+B)]^2-4[A_0(A+B)(A_0(1+A+B)-\beta)-\beta_0\alpha]>0,\ (l-Even,\quad k-odd)$$

$$(iii)\ \ (1+A+B)[(\beta-A_0(B-A-1)]^2-4(1-B)[4A_0(A_0(1+A-B)-\beta)+\alpha B_0(B-1)]>0,\quad (l,k-odd)$$

$$(iv)\ \ (1+A+B)[-\beta-A_0(1+A+B)]^2-4(A+B)[A_0(\beta+A_0)+(A+B)(A_0^2+\alpha B_0)]>0,\quad (l,k-\ even)$$

we will prove just the 1^{st}case, the remaining cases are the same.

Proof. (i) Let us assume the two cycle period of Eq.(1.1) will be in the form

$$...p,\ q,\ p,\ q...$$

$$s_{n+1}=s_{n-k} \tag{9}$$

we get, from Eq.(1.1)

$$\begin{aligned} p &= Aq+Bp+\left(\tfrac{\alpha+\beta q}{A_0+B_0q}\right) \\ q &= Ap+Bq+\left(\tfrac{\alpha+\beta p}{A_0+B_0p}\right) \end{aligned}$$

This transform to

$$A_0p+B_0pq = aA_0q+aB_0q^2+\alpha+\beta q \quad \text{and} \tag{10}$$
$$A_0q+B_0pq = aA_0p+aB_0p^2+\alpha+\beta p. \tag{11}$$

by subtracting (5.3) from (5.2) it gives,

$$A_0(p-q)+AB_0(p^2-q^2)=A_0B(p-q)-AA_0(p-q)-\beta(p-q)$$

Then since $p\neq q$, it follows that

$$p+q=\tfrac{A_0(B-A-1)-\beta}{AB_0}. \tag{12}$$

Again, adding (5.3) and (5.2) yields,

$$\begin{aligned} A_0(p+q)+2B_0pq &= (AA_0+A_0A)(p+q)+AB_0(p^2+q^2)+2BB_0pq+2\alpha+\beta(p+q), \\ AB_0(p^2+q^2) &= A_0(p+q)+2B_0pq-(AA_0+A_0B)(p+q)-2BB_0pq-2\alpha-\beta(p+q), \end{aligned}$$

$$AB_0(p^2+q^2) = (A_0 - AA_0 - A_0B - \beta)(p+q) + 2B_0pq - 2B_0pq - 2\alpha. \tag{13}$$

By using (5.4), (5.5) and the relation

$$p^2+q^2 = (p+q)^2 - 2pq, \qquad \text{for all } \ p,q \in R,$$

we get

$$AB_0((p+q)^2 - 2pq) = (A_0 - AA_0 - A_0B - \beta)(p+q) + 2B_0pq - 2BB_0pq - 2\alpha$$
$$\tfrac{(A_0(B-A-1)-\beta)^2}{AB_0} - 2AB_0pq = (A_0 - AA_0 - A_0B - \beta)\left(\tfrac{A_0(B-A-1)-\beta}{AB_0}\right) - 2\alpha.$$

So,

$$pq = -\tfrac{A_0(B-1)[A_0(B-1)-(A_0A+\beta)]+\alpha AB_0}{AB_0^2(B-A-1)}. \tag{14}$$

Now, it is obvious from Eq.(5.3) and Eq.(5.6) that, p and q are two distinct real roots of the quadratic equation

$$t^2 - \left(\tfrac{A_0(B-A-1)-\beta}{AB_0}\right)t - \left(\tfrac{A_0(B-1)[A_0(B-1)-(A_0A+\beta)]+\alpha AB_0}{AB_0^2(B-A-1)}\right) = 0,$$

or,

$$AB_0t^2 - (A_0B - AA_0 - A_0 - \beta)t - \left(\tfrac{A_0(B-1)[A_0(B-1)-(A_0A+\beta)]+\alpha AB_0}{B_0(B-A-1)}\right) = 0,$$

thus,

$$[A_0(B-1) - (A_0A+\beta)]^2 + \tfrac{4A\{A_0(B-1)[A_0(B-1)-(A_0A+\beta)]+\alpha AB_0}{(B-A-1)} > 0,$$

or

$$[A_0(B-1) - (A_0A+\beta)]^2(B-A-1) + 4A\{A_0(B-1)[A_0(B-1) - (A_0A+\beta)] + \alpha AB_0 > 0. \tag{15}$$

for $A+1 < B$ then the inequalities (i) holds.
Conversely suppose that inequality (i) is true. We will prove that Eq.(1.1) has a prime period two solution. Suppose that,

$$p = \frac{\xi+\delta}{2AB_0}$$

and

$$q = \frac{\xi-\delta}{2AB_0}$$

where $\delta = \sqrt{\xi^2 + \frac{4A[A_0A(B-1)+\alpha AB_0]}{(B-A-1)}}$ and $\xi = A_0(B-1) - (A_0A+\beta)$.
from (5.7) we get that $\delta^2 > 0$, therefore p and q are distinct real numbers, set

$$s_{-2} = p \quad \text{and} \quad s_{-1} = q.$$

We would like to show that

$$s_1 = q, \ \ and \quad s_2 = p$$

It follows from Eq.(1.1) that

$$\begin{aligned} s_1 &= Ap + Bq + \tfrac{\alpha+\beta p}{A_0+B_0p} \\ &= A\left(\tfrac{\xi+\delta}{2AB_0}\right) + B\left(\tfrac{\xi-\delta}{2AB_0}\right) + \tfrac{\alpha+\beta\left(\frac{\xi+\delta}{2AB_0}\right)}{A_0+B_0\left(\frac{\xi+\delta}{2AB_0}\right)} \end{aligned}$$

dividing numerator and denominator by $2AB_0$ we get

$$s_1 = A\left(\tfrac{\xi+\delta}{2AB_0}\right) + B\left(\tfrac{\xi-\delta}{2AB_0}\right) + \tfrac{2AB_0\alpha+\beta(\xi+\delta)}{2AA_0B_0+B_0(\xi+\delta)}$$

multiply denominator and numerator of right hand side by $2AA_0B_0 + B_0(\xi - \delta)$ and by computation we get

$$s_1 = q.$$

Similarly

$$s_2 = p.$$

Then by induction we get

$$s_{2n} = p \ \text{ and } \ s_{2n+1} = q \quad \text{for all } \ n \geq -\max\{l,\ k\}.$$

Thus Eq.(1.1) has prime period two solution. ∎

6. Global Stability of Eq.(1.1)

In this section we will study the global stability character of the solutions of Eq.(1.1).

Lemma 3. *For any values of the quotient $\frac{\alpha}{A_0}$ and $\frac{\beta}{B_0}$ the function $f(u,v,w)$ defined by Eq.(4.2) has monotonicity behaviour in its two arguments.*

Proof. The proof follows by easy computations and is omitted. ∎

Theorem 4. *The equilibrium point $\overline{s}$ of Eq.(1.1) is a global attractor ⇔*

$$i)\ \ A + B < 1. \tag{16}$$

$$ii)\ \ \beta A \leq \alpha B \text{ and } A < 1. \tag{17}$$

Proof. Let ζ, η are real numbers and assume that $g : [\zeta,\eta]^2 \to [\zeta,\eta]$ be a function defined by

$$g(u,v,w) = Au + Bv + \frac{\alpha + \beta w}{A_0 + B_0 w}$$

$$\begin{aligned} \frac{\partial f(u,v,w)}{\partial u} &= A, \quad \frac{\partial f(u,v,w)}{\partial v} = B \ \text{ and} \\ \frac{\partial f(u,v,w)}{\partial w} &= \frac{\beta A_0 - \alpha B_0}{(A_0 + B_0 w)^2} \end{aligned}$$

Now, two cases must be considered.
Case (1): Suppose that $\beta A_0 - \alpha B_0 < 0$, then we can easily see that the function $g(u,v,w)$ is increasing in u, v and decreasing in w.
Let (m, M) be a solution of the system $M = g(M,M,m)$ and $m = g(m,m,M)$. Then from Eq.(1.1), we can write

$$M = AM + BM + \tfrac{\alpha+\beta m}{A_0+B_0 m}\ , \quad m = Am + Bm + \tfrac{\alpha+\beta M}{A_0+B_0 M}$$

or

$$M(1-A-B) = \tfrac{\alpha+\beta m}{A_0+B_0 m}, \quad m(1-A-B) = \tfrac{\alpha+\beta M}{A_0+B_0 M}$$

then the equation

$$\begin{aligned} A_0(1-A-B)M + B_0(1-A-B)Mm &= \alpha + \beta m, \ \text{ and} \\ A_0(1-A-B)m + B_0(1-A-B)mM &= \alpha + \beta M. \end{aligned}$$

Then by subtracting we get,

$$(M-m)\{A_0(1-A-B)+\beta\}=0,$$

under the condition $A+B<1$,we see that

$$M=m$$

By using **Theorem B**, it follows that $\overline{s}$ is a global attractor of Eq.(1.1) and then the proof is completed.

Case (2): Suppose that $\beta A_0-\alpha B_0>0$ is true, let ζ,η are real numbers and assume that $g:[\zeta,\eta]^2\rightarrow[\zeta,\eta]$ be a function defined by $g(u,v,w)=Au+Bv+\frac{\alpha+\beta w}{A_0+B_0w}$, then we can easily see that the function $g(u,v,w)$ is increasing in u,v and w.

Let (m,M) be a solution of the system $M=g(M,M,M)$ and $m=g(m,m,m)$.Then from Eq.(1.1), we see that

Then from Eq.(1.1), we see that

$$M=AM+BM+\tfrac{\alpha+\beta M}{A_0+B_0M}\ ,\quad m=Am+Bm+\tfrac{\alpha+\beta m}{A_0+B_0m}$$

by subtracting we get

$$M(1-A-B)=\tfrac{\alpha+\beta M}{A_0+B_0M},\quad m(1-A-B)=\tfrac{\alpha+\beta m}{A_0+B_0m}$$

and

$$\begin{aligned}A_0(1-A-B)M+B_0(1-A-B)M^2 &= \alpha+\beta M,\\ A_0(1-A-B)m+B_0(1-A-B)m^2 &= \alpha+\beta m.\end{aligned}$$

Subtracting we obtain

$$\begin{aligned}A_0(M-m)(1-A-B)+B_0(M^2-m^2)(1-A-B) &= \beta(M-m),\\ (M-m)\{B_0(1-A-B)(m+M)+A_0(1-A-B)-\beta\} &= 0,\end{aligned}$$

under the condition $A+B<1$ and $A_0(1-A-B)>\beta$ we conclude that,

$$m=M.$$

It follows by Theorem C that $\overline{x}$ is a global attractor of Eq.(1.1) and then the proof is completed. ∎

7. Existence of Boundedness of Solutions of Eq.(1.1)

This section deals with the boundedness of solutions of Eq.(1.1).

Theorem 5. *Every solution of Eq.(1.1) is bounded and persist if $A+B<1$.*

Proof. Let $\{s_n\}_{n=-\max\{l,k\}}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) that

$$\begin{aligned}s_{n+1} &= As_n+Bs_{n-l}+\frac{\alpha+\beta s_{n-k}}{A_0+B_0s_{n-k}}\\ &= As_n+Bs_{n-l}+\frac{\alpha}{A_0+B_0s_{n-k}}+\frac{\beta s_{n-k}}{A_0+B_0s_{n-k}}\end{aligned}$$

Then

$$\begin{aligned}s_{n+1} &\leq As_n+Bs_{n-l}+\frac{\alpha}{A_0}+\frac{\beta s_{n-k}}{B_0s_{n-k}}\\ &= As_n+Bs_{n-l}+\frac{\alpha}{A_0}+\frac{\beta}{B_0}\quad for\ all\quad n\geq 0.\end{aligned}$$

By using comparison, the right hand side can be written as follows

$$y_{n+1}=Ay_n+By_{n-l}+\frac{\alpha}{A_0}+\frac{\beta}{B_0}.$$

So, we can write

$$y_n = a^n y_0 + \text{constant},$$

and this equation is locally asymptotically stable because $A+B<1$, and converges to the equilibrium point $\overline{y} = \frac{\alpha B_0+\beta A_0}{A_0 B_0(1-A-B)}$
Therefore

$$\lim_{n\to\infty} \sup s_n \leq \frac{\alpha B_0+\beta A_0}{A_0 B_0(1-A-B)}$$

Hence the solution is bounded. ∎

Theorem 6. *Every solution of Eq.(1.1) is unbounded if $A > 1$ or $B > 1$.*

Proof. Let $\{s_n\}_{n=-\max\{l,\ k\}}^{\infty}$ be a solution of Eq.(1.1). Then from Eq.(1.1) we see that

$$x_{n+1} = As_n + Bs_{n-l} + \frac{\alpha+\beta s_{n-k}}{A_0+B_0 s_{n-k}} > As_n \qquad for\ all \quad n \geq 0.$$

the right hand side can be written as follows

$$y_{n+1} = Ay_n \quad \Rightarrow \quad y_n = A^n y_0,$$

and this equation is unbounded because $A > 1$ and $\lim_{n\to\infty} y_n = \infty$. Then by using ratio test $\{s_n\}_{n=-\max\{l,k\}}^{\infty}$is unbounded from above.
Similarly from Eq.(1.1) we see that

$$x_{n+1} = As_n + Bs_{n-l} + \frac{\alpha+\beta s_{n-k}}{A_0+B_0 s_{n-k}} > Bs_{n-l} \qquad for\ all \quad n \geq 0.$$

The right hand side can be written as follows

$$\begin{aligned} y_{n+1} &= By_{n-l} \\ &\Rightarrow \quad y_{2n-l} = B^n y_{-l}, \ \ and \ \ y_{2n} = B^n y_0 \end{aligned}$$

and this equation is unstable because $B > 1$, and $\lim_{n\to\infty} y_{2n-l} = \lim_{n\to\infty} y_n = \infty$. Then by using ratio test $\{s_n\}_{n=-\max\{l,k\}}^{\infty}$is unbounded from above. ∎

8. Numerical Examples

Here we would like to exhibit some numerical examples in order to show the variety of dynamics happening in the non-linear difference equation Eq. (1.1). For positive parameters and initial values, the dynamics of the Eq.(1.1.) is limited to local asymptotic stability of the positive equilibrium and periodic, bounded solutions. It does not show chaotic, aperiodic, fractal-like solutions. It is seen that if we consider the negative values of the controlling parameters A and/or B, the Eq.(1.1) exhibits very complicated dynamics viz. fractal-like and chaotic and periodic.

Example 1. *Consider* $A = 0.0072$, $B = -0.9804$, $\alpha = 0.7264$, $\beta = 0.4899$, $A_0 = 0.6355$, $B_0 = 0.5154$, $k = 28$ *&* $l = 10$ *then for any initial values taken from the interval* $(0, 1)$*, the solution of the Eq. (1.1) exhibit fractal-like trajectories as shown in Fig. 3.*

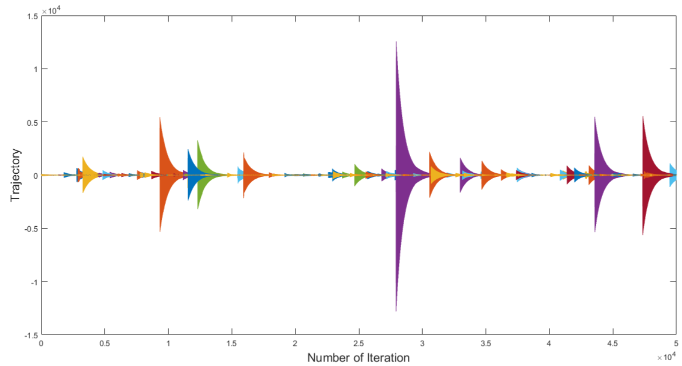

FIGURE 3. Fractal-like trajectories

Example 2. *Consider* $A = 0.6601$, $B = -0.4006$, $\alpha = 0.5699$, $\beta = 0.9087$, $A_0 = 0.3423$, $B_0 = 0.8499$, $k = 28$ *&* $l = 8$ *then for any initial values taken from the interval* $(0, 1)$*, the solution of the Eq. (1.1) exhibit fractal-like trajectories as shown in Fig. 4.*

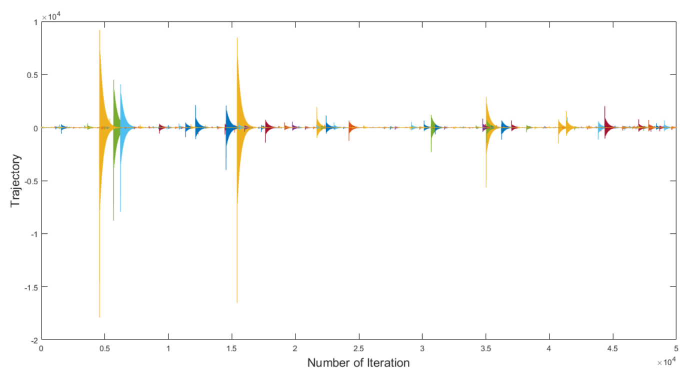

FIGURE 4. Fractal-like trajectories

Example 3. *Consider* $A = 0.0.8602$, $B = -0.6432$, $\alpha = 0.2558$, $\beta = 0.3436$, $A_0 = 0.6028$, $B_0 = 0.1760$, $k = 6$ *& $l = 2$ then for any initial values taken from the interval $(0, 1)$, the solution of the Eq. (1.1) exhibit fractal-like trajectories as shown in Fig. 5.*

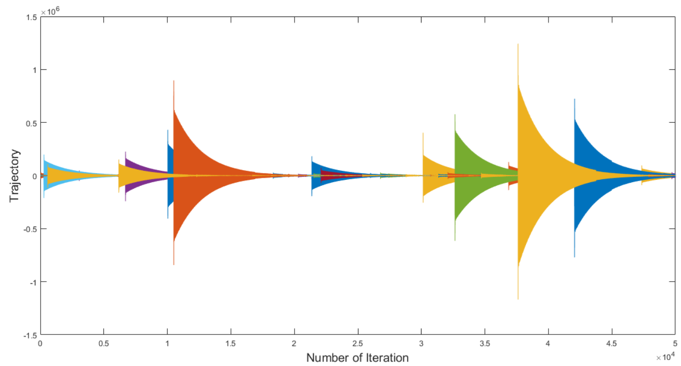

FIGURE 5. Fractal-like trajectories

Example 4. *Consider* $A = -0.4531$, $B = 0.6058$, $\alpha = 0.5845$, $\beta = 0.3034$, $A_0 = 0.1249$, $B_0 = 0.8747$, $k = 26$ *& $l = 6$ then for any initial values taken from the interval $(0, 1)$, the solution of the Eq. (1.1) exhibit fractal-like trajectories as shown in Fig. 6.*

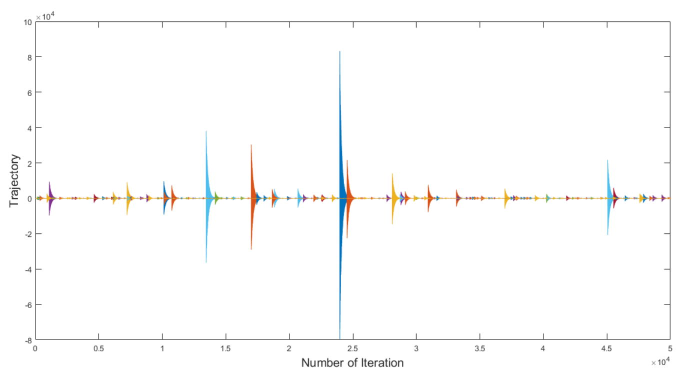

FIGURE 6. Fractal-like trajectories

Example 5. *Consider $A = 0.1714$, $B = -0.7882$, $\alpha = 0.8206$, $\beta = 0.3087$, $A_0 = 0.1027$, $B_0 = 0.1826$, $k = 30$ & $l = 9$ then for any initial values taken from the interval $(0,1)$, the solution of the Eq. (1.1) exhibit chaotic trajectories as shown in Fig. 7.*

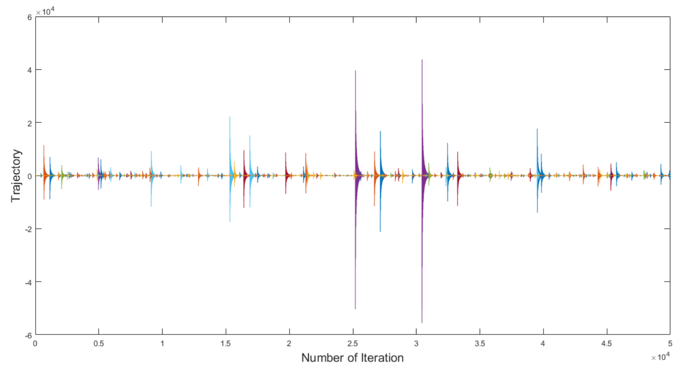

FIGURE 7. Chaotic trajectories

Example 6. *Consider $A = 0.2161$, $B = -0.7231$, $\alpha = 0.6634$, $\beta = 0.4058$, $A_0 = 0.1341$, $B_0 = 0.6239$, $k = 14$ & $l = 1$ then for any initial values taken from the interval $(0,1)$, the solution of the Eq. (1.1) exhibit chaotic trajectories as shown in Fig. 8.*

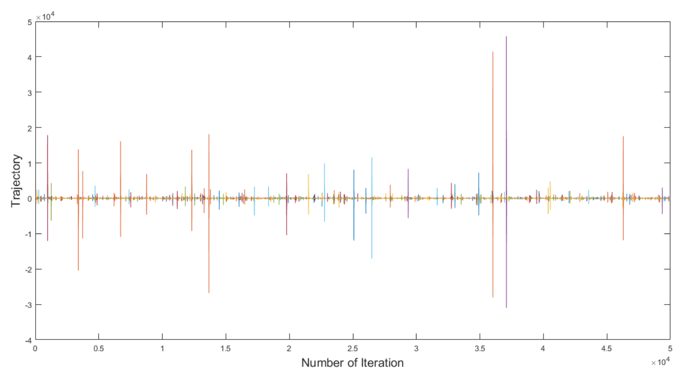

FIGURE 8. Chaotic trajectories

Example 7. *Consider* $A = 0.6592$, $B = -0.7230$, $\alpha = 0.0657$, $\beta = 0.5430$, $A_0 = 0.1904$, $B_0 = 0.1164$, $k = 4$ *&* $l = 1$ *then for any initial values taken from the interval* $(0, 1)$, *the solution of the Eq. (1.1) exhibit chaotic trajectories as shown in Fig. 9.*

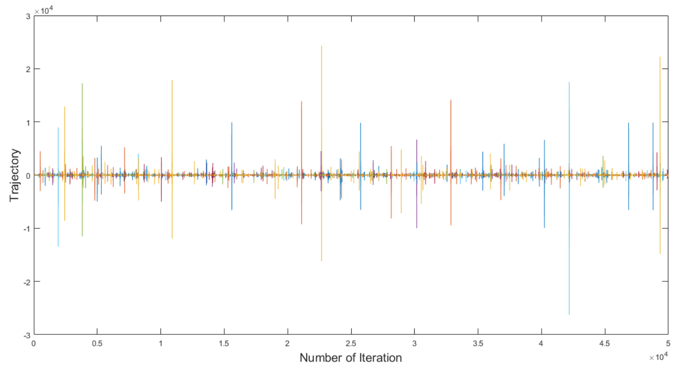

FIGURE 9. Chaotic trajectories

Example 8. *Consider* $A = 0.1663$, $B = -0.6324$, $\alpha = 0.8438$, $\beta = 0.9410$, $A_0 = 0.0857$, $B_0 = 0.4363$, $k = 20$ *&* $l = 5$ *then for any initial values taken from the interval* $(0, 1)$, *the solution of the Eq. (1.1) exhibit chaotic trajectories as shown in Fig. 10.*

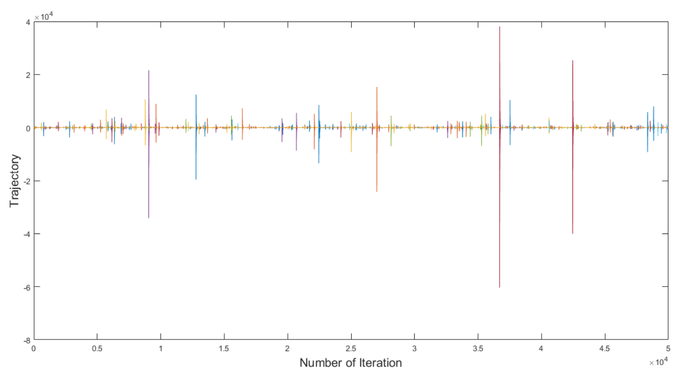

FIGURE 10. Chaotic trajectories

Example 9. *Consider* $A = -0.6139$, $B = -0.4973$, $\alpha = 0.8195$, $\beta = 0.2645$, $A_0 = 0.6023$, $B_0 = 0.9178$, $k = 12$ *& $l = 3$ then for any initial values taken from the interval $(0, 1)$, the solution of the Eq. (1.1) exhibit periodic trajectories of period 5 as shown in Fig. 11.*

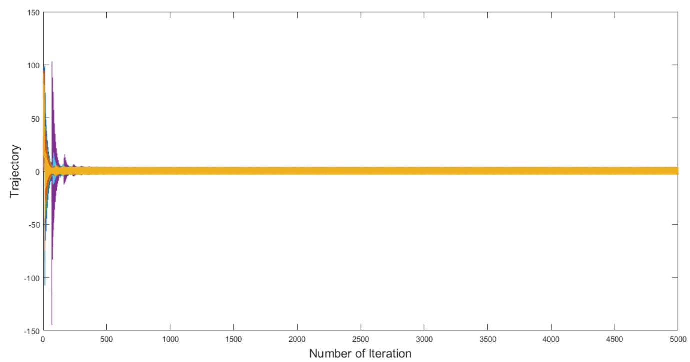

FIGURE 11. Chaotic trajectories

Example 10. *Consider* $A = -0.8900$, $B = -0.3202$, $\alpha = 0.6870$, $\beta = 0.1860$, $A_0 = 0.4345$, $B_0 = 0.6596$, $k = 22$ *& $l = 5$ then for any initial values taken from the interval $(0, 1)$, the solution of the Eq. (1.1) exhibit periodic trajectories of period 9 as shown in Fig. 12.*

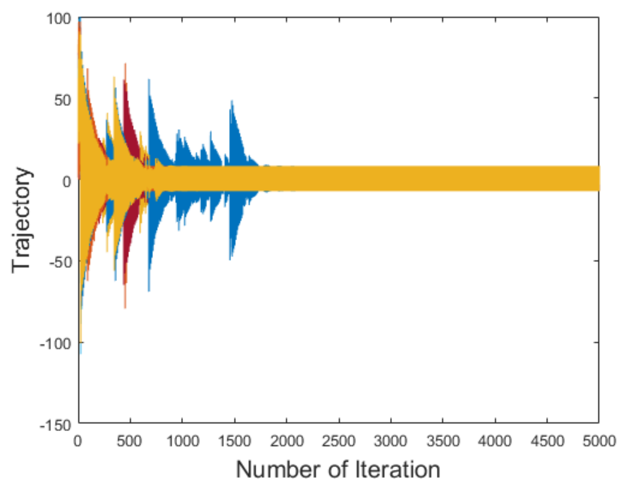

FIGURE 12. Chaotic trajectories

9. Conclusion

This work is related to the qualitative behaviour of a rational difference equation, which may be considered as generalized equation studied in [36]. Thus our results, considerably extend some previous investigations in literature. Firstly existence and uniqueness of positive equilibrium point is prove. Then it investigated that Eq. (1.1) is bounded and persists. We proved that the Eq. (1.1) has a unique positive equilibrium point, which is locally asymptotically stable. The method of linearization is used to prove the local asymptotic stability of unique equilibrium point. Linear stability analysis shows that the positive steady-state of Eq. (1.1) is asymptotically stable and there exist positive prime period 2 solution of Eq. (1.1) under certain parametric conditions and the chosen value of the delays $(l,\ k)$. The main objective of the theory of difference equations is to predict the global behaviour of an equation under consideration based on the knowledge of its present state. At the end some numerical examples are given in order to show the completeness of dynamics when the parameters A and B are extended to the negative real numbers.

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