ISSN: 2832-4293

Pan-American Journal of Mathematics 5 (2026), 15
https://doi.org/10.28919/cpr-pajm/5-15

TWO REGULARIZATION METHODS FOR AN UNKNOWN SOURCE PROBLEM FOR THE TEMPERED FRACTIONAL DIFFUSION EQUATION

LE DINH LONG

ABSTRACT. In this work, we focus on the inverse source problem for tempered fractional diffusion equation. The problem is ill-posed. We introduce a modified quasi boundary method and the fractional Landweber method to tackle this issue. In the convergence results, we present the errors under two choices: the a priori and the a posteriori strategies. Finally, we provide numerical example to demonstrate the efficacy and stability of the proposed methodologies.

1. INTRODUCTION

Fractional diffusion equations have garnered significant attention across scientific disciplines due to their capacity to elucidate anomalous diffusion phenomena. Within the realm of physics, these equations serve as invaluable tools for modeling non-local transport dynamics in heterogeneous materials, including porous media and fractal structures. In financial studies, fractional diffusion equations are instrumental in capturing the intricate dynamics of asset prices and volatility, thus advancing our comprehension of market behaviors beyond conventional Brownian motion models. Furthermore, in biophysics research, fractional diffusion equations are leveraged to investigate the nuanced mechanisms of molecular transport within biological systems, offering critical insights into phenomena such as protein diffusion within cellular environments. Recent studies have focused on various aspects of fractional differential equations and inverse problems. Binh et al. [2] investigated continuity properties for differential equations involving the Caputo-Fabrizio derivative. Minh and Can [3] analyzed the dependence of solutions on the order of the Caputo–Hadamard derivative and provided several illustrative examples. Triet et al. [4] proposed new regularization methods and error estimates for terminal value problems associated with elliptic equations. In addition, Binh et al. [5] studied nonlinear Sobolev equations with Caputo fractional operators and exponential nonlinearities. Can et al. [6] considered parabolic equations with fractional memory terms and established several results on existence and qualitative properties of solutions. These works contribute significantly to the development of fractional calculus, nonlinear evolution equations, and regularization theory.

In this paper, we consider the following diffusion equation associated with fractional derivative:

FACULTY OF INFORMATION TECHNOLOGY, INDUSTRIAL UNIVERSITY OF HO CHI MINH CITY, HO CHI MINH CITY, VIETNAM
Email address: ledinhlong@iuh.edu.vn.
Submitted on May 29, 2026.
2020 *Mathematics Subject Classification.* 35R30.
Key words and phrases. inverse source problem; tempered fractional diffusion equation; modified quasi boundary method; fractional Landweber method; priori parameter choice rule; posteriori parameter choice rule.

(1)
$$\begin{cases} \mathbb{D}_t^{\gamma,\kappa}u(x,t)-\Delta u(x,t)=\varphi(t)f(x), & \text{in } \Omega\times(0,T],\\ u(x,t)=0 & \text{in } \partial\Omega\times(0,T],\\ u(x,0)=0, & \text{in } \Omega. \end{cases}$$

with the following nonlocal condition

(2)
$$\int_0^T u(x,t)dt=g(x), \qquad \text{in } \Omega.$$

where $\kappa>0$ and $\gamma\in(0,1)$. Here $\Omega\in\mathcal{R}^d,(d\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. Here $\mathbb{D}_t^{\gamma,\kappa}$ is called tempered Caputo of order α which is defined by (see p. 430 [1])

(3)
$$\mathbb{D}_t^{\gamma,\kappa}z(t)=\frac{e^{-\kappa t}}{\Gamma(1-\gamma)}\int_0^t(t-s)^{-\gamma}\frac{d}{ds}\big(e^{\kappa s}z(s)\big)ds,$$

where Γ is the Gamma function.

In traditional fractional calculus, the fractional order derivative of a function is defined using fractional orders. However, in tempered fractional calculus, the fractional order derivatives are modified by a tempering function, which can be used to control the behavior of the derivatives at different scales.

Tempered fractional derivatives find applications in various fields, including signal processing, image processing, and physics, where they provide a flexible tool for modeling systems with memory and complex dynamics. They allow for a more nuanced description of phenomena that exhibit both short-term and long-term dependencies, making them valuable in the analysis of real-world processes.

The tempered fractional derivative was introduced in [7] and [8]. Recent applications in sub-diffusion examined in references [11] [13]. The concept of combining fractional operators with exponential functions has been explored in [8] [9] [10] . Serving as an extension of fractional calculus, tempered fractional calculus has emerged to address various fields such as poroelasticity [14], geophysical flows [15], ground water hydrology [16] . In recent times, there has been a surge of interest in tempered fractional differential equations. For instance, numerical methods have been discussed in [9] [17] [18], the initial value problem has been investigated in [19], tempered terminal value problems have been addressed in [20] [21], boundary value problems have been explored in [1], and discrete tempered fractional calculus of variation has been studied in reference [7].

As far as we know, in the case of $\gamma=1,\kappa=0$, the inverse source problem (1) has been studied by many mathematicians such as [23] [24], etc. Moreover, there are many works in the case of $\gamma\neq 1,\kappa=0$, for example, Rudolf Gorenflo et al. in [27] with the research on time fractional diffusion equation in the fractional Sobolev spaces, the study of M. Kh. Beshtokov on boundary value problems for a pseudoparabolic equation with the Caputo fractional derivative in [28], or Y. Wang et al. with asymptotic behavior of stochastic lattice systems with a Caputo fractional time derivative in [29]

Nevertheless, to the best of our knowledge, there are very few studies on the fractional inverse source problem, for example, Sakamoto et al. obtained a Lipschitz stability for $\varphi(t)$ in [22], Zhang et al. in [25] used the Fourier truncation method and Wang et al. in [26] based on the Tikhonov regularization to deal with an inverse source problem for problem (1) in the case $\varphi(t)=1$. Furthermore, works on fractional diffusion equations associated with tempered derivative are even rarer. In this study, we concentrate on recovering and regularizing the source term for tempered fractional diffusion equation.

Supposing that the time-dependent source term φ^ϵ is given. The source term $f(x)$ is determined from a noisy final data g^ϵ, and φ^ϵ which satisfy

(4)
$$\|g-g^\epsilon\|_{L^2(\Omega)}+\|\varphi-\varphi^\epsilon\|_{L^\infty(0,T)}\leq\epsilon.$$

This paper is organized as follows. Section 2 gives some preliminaries. In Section 3, we show the sought solution of problem (1), and an example describes the mismatch of the problem and the intercept of the source function in space $L^2(\Omega)$. In section 4, we present a modified quasi boundary value mehtod. In Section 5, we show the Fractional Landweber method to solve the problem (1) and show the convergent rate under a priori parameter choice rule and a posteriori parameter choice rule, and final section, we present a simple numerical example to illustrate the results in our theory.

2. Preliminaries

Let us recall the spectral problem for the fractional Laplace operator on the bounded domain $\mathcal{D}$ as follows

$$\begin{cases} (-\Delta)\phi_m(x) = \xi_m \phi_m(x), & x \in \Omega, \forall m \in \mathbb{N}, \\ \phi_m(x) = 0, & x \in \partial\Omega, \forall m \in \mathbb{N}, \end{cases} \tag{5}$$

where the sequence of positive eigenvalues $\{\phi_m\}_{m\in N}$ satisfy

$$\xi_1 \le \xi_2 \le \dots \le \xi_m \nearrow \infty,$$

whose corresponding set of real eigenfunctions $\{\phi_m\}_{m\in N}$ is orthogonal and complete.

Definition 2.1. *[22] For $\gamma > 0$, and a arbitrary constant $\beta \in \mathbb{R}$, the Mittag-Leffler function is defined by*

$$E_{\gamma,\beta}(z) = \sum_{m=1}^{+\infty} \frac{z^m}{\Gamma(\gamma m + \beta)}, \ z \in \mathbb{C}, \tag{6}$$

where Γ is the Gamma function.

Definition 2.2. *[22] For any $\tau > 0$, we define the fractional Hilbert scale space by*

$$\mathscr{H}^\tau(\Omega) = \Big\{ w \in L^2(\Omega) : \|w\|^2_{\mathscr{H}^\tau(\Omega)} = \sum_{m=1}^{+\infty} \xi_m^{2\tau} \big|\langle w, \phi_m \rangle\big|^2 < \infty \Big\}.$$

The space $\mathscr{H}^\tau(\Omega)$ is a Hilbert space, equipped with the norm

$$\|f\|_{\mathscr{H}^\tau(\mathcal{D})} = \Big(\sum_{m=1}^{+\infty} \xi_m^{2\tau} \big|\langle w, \phi_m \rangle\big| \Big)^{\frac{1}{2}},$$

Lemma 2.1. *If $0 < \gamma < 1$, $\beta \in \mathbb{R}$, then*

$$\frac{\mathcal{M}_0}{1+|z|} \le \big|E_{\gamma,\beta}(z)\big| \le \frac{\mathcal{M}_1}{1+|z|}, \tag{7}$$

for all $z < 0$, where $\mathcal{M}_0$ and $\mathcal{M}_1$ are positive constants depending only on γ, κ. In addition, we obtain

$$\frac{\mathscr{M}_0}{1+|z|} \le E_{\gamma,1}(z) \le \frac{\mathscr{M}_1}{1+|z|}. \tag{8}$$

Lemma 2.2. *(Hölder's inequality for negative exponents, see [15]) Let $p' < 0$, and $p \in \mathbb{R}$ be such that $\frac{1}{p'} + \frac{1}{p} = 1$. For $f(x) > 0,\ g(x) \ge 0,\ \forall x \in \Omega$ are Lebesgue measurable functions. Then*

$$\int_\Omega fg dx \ge \Big(\int_\Omega |f|^{p'} dx \Big)^{\frac{1}{p'}} \Big(\int_\Omega |g|^p dx \Big)^{\frac{1}{p}}, \tag{9}$$

3. The Mild Solution of Problem (1)

Theorem 3.1. *The source function f is defined by*

$$f(x) = \sum_{m=1}^{+\infty} \frac{g_m}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)} \phi_m(x), \tag{10}$$

where

$$\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) = \int_0^T \int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma) \varphi(s) ds dt. \tag{11}$$

Proof. If the inner product on L^2 is denoted by $\langle \cdot \rangle$, then the Fourier series of a function u in L^2 can be formulated as

$$u(x,t) = \sum_{m=1}^{+\infty} \langle u(t), \phi_m \rangle \phi_m(x).$$

Noting that $u_m(0) = 0$, we derive that

$$\begin{aligned} u_m(t) &= e^{-\kappa t} E_{\gamma,1}(-\xi_m t^\gamma) u_m(0) + \int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma) \varphi(s) \langle f, \phi_m \rangle ds \\ &= \int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma) \varphi(s) \langle f, \phi_m \rangle ds, \end{aligned} \tag{12}$$

From (12), the nonlocal condition (2) allows us to give that

$$\int_0^T u_m(t) dt = g_m. \tag{13}$$

It follows from (12) that

$$\langle f, \phi_m \rangle = \langle g, \phi_m \rangle \Big(\int_0^T \Big(\int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma) \varphi(s) ds \Big) dt \Big)^{-1}, \tag{14}$$

by denoting

$$\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) = \int_0^T \Big(\int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma) \varphi(s) ds \Big) dt, \tag{15}$$

which gives (10). □

Lemma 3.1. *Let $\varphi \in L^\infty(0,T)$ such that $0 < \varphi_0 \le \varphi(t) \le \varphi_1$. Let $0 < \gamma < 1$ and $\kappa > 0$, then we obtain*

$$\frac{\mathscr{B}_0}{\xi_m} \le \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) \le \frac{\varphi_1 T}{\xi_m}, \tag{16}$$

and

$$\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) \le \frac{\mathcal{M}_1 \varphi_1 T^{\gamma-\gamma\zeta+1}}{(\gamma-\gamma\zeta+1)(\gamma-\gamma\zeta)} \lambda_j^{-\zeta}, 0 < \zeta < 1. \tag{17}$$

where by

$$\mathscr{B}_0 = \mathcal{M}_0 \varphi_0 T^{\frac{p-1}{p}} \left(\frac{1-e^{-p\kappa T}}{p\kappa} \right)^{\frac{1}{p}} \big(1 - E_{\gamma,1}(-\xi_1 T^\gamma)\big),\ 0 < p < 1. \tag{18}$$

Proof.

- We have

$$E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma) \le \frac{\mathcal{M}_1}{1+\xi_m (t-s)^\gamma} \le \mathcal{M}_1 \xi_m^{-\zeta} (t-s)^{-\gamma\zeta},\ 0 < \zeta < 1. \tag{19}$$

Since $|\varphi(s)| \le \varphi_1$, one has

$$\begin{aligned}&\int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma)\varphi(s)ds \\ &\quad \le \mathcal{M}_1 \varphi_1 \xi_m^{-\zeta} \int_0^t (t-s)^{\gamma-1-\gamma\zeta} = \mathcal{M}_1 \varphi_1 \xi_m^{-\zeta} \frac{t^{\gamma-\gamma\zeta}}{\gamma-\gamma\zeta}. \end{aligned} \tag{20}$$

This implies that

$$\begin{aligned}\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) &= \int_0^T \int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma)\varphi(s)dsdt \\ &\le \mathcal{M}_1 \varphi_1 \xi_j^{-\zeta} \int_0^T \frac{t^{\gamma-\gamma\zeta}}{\gamma-\gamma\zeta} dt = \frac{\mathcal{M}_1 \varphi_1 T^{\gamma-\gamma\zeta+1}}{(\gamma-\gamma\zeta+1)(\gamma-\gamma\zeta)} \xi_m^{-\zeta}. \end{aligned} \tag{21}$$

- Since $\varphi(s) \le \varphi_1$ and $e^{-\kappa(t-s)} \le 1$ and noting that $E_{\gamma,\gamma}(-z) > 0$ for any $z > 0$, we find that

$$\begin{aligned}&\int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma)\varphi(s)ds \\ &\qquad \le \varphi_1 \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma)ds \\ &\qquad = -\frac{\varphi_1}{\xi_m} \int_0^t \frac{d}{ds} E_{\gamma,1}(-\xi_m s^\gamma) ds = \frac{\mathcal{M}_1}{\xi_m} \Big(1 - E_{\gamma,1}(-\xi_m t^\gamma) \Big), \end{aligned} \tag{22}$$

where we have used the fact that

$$\frac{d}{ds} E_{\gamma,1}\big(-\xi s^\gamma\big) = -\xi s^{\gamma-1} E_{\gamma,\gamma}\big(-\xi s^\gamma\big).$$

This follows from (22) that

$$\begin{aligned}\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) &= \int_0^T \int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma)\varphi(s)dsdt \\ &\le \frac{\varphi_1}{\xi_m} \int_0^T \Big(1 - E_{\gamma,1}(-\xi_m t^\gamma) \Big) dt \le \frac{\varphi_1 T}{\xi_m}. \end{aligned} \tag{23}$$

- Since $\varphi(s) \ge \varphi_0$ and noting that $E_{\gamma,\gamma}(-z) > 0$ for any $z > 0$, we find that

$$\begin{aligned}&\int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma)\varphi(s)ds \\ &\qquad \ge \varphi_0 e^{-\kappa t} \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma)ds \\ &\qquad = -\frac{\varphi_0 e^{-\kappa t}}{\xi_m} \int_0^t \frac{d}{ds} E_{\gamma,1}(-\xi_m s^\gamma) ds = \frac{\varphi_0 e^{-\kappa t}}{\xi_m} \Big(1 - E_{\gamma,1}(-\xi_m t^\gamma) \Big). \end{aligned} \tag{24}$$

This implies that

$$\begin{aligned}\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) &= \int_0^T \int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma)\varphi(s)dsdt \\ &\ge \frac{\varphi_0 e^{-\kappa t}}{\xi_m} \int_0^T e^{-\kappa t} \Big(1 - E_{\gamma,1}(-\xi_m t^\gamma) \Big) dt. \end{aligned} \tag{25}$$

Let us set the following term

$$\mathscr{L}(T) = \int_0^T e^{-\kappa t} \Big(1 - E_{\gamma,1}(-\xi_m t^\gamma) \Big) dt. \tag{26}$$

Let $0 < p < 1$. Let $p' = \dfrac{-p}{1-p} < 0$, thank to Hölder inequality, we receive

$$\begin{aligned}\mathscr{L}(T) &\geq \Big(\int_0^T e^{-p\kappa t} dt \Big)^{\frac{1}{p}} \Big(\int_0^T \Big(1 - E_{\gamma,1}(-\xi_m t^\gamma)\Big)^{p'} dt \Big)^{\frac{1}{p'}} \\ &= \Big(\frac{1 - e^{-p\kappa T}}{p\kappa} \Big)^{\frac{1}{p}} \Big(\int_0^T \Big(1 - E_{\gamma,1}(-\xi_m t^\gamma)\Big)^{p'} dt \Big)^{\frac{1}{p'}}.\end{aligned} \tag{27}$$

It is obvious to see that

$$1 - E_{\gamma,1}(-\xi_m t^\gamma) \leq 1 - E_{\gamma,1}(-\xi_m T^\gamma). \tag{28}$$

Thus, since $p' < 0$, we know that

$$\Big(1 - E_{\gamma,1}(-\xi_m t^\gamma)\Big)^{p'} \geq \Big(1 - E_{\gamma,1}(-\xi_m T^\gamma)\Big)^{p'}. \tag{29}$$

This implies that

$$\Big(\int_0^T \Big(1 - E_{\gamma,1}(-\xi_m t^\gamma)\Big)^{p'} dt \Big)^{\frac{1}{p'}} \geq \Big(1 - E_{\gamma,1}(-\xi_m T^\gamma)\Big) T^{\frac{1}{p'}}. \tag{30}$$

Combining (25) (27) and (30), we find that

$$\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) \geq \frac{\varphi_0}{\xi_m} \mathscr{L}(T) \geq \frac{\varphi_0}{\xi_m} T^{\frac{p-1}{p}} \Big(\frac{1 - e^{-p\kappa T}}{p\kappa} \Big)^{\frac{1}{p}} \Big(1 - E_{\gamma,1}(-\xi_m T^\gamma)\Big). \tag{31}$$

Since $E_{\gamma,1}(-\xi_m T^\gamma) \leq E_{\gamma,1}(-\xi_1 T^\gamma)$, one obtains $\big(1 - E_{\gamma,1}(-\xi_m T^\gamma)\big) \geq \big(1 - E_{\gamma,1}(-\xi_1 T^\gamma)\big)$. Thus, we have

$$\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) \geq \frac{\mathscr{B}_0}{\xi_m}. \tag{32}$$

Here

$$\mathscr{B}_0 = \varphi_0 T^{\frac{p-1}{p}} \Big(\frac{1 - e^{-p\kappa T}}{p\kappa} \Big)^{\frac{1}{p}} \Big(1 - E_{\gamma,1}(-\xi_1 T^\gamma)\Big). \tag{33}$$

□

Lemma 3.2. *Assume that $\varphi_0 \leq \varphi(t) \leq \varphi_1$ $\forall t \in [0,T]$. Let choose $\epsilon \in \Big(0, \dfrac{\varphi_0}{4}\Big)$, we obtain*

$$\frac{\varphi_0}{4} \leq \big|\varphi^\epsilon(t)\big| \leq \mathscr{P}(\varphi_0,\varphi_1). \tag{34}$$

Here, φ^ϵ is the measured data which is defined later.

Proof. We know that

$$\big|\varphi(t)\big| \leq \big|\varphi^\epsilon(t)\big| + \big|\varphi(t) - \varphi^\epsilon(t)\big| \leq \big|\varphi^\epsilon(t)\big| + \big\|\varphi^\epsilon - \varphi\big\|_{L^\infty(0,T)} \leq \big|\varphi^\epsilon(t)\big| + \epsilon. \tag{35}$$

From (35), we obtain

$$\big|\varphi^\epsilon(t)\big| \geq |\varphi(t)| - \epsilon \geq \varphi_0 - \epsilon \geq \frac{\varphi_0}{4}. \tag{36}$$

Similarly, we get

$$\big|\varphi^\epsilon(t)\big| \leq \varphi_1 + \epsilon < \varphi_1 + \frac{\varphi_0}{4}. \tag{37}$$

Denoting $\mathscr{P}(\varphi_0,\varphi_1) = \varphi_1 + \dfrac{\varphi_0}{4}$, combining (36) and (37) which leads to (34) holds. □

3.1. **Ill-posedness of the problem** (1)**.** In this work, we prove that the Problem (1) is an ill-posed problem in the sense of Hadamard.

Theorem 3.2. *The source problem* (1) *is non-well-posed.*

Proof. We define a linear operator as follows:

$$\mathcal{K}f(x) = \sum_{n=1}^{\infty} \mathbf{Q}_m^{\gamma;\kappa}(T,\varphi)\langle f, \phi_m\rangle \phi_m(x) = \int_\Omega g(x,\xi)f(\xi)d\xi, \tag{38}$$

where

$$g(x,\xi) = \sum_{m=1}^{+\infty} \mathbf{Q}_m^{\gamma;\kappa}(T,\varphi)\phi_m(x)\phi_m(\xi). \tag{39}$$

Since $g(x,\xi) = g(\xi,x)$, we know that $\mathcal{K}$ is self-adjoint operator. Next, we prove its compactness. To do this, defining the finite rank operators $\mathcal{K}_\mathcal{M}$ as follows

$$\mathcal{K}_\mathcal{M} f(x) = \sum_{m=1}^{\mathcal{M}} \mathbf{Q}_m^{\gamma;\kappa}(T,\varphi)\langle f, \phi_m\rangle \phi_m(x). \tag{40}$$

From (38) and (40), we obtain

$$\begin{aligned}\big\|\mathcal{K}_\mathcal{M} f - \mathcal{K}f\big\|_2^2 &= \sum_{m=\mathcal{M}+1}^{+\infty} \big(\mathbf{Q}_m^{\gamma;\kappa}(T,\varphi)\big)^2 \big|\langle f,\phi_m\rangle\big|^2 \\ &\le \varphi_1^2 \sum_{n=\mathcal{M}+1}^{+\infty} \xi_m^{-2\tau}\big|\langle f,\phi_m\rangle\big|^2 \le \varphi_1^2 \xi_\mathcal{M}^{-2\tau} \sum_{n=\mathcal{M}+1}^{\infty} \big|\langle f,\phi_m\rangle\big|^2.\end{aligned} \tag{41}$$

From (41), this implies that

$$\big\|\mathcal{K}_\mathcal{M} f - \mathcal{K}f\big\|_2 \le \xi_M^{-\tau}\varphi_1 \big\|f\big\|_2. \tag{42}$$

Hence, $\big\|\mathcal{K}_\mathcal{M} - \mathcal{K}\big\|_2 \to 0$ in the sense of operator norm in $L(L^2(\Omega); L^2(\Omega))$ as $\mathcal{M} \to \infty$. Moreover, $\mathcal{K}$ is a compact operator. Next, the singular values for the linear self-adjoint compact operator $\mathcal{K}$ are

$$\Xi_m = \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi), \tag{43}$$

and corresponding eigenvectors are ϕ_k which is known as an orthonormal basis in $L^2(\Omega)$. From (38), the problem (1) we introduced above can be formulated as an operator equation

$$\mathcal{K}f(x) = g(x), \tag{44}$$

see Kirsch [12]. Next, we show an example to illustrate the ill-posedness of problem (1). Let us choose the input final data $\widetilde{g}_j := \frac{\phi_j}{\sqrt{\xi_j}}$. We assume that to choice $g = 0$, then the formula (10), the source term corresponding to g is $f = 0$. An error in $L^2(\Omega)$ norm between two input final data is

$$\big\|\widetilde{g}_j - g\big\|_2 = \big\|\xi_j^{-1/2}\phi_j\big\|_2 = \xi_j^{-1/2} \to 0, \text{ as } j \to \infty. \tag{45}$$

And the source term corresponding to $\widetilde{g}_j$ is

$$\widetilde{f}_j(x) = \sum_{m=1}^{+\infty} \frac{\langle \widetilde{g}_j, \phi_m\rangle \phi_m(x)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)} = \sum_{m=1}^{+\infty} \frac{\langle \xi_j^{-1/2}\phi_j, \phi_m\rangle \phi_m(x)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}. \tag{46}$$

Thank to Lemma 3.1, then the error estimation between the f and $\widetilde{f}_j$

$$\big\|\widetilde{f}_j - f\big\|_2 \ge \frac{\xi_j^{1/2}}{\varphi_1 T}. \tag{47}$$

Combining (45) and (1), we conclude that

$$\lim_{j\to+\infty}\left\|\widetilde{f}_j - f_j\right\|_2 \geq \lim_{j\to+\infty}\frac{\xi_j^{1/2}}{\varphi_1 T}\to+\infty. \tag{48}$$

By combining (45) and (48), we obtain that the inverse source problem is non-well-posed. $\square$

3.2. **Conditional stability of source term f.** In this section, we introduce a conditional stability by the following theorem.

Theorem 3.3. *Let*

$$\|f\|_{\mathscr{H}^\tau(\Omega)} \leq \mathcal{E}, \tag{49}$$

for $\mathcal{E}>0$ then

$$\|f\|_2 \leq |\varphi_0 T|^{-\frac{\tau}{\tau+1}}\|f\|_{\mathscr{H}^\tau(\Omega)}^{\frac{1}{\tau+1}}\|g\|_2^{\frac{2\tau}{\tau+1}}, \tag{50}$$

Proof. Combining (10) and Hölder inequality, we have:

$$\begin{aligned}
\|f\|_2^2 &= \sum_{m=1}^{+\infty}\left|\frac{\langle g,\phi_m\rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\right|^2 = \sum_{m=1}^{+\infty}\frac{\left|\langle g,\phi_m\rangle\right|^{\frac{2}{\tau+1}}\left|\langle g,\phi_m\rangle\right|^{\frac{2\tau}{\tau+1}}}{\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2}\\
&\leq \left(\sum_{m=1}^{+\infty}\frac{\left|\langle g,\phi_m\rangle\right|^2}{\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^{2\tau+2}}\right)^{\frac{1}{\tau+1}}\left(\sum_{m=1}^{+\infty}\left|\langle g,\phi_m\rangle\right|^2\right)^{\frac{\tau}{\tau+1}}\\
&\leq \left(\sum_{m=1}^{+\infty}\frac{\left|\langle f,\phi_m\rangle\right|^2}{\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^{2\tau}}\right)^{\frac{1}{\tau+1}}\|g\|_2^{\frac{2\tau}{\tau+1}}.
\end{aligned} \tag{51}$$

Using Lemma 3.2, we have $\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^{2\tau} \geq \frac{|\varphi_0 T|^{2\tau}}{\xi_m^{2\tau}}$, and this inequality leads to

$$\sum_{m=1}^{+\infty}\frac{\left|\langle f,\phi_m\rangle\right|^2}{\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^{2\tau}} \leq \sum_{m=1}^{+\infty}\frac{\xi_m^{2\tau}\left|\langle f,\phi_m\rangle\right|^2}{|\varphi_0 T|^{2\tau}} = \frac{\|f\|^2_{\mathscr{H}^\tau(\Omega)}}{|\varphi_0 T|^{2\tau}}. \tag{52}$$

Combining (51) and (52), we get

$$\|f\|_2^2 \leq |\varphi_0 T|^{-\frac{2\tau}{\tau+1}}\|f\|_{\mathscr{H}^\tau(\Omega)}^{\frac{2}{\tau+1}}\|g\|_2^{\frac{2\tau}{\tau+1}}. \tag{53}$$

$\square$

Lemma 3.3. *For constants $\tau>0, \varsigma>0, \alpha>0, r\geqslant\xi_1>0$, we have*

$$G(r) = \frac{\varsigma r^{2-\tau}}{\varsigma r^2+\alpha} \leq \begin{cases}\mathcal{C}_1\varsigma^{\frac{\tau}{2}}, & 0<\tau<2,\\ \mathcal{C}_2\varsigma, & \tau\geqslant 2,\end{cases} \tag{54}$$

where $\mathcal{C}_1=\mathcal{C}_1(\tau,\alpha)>0, \mathcal{C}_2=\mathcal{C}_2(\tau,\alpha,\xi_1)>0$ are independent of r.

Proof.
- If $0<\tau<2$, then we have $G(r)\leqslant\sup_{r\in(0,+\infty)}G(r)\leqslant G(r_0)$ where $r_0\in(0,+\infty)$, to solve $G'(r_0)=0$, we can find that $r_0=\left(\frac{\alpha(2-k)}{\varsigma k}\right)^{\frac{1}{2}}>0$, thus we have

$$G(r)\leqslant G(r_0) = \frac{\left(\frac{2-k}{k}\alpha\right)^{\frac{2-k}{2}}}{\left(\frac{2-k}{k}\right)\alpha+\alpha}\varsigma^{\frac{\tau}{2}} = \mathcal{C}_1(\tau,\alpha)\varsigma^{\frac{\tau}{2}}. \tag{55}$$

- If $\tau\geqslant 2$, then we have

$$G(r)\leqslant\frac{\varsigma}{\varsigma r^2+\alpha}\frac{1}{r^{\tau-2}}\leqslant\frac{\varsigma}{\gamma}\frac{1}{\xi_1^{\tau-2}} =: \mathcal{C}_2(\tau,\alpha,\xi_1)\varsigma. \tag{56}$$

$\square$

Lemma 3.4. *For constants $\tau > 0, \varsigma > 0, \alpha > 0, r \geqslant \xi_1 > 0$, we have*

$$F(r) = \frac{\varsigma r^{1-\tau}}{\varsigma r^2 + \alpha} \leqslant \begin{cases} \mathcal{C}_3 \varsigma^{\frac{\tau+1}{2}}, & 0 < \tau < 1, \\ \mathcal{C}_4 \varsigma, & \tau \geqslant 1, \end{cases} \tag{57}$$

where $\mathcal{C}_3 = \mathcal{C}_3(\tau, \alpha) > 0, \mathcal{C}_4 = \mathcal{C}_4\big(\tau, \alpha, \xi_1\big) > 0$ are independent of r.

Proof.

- If $0 < \tau < 1$, then we have $F(r) \leqslant \sup_{r \in (0,+\infty)} F(r) \leqslant F(r_0)$, where $r_0 \in (0, +\infty)$ such that $F'(r_0) = 0$, we can find that $r_0 = \Big(\frac{\alpha(1-\tau)}{\varsigma(1+\tau)}\Big)^{\frac{1}{2}} > 0$, thus we have

$$F(r) \leqslant \frac{\big(\alpha \frac{1-\tau}{1+\tau}\big)^{\frac{1-\tau}{2}} (1+\tau)}{2\alpha} \varsigma^{\frac{\tau+1}{2}} =: \mathcal{C}_3(\tau, \alpha) \varsigma^{\frac{\tau+1}{2}}. \tag{58}$$

- If $\tau \geqslant 1$, then for $r \geqslant \xi_1 > 0$, we have

$$F(r) \leqslant \frac{\varsigma}{\varsigma r^2 + \alpha} \frac{1}{r^{\tau-1}} \leqslant \frac{\varsigma}{\alpha} \frac{1}{\xi_1^{\tau-1}} =: \mathcal{C}_4\big(\tau, \alpha, \xi_1\big)\varsigma. \tag{59}$$

□

4. A Modified Quasi Boundary Value Method

In this section, we propose a modified quasi-boundary value method to solve problem (1).

$$\begin{cases} \mathbb{D}_t^{\gamma,\kappa} u^\epsilon_{\beta(\epsilon)}(x,t) - \Delta u^\epsilon_{\beta(\epsilon)}(x,t) = \varphi^\epsilon(t) f^\epsilon_{\beta(\epsilon)}(x), & \text{in } \Omega \times (0,T], \\ u^\epsilon_{\beta(\epsilon)}(x,t) = 0 & \text{in } \partial\Omega \times (0,T], \\ u^\epsilon_{\beta(\epsilon)}(x,0) = 0, & \text{in } \Omega \\ \displaystyle\int_0^T u^\epsilon_{\beta(\epsilon)}(x,t)dt = g^\epsilon(x) + \beta(\epsilon)(\Delta f^\epsilon_{\beta(\epsilon)})(x), & \text{in } \Omega. \end{cases}$$

Let $\Big\{u^\epsilon_{\beta(\epsilon),m}(x,t), f^\epsilon_{\beta(\epsilon),m}(x)\Big\}$ be the solution of the following regularized problem, where $\beta(\epsilon) > 0$ is a regularization parameter. In following subsections, we will prove that $f^\epsilon_{\beta(\epsilon)}(x)$ is a regularized solution for the source term $f(x)$ in (10) and (11). By the separation of variables, we know $u^\epsilon_{\beta(\epsilon)}(x,t)$ has the following form

$$\big\langle u^\epsilon_{\beta(\epsilon)}(\cdot,t), \phi_m \big\rangle = \big\langle f^\epsilon_{\beta(\epsilon)}, \phi_m \big\rangle \int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma) \varphi^\epsilon(s) ds, \tag{60}$$

From the condition non-local, we get

$$\big\langle f^\epsilon_{\beta(\epsilon)}, \phi_m \big\rangle \bigg(\int_0^T \Big(\int_0^t (t-s)^{\gamma-1} e^{-\kappa(t-s)} E_{\gamma,\gamma}(-\xi_m (t-s)^\gamma) \varphi^\epsilon(s) ds \Big) dt + \beta(\epsilon)\xi_m \bigg) = \big\langle g^\epsilon, \phi_m \big\rangle. \tag{61}$$

From (61), we rewrite this formula as follows

$$\big\langle f^\epsilon_{\beta(\epsilon)}, \phi_m \big\rangle = \frac{\big\langle g^\epsilon, \phi_m \big\rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T, \varphi^\epsilon) + \beta(\epsilon)\xi_m}. \tag{62}$$

This leads to

$$f^\epsilon_{\beta(\epsilon)}(x) = \sum_{m=1}^{+\infty} \frac{\big\langle g^\epsilon, \phi_m \big\rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T, \varphi^\epsilon) + \beta(\epsilon)\xi_m} \phi_m(x), \tag{63}$$

and

$$f_{\beta(\epsilon)}(x) = \sum_{m=1}^{+\infty} \frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} \phi_m(x). \tag{64}$$

In the following, we give two convergence rate estimates for $\|f^\epsilon_{\beta(\epsilon)} - f\|_2$ by using *an a priori* and *an a posteriori* choice rule for the regularization parameter.

4.1. **A priori parameter choice rule.**

Theorem 4.1. *Let* $\varphi(t) \in L^\infty(0,T)$ *satisfying* (4). *Suppose the a priori condition* (49) *and the noise assumption* $\|\varphi - \varphi^\epsilon\|_{L^\infty(0,T)} + \|g^\epsilon - g\|_{L^2(\Omega)} \le \epsilon$ *hold, then,*

- *If* $0 < \tau < 2$, *and* $\beta(\epsilon) = \left(\frac{\epsilon}{\mathcal{E}}\right)^{\frac{2}{\tau+1}}$, *we have*

$$\|f^\epsilon_{\beta(\epsilon)} - f\|_2 \text{ ***is of order*** } \epsilon^{\frac{\tau}{\tau+1}}; \tag{65}$$

- *If* $\tau \geqslant 2$, *and* $\beta(\epsilon) = \left(\frac{\epsilon}{\mathcal{E}}\right)^{\frac{2}{3}}$, *we have*

$$\|f^\epsilon_{\beta(\epsilon)} - f\|_2 \text{ ***is of order*** } \epsilon^{\frac{2}{3}}, \tag{66}$$

Proof. By the triangle inequality, we know that

$$\|f^\epsilon_{\beta(\epsilon)} - f\|_2 \leqslant \|f^\epsilon_{\beta(\epsilon)} - f_{\beta(\epsilon)}\|_2 + \|f_{\beta(\epsilon)} - f\|_2. \tag{67}$$

We firstly give an estimate for the first term. From (63), (64), we have

$$\begin{aligned}
\|f^\epsilon_{\beta(\epsilon)} - f_{\beta(\epsilon)}\|_2^2 &= \sum_{m=1}^{+\infty} \Big(\frac{\langle g^\epsilon, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon) + \beta(\epsilon)\xi_m} - \frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} \Big)^2 \\
&\le 2\sum_{m=1}^{+\infty} \left(\frac{\langle g^\epsilon, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon) + \beta(\epsilon)\xi_m} - \frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon) + \beta(\epsilon)\xi_m} \right)^2 \\
&\quad + 2\sum_{m=1}^{+\infty} \left(\frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon) + \beta(\epsilon)\xi_m} - \frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} \right)^2 \\
&\le 2\sum_{m=1}^{+\infty} \big[\mathcal{A}(m)\big]^2 \big|\langle g^\epsilon - g, \phi_m \rangle\big|^2 + 2\sum_{m=1}^{+\infty} \big[\mathcal{B}(m)\big]^2 \left| \frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)} \right|^2, \qquad (68)
\end{aligned}$$

whereby

$$\mathcal{A}(m) = \frac{1}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon) + \beta(\epsilon)\xi_m},\ \mathcal{B}_m = \frac{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon - \varphi)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon) + \beta(\epsilon)\xi_m} \tag{69}$$

By Lemma 2.6 and (3.9), we get

$$\mathcal{A}(m) \leqslant \left(\frac{\mathscr{B}_0}{\xi_m} + \beta(\epsilon)\xi_m \right)^{-1} \leqslant \frac{1}{2}\big(\mathscr{B}_0 \beta(\epsilon)\big)^{-\frac{1}{2}}, \tag{70}$$

and

$$\mathcal{B}(m) \le \frac{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon - \varphi)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon) + \beta(\epsilon)\xi_m} \le \epsilon \frac{T}{\mathscr{B}_0}. \tag{71}$$

Substituting (69) and (70), (71) into (68), we obtain

$$\|f^\epsilon_{\beta(\epsilon)} - f_{\beta(\epsilon)}\|_2 \leqslant \frac{1}{2}\epsilon\big(\mathscr{B}_0\beta(\epsilon)\big)^{-\frac{1}{2}} + \epsilon\frac{T}{\mathscr{B}_0}\|f\|_2. \tag{72}$$

Now we estimate the second term in (67), we can deduce that

$$f_{\beta(\epsilon)}(x) - f(x) = \sum_{m=1}^{+\infty} \Big(\frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} - \frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)} \Big) \phi_m(x)$$

$$= \sum_{m=1}^{+\infty} \frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)} \frac{-\beta(\epsilon)\xi_m}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)+\beta(\epsilon)\xi_m} \phi_m(x). \tag{73}$$

Applying the a priori bound condition (49), we obtain

$$\begin{aligned} \left\| f_{\beta(\epsilon)} - f \right\|_2^2 &= \sum_{m=1}^{+\infty} \frac{\xi_m^{2\tau} \left| \langle g, \phi_m \rangle \right|^2}{\left| \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) \right|^2} \left(\frac{\beta(\epsilon)\xi_m}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)+\beta(\epsilon)\xi_m} \right)^2 \frac{1}{\xi_m^{2\tau}} \\ &\leqslant \mathcal{E}^2 \big(\sup_{m\in\mathbb{N}} \mathcal{C}(m) \big)^2, \end{aligned} \tag{74}$$

where

$$\mathcal{C}(m) = \frac{\beta(\epsilon)\xi_m^{1-\tau}}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)+\beta(\epsilon)\xi_m}. \tag{75}$$

By Lemmas 3.4 and 3.3, we have

$$\mathcal{C}(m) \leqslant \frac{\beta(\epsilon)\xi_m^{2-\tau}}{\beta(\epsilon)\xi_m^2 + \mathscr{B}_0} \leqslant \begin{cases} \mathcal{C}_1\left(k, \mathscr{B}_0\right) [\beta(\epsilon)]^{\frac{\tau}{2}}, & 0<\tau<2, \\ \mathcal{C}_2\left(\tau, \mathscr{B}_0, \xi_1\right) \beta, & \tau \geqslant 2. \end{cases} \tag{76}$$

Substituting the above inequality into (74), and using the (67), we obtain

$$\left\| f^{\epsilon}_{\beta(\epsilon)} - f \right\|_2 \leqslant \frac{1}{2}\epsilon\big(\mathscr{B}_0\beta(\epsilon)\big)^{-\frac{1}{2}} + \epsilon\frac{T}{\mathscr{B}_0}\left\|f\right\|_2 + \begin{cases} \mathcal{C}_1(\tau,\mathscr{B}_0)\mathcal{E}[\beta(\epsilon)]^{\frac{\tau}{2}}, & 0<\tau<2, \\ \mathcal{C}_2(\tau,\mathscr{B}_0,\xi_1)\mathcal{E}[\beta(\epsilon)], & \tau\geqslant 2. \end{cases} \tag{77}$$

Combining (72) and (77), we conclude that

- If $0<\tau<2$, $\beta(\epsilon) = \left(\frac{\epsilon}{\mathcal{E}}\right)^{\frac{2}{\tau+1}}$, then we have

$$\left\| f^{\epsilon}_{\beta(\epsilon)} - f \right\|_2 \leqslant \left(\frac{1}{2}\mathcal{E}^{\frac{1}{\tau+1}}\mathscr{B}_0^{-1/2} + \epsilon^{\frac{1}{\tau+1}}\frac{T}{\mathscr{B}_0}\|f\|_2 + \mathcal{C}_1(\tau,\mathscr{B}_0)\mathcal{E}^{\frac{1}{\tau+1}} \right)\epsilon^{\frac{\tau}{\tau+1}}. \tag{78}$$

- If $\tau>2$, $\beta(\epsilon) = \left(\frac{\epsilon}{\mathcal{E}}\right)^{\frac{2}{3}}$, then we have

$$\left\| f^{\epsilon}_{\beta(\epsilon)} - f \right\|_2 \leqslant \left(\frac{1}{2}\mathcal{E}^{\frac{1}{3}}\mathscr{B}_0^{-1/2} + \epsilon^{\frac{1}{3}}\frac{T}{\mathscr{B}_0}\|f\|_2 + \mathcal{C}_1(\tau,\mathscr{B}_0)\mathcal{E}^{\frac{1}{3}} \right)\epsilon^{\frac{2}{3}}. \tag{79}$$

The proof is completed. □

4.2. **A posteriori parameter choice rule.** In this subsection, we use an a posterior regularization parameter choice, i.e., Morozov's discrepancy principle to choose the regularization parameter $\beta(\epsilon)$ in [26]. Morozov's discrepancy principle for our case is to find $\beta(\epsilon)$ such that

$$\left\| \mathcal{K} f^{\epsilon}_{\beta(\epsilon)} - g^{\epsilon} \right\|_2 = \delta\epsilon. \tag{80}$$

where $\delta>1$ is a constant. According to the following lemma, we know there exists a unique solution for (80) if $\left\|g^{\epsilon}\right\|_2 > \delta\epsilon > 0$.

Lemma 4.1. *Set* $\Xi(\beta(\epsilon)) = \left\| \mathcal{K} f^{\epsilon}_{\beta(\epsilon)} - g^{\epsilon} \right\|_2$. *If* $\|g^{\epsilon}\|_2 > \epsilon > 0$, *then the following results hold:*

- $\Xi(\beta(\epsilon))$ *is a continuous function;*
- $\lim_{\beta(\epsilon)\to 0} \Xi(\beta(\epsilon)) = 0$;
- $\lim_{\beta(\epsilon)\to +\infty} \Xi(\beta(\epsilon)) = \|g^{\epsilon}\|_2$;
- $\Xi(\beta(\epsilon))$ *is a strictly increasing function over* $(0,\infty)$.

Proof. The proofs are straightforward results by the expression of

$$\Xi(\beta(\epsilon)) = \left(\sum_{m=1}^{+\infty} \Big(\frac{\beta(\epsilon)\xi_m}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} \Big)^2 \big| \langle g^\epsilon, \phi_m \rangle \big|^2 \right)^{\frac{1}{2}}. \tag{81}$$

□

Theorem 4.2. *Let $\varphi(t) \in L^\infty(0,T)$ satisfying $\varphi(t) \geqslant \varphi_0 > 0, t \in [0,T]$. Suppose the a priori condition* (49) *and the noise assumption* (4) *hold, and there exists $\delta > 1$ such that $\|g^\epsilon\|_2 > \delta\epsilon > 0$. The regularization parameter $\beta(\epsilon) > 0$ is chosen by Morozov's discrepancy principle* (80)*. Then*

- *If $0 < \tau < 1$, we have a convergence rate estimate*

$$\big\| f^\epsilon_{\beta(\epsilon)} - f \big\|_2 \textit{ is of order } \epsilon^{\frac{\tau}{\tau+1}}. \tag{82}$$

- *If $\tau \geqslant 1$, we have a convergence rate estimate*

$$\big\| f^\epsilon_{\beta(\epsilon)} - f \big\|_2 \textit{ is of order } \epsilon^{\frac{1}{2}}. \tag{83}$$

Proof. Similar to (67), we have

$$\big\| f^\epsilon_{\beta(\epsilon)} - f \big\|_2 \leqslant \big\| f^\epsilon_{\beta(\epsilon)} - f_{\beta(\epsilon)} \big\|_2 + \big\| f_{\beta(\epsilon)} - f \big\|_2 \tag{84}$$

We firstly give an estimate for the second term. From (73), we know that

$$\begin{aligned} \mathcal{K}\big(f_{\beta(\epsilon)} - f\big) &= \sum_{m=1}^{+\infty} \langle g, \phi_m \rangle \frac{-\beta(\epsilon)\xi_m}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} \phi_m(x) \\ &= \sum_{m=1}^{+\infty} \langle g - g^\epsilon, \phi_m \rangle \frac{-\beta(\epsilon)\xi_m}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} \phi_m(x) \\ &\quad + \sum_{m=1}^{+\infty} \langle g^\epsilon, \phi_m \rangle \frac{-\beta(\epsilon)\xi_m}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} \phi_m(x). \end{aligned} \tag{85}$$

Using (85) and (80), we get

$$\big\| \mathcal{K}\big(f_{\beta(\epsilon)} - f\big) \big\|_2 \leqslant (\delta + 1)\epsilon. \tag{86}$$

Applying the a priori bound condition for $f(x)$, we obtain

$$\begin{aligned} \big\| f_{\beta(\epsilon)} - f \big\|_{\mathscr{H}^\tau(\Omega)} &= \left(\sum_{m=1}^{+\infty} \Big(\frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)} \frac{-\beta(\epsilon)\xi_m}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} \xi_m^\tau \Big)^2 \right)^{\frac{1}{2}} \\ &\quad \times \Big(\sum_{m=1}^{+\infty} \Big(\frac{\langle g, \phi_m \rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)} \Big)^2 \xi_m^{2\tau} \Big)^{\frac{1}{2}} \leqslant \mathcal{E}. \end{aligned} \tag{87}$$

By Theorem 3.3, we receive

$$\big\| f_{\beta(\epsilon)} - f \big\|_2 \leqslant |\varphi_0 T|^{-\frac{\tau}{\tau+1}} (\delta + 1)^{\frac{\tau}{\tau+2}} \mathcal{E}^{\frac{1}{\tau+1}} \epsilon^{\frac{\tau}{\tau+1}}, \quad \forall \tau > 0. \tag{88}$$

Similar to (72), we know that

$$\big\| f^\epsilon_{\beta(\epsilon)} - f_{\beta(\epsilon)} \big\|_2 \leq \frac{1}{2} \epsilon \big(\mathscr{B}_0 \beta(\epsilon) \big)^{-\frac{1}{2}} + \epsilon \frac{T}{\mathscr{B}_0} \big\| f \big\|_2. \tag{89}$$

From (81), there holds

$$\delta\epsilon = \Big\| \sum_{m=1}^{+\infty} \frac{\beta(\epsilon)\xi_m}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) + \beta(\epsilon)\xi_m} \langle g^\epsilon, \phi_m \rangle \phi_m(x) \Big\|_2 \leqslant \epsilon + \mathcal{H}. \tag{90}$$

Using again the a priori bound condition for f, we obtain

$$\mathcal{H} = \Big\| \sum_{m=1}^{+\infty} \frac{\beta(\epsilon)\xi_m \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)+\beta(\epsilon)\xi_m} \frac{1}{\xi_m^\tau} \frac{\langle g,\phi_m\rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)} \xi_m^\tau \phi_m(x) \Big\|_2 \leqslant \mathcal{E} \sup_{m\in\mathbb{N}} \mathcal{D}(m), \tag{91}$$

whereby

$$\mathcal{D}(m) = \frac{\beta(\epsilon)\xi_m \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)+\beta(\epsilon)\xi_m} \frac{1}{\xi_m^\tau} \leqslant \varphi_1 T \frac{\beta(\epsilon)}{\frac{\mathscr{B}_0}{\xi_m}+\beta(\epsilon)\xi_m} \frac{1}{\xi^\tau} \leqslant \varphi_1 T \frac{\beta(\epsilon)\xi_m^{1-\tau}}{\mathscr{B}_0+\beta(\epsilon)\xi_m^2}, \tag{92}$$

By Lemma 3.4, (92) becomes

$$\mathcal{D}(m) \leqslant \begin{cases} \varphi_1 T \mathcal{C}_3(\tau,\mathscr{B}_0)\,[\beta(\epsilon)]^{\frac{\tau+1}{2}}, & 0<\tau<1, \\ \varphi_1 T \mathcal{C}_4(\tau,\mathscr{B}_0,\xi_1)\,[\beta(\epsilon)], & \tau \geqslant 1. \end{cases} \tag{93}$$

Combining (90)-(93), it yields

$$(\delta-1)\epsilon \leqslant \begin{cases} \varphi_1 T \mathcal{C}_3(\tau,\mathscr{B}_0)\mathcal{E}[\beta(\epsilon)]^{\frac{\tau+1}{2}}, & 0<\tau<1, \\ \varphi_1 T \mathcal{C}_4(\tau,\mathscr{B}_0,\xi_1)\mathcal{E}\beta(\epsilon), & \tau \geqslant 1. \end{cases} \tag{94}$$

- If $0<\tau<1$, and $\dfrac{1}{\beta(\epsilon)} \leqslant \left(\dfrac{\varphi_1 T\mathcal{C}_3(\tau,\mathscr{B}_0)}{\delta-1}\right)^{\frac{2}{\tau+1}} \left(\dfrac{\mathcal{E}}{\epsilon}\right)^{\frac{2}{\tau+1}}$, then we have

$$\begin{aligned} \Big\| f^\epsilon_{\beta(\epsilon)} - f \Big\|_2 &\leqslant |\varphi_0 T|^{-\frac{\tau}{\tau+1}} (\delta+1)^{\frac{\tau}{\tau+2}} \mathcal{E}^{\frac{1}{\tau+1}} \epsilon^{\frac{\tau}{\tau+1}} \\ &\quad + \frac{1}{2}\mathscr{B}_0^{-\frac{1}{2}} \mathcal{E}^{\frac{1}{\tau+1}} \left(\frac{\varphi_1 T\mathcal{C}_3(\tau,\mathscr{B}_0)}{\delta-1}\right)^{\frac{1}{\tau+1}} \epsilon^{\frac{\tau}{\tau+1}} + \epsilon\frac{T}{\mathscr{B}_0}\|f\|_2. \end{aligned} \tag{95}$$

- If $\tau>1$, and $\dfrac{1}{\beta(\epsilon)} \leqslant \dfrac{\varphi_1 T\mathcal{C}_4(\tau,\mathscr{B}_0,\xi_1)}{\tau-1}\dfrac{\mathcal{E}}{\epsilon}$, then we have

$$\begin{aligned} \| f^\epsilon_{\beta(\epsilon)} - f \|_2 &\leqslant |\varphi_0 T|^{-\frac{1}{2}} (\delta+1)^{\frac{1}{2}} \mathcal{E}^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \\ &\quad + \frac{1}{2}\mathscr{B}_0^{-\frac{1}{2}} \mathcal{E}^{\frac{1}{2}} \left(\frac{\varphi_1 T\mathcal{C}_3(\tau,\mathscr{B}_0)}{\delta-1}\right)^{\frac{1}{2}} \epsilon^{\frac{1}{2}} + \epsilon\frac{T}{\mathscr{B}_0}\|f\|_2. \end{aligned} \tag{96}$$

□

5. Fractional Landweber Method

In this section, we propose a fractional Landweber iterative regularization method to solve the problem (1). Let $\mathcal{K}^*$ be the adjoint of the linear compact operator $\mathcal{K}$. Since $\mathcal{K}$ is self-adjoint operator and $\{\phi_m\}_{m=1}^\infty$ are orthonormal in $L^2(\Omega)$, it yields

$$\mathcal{K}^*\mathcal{K}\phi_m(\xi) = \left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2 \phi_m(\xi), \tag{97}$$

Hence, the singular values of $\mathcal{K}$ is $\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)$. Further, we can verify

$$\mathcal{K}\phi_m(x) = \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\phi_m(x),\ \mathcal{K}^*\phi_m(x) = \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\phi_m(x). \tag{98}$$

Therefore, the singular system of $\mathcal{K}$ is $\left(\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi), \phi_m, \phi_m\right)$. In general, the iterative Landweber method defines a regularized solution

$$f^{n,\epsilon}(x) = b\sum_{j=0}^{n-1} \left(I - b\mathcal{K}^*\mathcal{K}\right)^j \mathcal{K}^* g^\epsilon(x), \quad n=1,2,3,\ldots, \tag{99}$$

where n is the regularization parameter, b is called the relaxation factor and satisfies $0 < b < \left|\mathcal{K}\right|^{-2}$. In order to introduce the Landweber iterative method, defining a filter-based regularization operators

$$\mathcal{R}_m g := \sum_{\mathbf{Q}_m^{\gamma,\kappa}(T)>0} F_n\big(\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big)\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)^{-1} g_m\phi_m, \tag{100}$$

with the filter function $F_n\big(\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big) := 1 - \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n$ and the singular system $\big(\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi), \phi_m, \phi_m\big)$ of a linear compact operator $\mathcal{K}$. Here, the operator $\mathcal{R}_m : L^2(\Omega) \to L^2(\Omega)$ defines an regularization method. Whether a filter-based method is a regularization method at all can be checked by the following conditions (101).

$$\begin{aligned}
&\sup_{m\geq 1} \left|F_\beta\big(\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big)\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)^{-1}\right| = \mathcal{C}(\beta) < \infty,\\
&\lim_{\beta\to 0} \left|F_\beta\left(\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right)\right| = 1, \text{ pointwise in } \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi),\\
&\Big|F_\beta\big(\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big)\Big| \leq \mathcal{C}, \quad \forall\beta, \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi),
\end{aligned} \tag{101}$$

where β is the regularization parameter, $\mathbf{Q}_m$ are the singular values. From (101) we can obtain

$$f^{n,\epsilon}(x) = \mathcal{R}_m g^\epsilon(x) = \sum_{m=1}^{+\infty} \frac{1 - \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon)} \langle g^\epsilon, \phi_m\rangle\phi_m(x). \tag{102}$$

- If the filter function $F_n^z = \bigg(1 - \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n\bigg)^z$ for $\frac{1}{2} < z < 1$ in (100), it is called fractional Landweber iterative regularization method, see [9].
- The fractional Landweber iterative regularization solution with exact data and noisy data are given by

$$f_z^n(x) = \sum_{m=1}^{+\infty} \frac{\Big[1 - \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n\Big]^z}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)} \langle g, \phi_m\rangle\phi_m(x), \quad \frac{1}{2} < z < 1, \tag{103}$$

and

$$f_z^{n,\epsilon}(x) = \sum_{m=1}^{+\infty} \frac{\Big[1 - \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n\Big]^z}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon)} \langle g^\epsilon, \phi_m\rangle\phi_m(x), \quad \frac{1}{2} < z < 1, \quad n = 1,2,3,\ldots \tag{104}$$

  - Case $z = 1$ in equations (103) and (104), it is called the classic Landweber iterative regularization method .

5.1. **A Priori Parameter Choice.** Firstly, based on the priori regularization parameter selection rule, we give a priori convergence error estimate between $f_z^{n,\epsilon}$ and f.

Theorem 5.1. *Let $f(x)$ given by* (10) *be the exact solution of problem* (1). *Let $f_z^{n,\epsilon}(x)$ given by* (104) *be the fractional Landweber iterative approximation solution. Suppose the assumption* (49) *is satisfied. Choosing the regularization parameter $n = \lfloor b \rfloor$, where*

$$b = \frac{1}{2}\Big(\frac{\mathcal{E}}{\epsilon}\Big)^{\frac{2}{\tau+1}}, \tag{105}$$

then we have

$$\|f_z^{n,\epsilon} - f\|_2 \leq \textit{\textbf{is of order}}\ \epsilon^{\frac{\tau}{\tau+1}}\cdot \tag{106}$$

where $\lfloor b \rfloor$ denotes the largest integer less than or equal to b

Proof. Using the triangle inequality, we obtain

$$\left\|f_z^{n,\epsilon}-f\right\|_2 \le \left\|f_z^{n,\epsilon}-f_z^n\right\|_2+\left\|f_z^n-f\right\|_2. \tag{107}$$

From (103) and (104), we receive

$$\begin{aligned}
&\left\|f_z^{n,\epsilon}-f_z^n\right\|_2\\
&=\Bigg\|\sum_{m=1}^{+\infty}\left[1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n\right]^z\frac{\langle g^\epsilon,\phi_m\rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon)}\phi_m(\cdot)\\
&\qquad-\sum_{m=1}^{+\infty}\left[1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n\right]^z\frac{\langle g,\phi_m\rangle\phi_m(\cdot)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\Bigg\|_2\\
&=\Bigg\|\sum_{m=1}^{+\infty}\left[1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n\right]^z\frac{\langle g^\epsilon-g,\phi_m\rangle}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon)}\phi_m(\cdot)\\
&\qquad-\sum_{m=1}^{+\infty}\left[1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n\right]^z\left(\frac{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi-\varphi^\epsilon)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon)}\right)\frac{\langle g,\phi_m\rangle\phi_m(\cdot)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\Bigg\|_2\\
&\le\mathcal{I}_1+\mathcal{I}_2, \qquad (108)
\end{aligned}$$

whereby

$$\begin{aligned}
\mathcal{I}_1&=\sum_{m=1}^{+\infty}\frac{\left[1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n\right]^z}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\frac{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon)}\langle g^\epsilon-g,\phi_m\rangle\phi_m(\cdot),\\
\mathcal{I}_2&=\sum_{m=1}^{+\infty}\frac{\left[1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n\right]^z}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\left(\frac{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi-\varphi^\epsilon)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi^\epsilon)}\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right)\frac{\langle g,\phi_m\rangle\phi_m(\cdot)}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}. \qquad (109)
\end{aligned}$$

Let us denote

$$\mathcal{H}(m):=\frac{\left[1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n\right]^z}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}. \tag{110}$$

Due to Bernoulli inequality, we obtain

$$\frac{1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T)\right|^2\right)^n}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\le\left(\frac{1-\left(1-b\left|\mathbf{Q}_n^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\right)^{\frac{1}{2z}}\le(bn)^{\frac{1}{2}}, \tag{111}$$

then we have estimate of $\mathcal{I}_1$ and $\mathcal{I}_2$ as follows:

$$\mathcal{I}_1\le(bn)^{\frac{1}{2}}\frac{\varphi_1}{\mathcal{P}(\varphi_0,\varphi_1)}\epsilon,\ \mathcal{I}_2\le(bn)^{\frac{1}{2}}\epsilon\frac{\varphi_1}{\mathcal{P}(\varphi_0,\varphi_1)}\|f\|_2 \tag{112}$$

Combining (108) to (112), we conclude that

$$\left\|f_z^{n,\epsilon}-f_z^n\right\|_2\le(bn)^{\frac{1}{2}}\epsilon\mathcal{Q}(\varphi_0,\varphi_1,f). \tag{113}$$

Next, we have

$$\begin{aligned}
\left\|f_z^n-f\right\|_2&=\Bigg\|\sum_{m=1}^{+\infty}\frac{\left[1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n\right]^z}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\langle g,\phi_m\rangle\phi_m(\cdot)-\sum_{m=1}^{+\infty}\frac{1}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\langle g,\phi_m\rangle\phi_m(\cdot)\Bigg\|_2\\
&=\Bigg\|\sum_{m=1}^{+\infty}\frac{\left\{1-\left[1-\left(1-b\left|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\right|^2\right)^n\right]^z\right\}}{\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)}\langle g,\phi_m\rangle\phi_m(\cdot)\Bigg\|_2
\end{aligned}$$

$$= \left\| \sum_{m=1}^{+\infty} \left\{ 1 - \left[1 - \left(1 - b \left| \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) \right|^2 \right)^n \right]^z \right\} \xi_m^{-\tau} \xi_m^{\tau} \langle f, \phi_m \rangle \phi_m(\cdot) \right\|_2$$

$$\le \sup_{m \ge 1} \left\{ \left\{ 1 - \left[1 - \left(1 - b \left| \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) \right|^2 \right)^m \right]^z \right\} \xi_m^{-\tau} \right\} \mathcal{E} \le \sup_{m \ge 1} \mathcal{Z}(m) \mathcal{E}, \tag{114}$$

where $\mathcal{Z}(m) := \left(1 - b \left| \mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) \right|^2 \right)^n (\xi_m)^{-\tau}$. According to Lemmas 34, 3.2, we obtain

$$\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi) \ge \frac{\mathscr{B}_0}{\xi_m}, \tag{115}$$

then

$$\mathcal{Z}(m) \le \left(1 - b \left| \frac{\mathscr{B}_0}{\xi_m} \right|^2 \right)^n (\xi_m)^{-\tau}. \tag{116}$$

Let $\mathcal{Z}(r) = \left(1 - b \left| \frac{\mathscr{B}_0}{r} \right|^2 \right)^n r^{-\tau}, r := \xi_m$. Suppose r_0 satisfies $\mathcal{Z}'(r_0) = 0$, we can find that

$$r_0 = \left(\frac{b\mathscr{B}_0(2n+\tau)}{\tau} \right)^{\frac{1}{2}}, \tag{117}$$

then

$$\mathcal{Z}(r) \le \mathcal{Z}(r_0) = \left(1 - \frac{1}{2n+\tau} \right)^n \left(\frac{b\mathscr{B}_0(2n+\tau)}{\tau} \right)^{-\frac{\tau}{2}} \le \left(\frac{\tau}{b\mathscr{B}_0} \right)^{\frac{\tau}{2}} (2n+\tau)^{-\frac{\tau}{2}}, \tag{118}$$

thus

$$\left\| f_z^n - f \right\|_2 \le \left(\frac{\tau}{b\mathscr{B}_0} \right)^{\frac{\tau}{2}} (2n+\tau)^{-\frac{\tau}{2}} \mathcal{E}. \tag{119}$$

Combining (113) to (119), we obtain

$$\left\| f_z^{n,\epsilon} - f \right\|_2 \le \left(\frac{\tau}{b\mathscr{B}_0} \right)^{\frac{\tau}{2}} (2n+\tau)^{-\frac{\tau}{2}} \mathcal{E} + (bn)^{\frac{1}{2}} \epsilon \mathcal{Q}(\varphi_0, \varphi_1, f) \le C_2 \mathcal{E}^{\frac{1}{\tau+1}} \epsilon^{\frac{\tau}{\tau+1}}, \tag{120}$$

where $\mathcal{C}_2 := \left(\frac{b}{2}\right)^{\frac{1}{2}} + \left(\frac{\tau}{b\mathscr{B}_0}\right)^{\frac{\tau}{2}}$. The proof of Theorem 5.1 is completed. □

5.2. **A posteriori parameter choice rule.** Next, we mainly use the Morozov's discrepancy principle to obtain a posteriori convergence error estimate between $f_z^{n,\epsilon}$ and f. Assume $\delta > 1$ is given a fixed constant. Stop the algorithm at the first occurrence of $n = n(\delta) \in \mathbb{N}_0$ with the first occurrence of n with

$$\left\| \mathcal{K} f_z^{n,\epsilon} - g^\epsilon \right\|_2 \le \delta \epsilon, \tag{121}$$

where $\|g^\epsilon\| \ge \delta\epsilon$ is constant.

Lemma 5.1. *Let $\zeta(m) = \|K f_z^{n,\epsilon} - g^\epsilon\|_2$, then we have the following conclusions*

- *$\zeta(m)$ is a continuous function;*
- *$\lim_{m\to 0} \zeta(m) = \|g^\epsilon\|_2$;*
- *$\lim_{m\to +\infty} \varsigma(m) = 0$;*
- *$\zeta(m)$ is a strictly decreasing function for any $m \in (0, +\infty)$.*

Proof. The proof of Lemma 5.1 can be obtained by formula (121), so it is omitted here. Further, we have the following lemma. □

Lemma 5.2. *For $\delta > 1$, combining the fractional Landweber iteration method with stopping rule* (121)*, we can obtain the regularization parameter $n = n(\epsilon)$ satisfies*

$$n(\epsilon) \le \left(\frac{\tau+1}{2b\mathscr{B}_0^2} \right) \left(\frac{\varphi_1 T}{\delta - 1} \right)^{\frac{2}{\tau+1}} \left(\frac{\mathcal{E}}{\epsilon} \right)^{\frac{2}{\tau+1}}. \tag{122}$$

Proof. Notice $\left|1-b|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)|^2\right|<1$, combining (4) and (121), we have

$$\begin{aligned}
\delta\epsilon &\le \left\|\mathcal{K}f_z^{n-1,\epsilon}-g^\epsilon\right\|_2\\
&=\Big\|\sum_{m=1}^{+\infty}(1-\left[1-\left(1-b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\right)^{n-1}\right]^z\langle g,\phi_m\rangle\phi_m(\cdot)\Big\|_2.\\
&\le\Big\|\sum_{m=1}^{+\infty}\left(1-b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\right)^{n-1}\langle g^\epsilon-g,\phi_m\rangle\phi_m(\cdot)\Big\|_2\\
&\qquad+\Big\|\sum_{m=1}^{+\infty}\left(1-b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\right)^{n-1}\langle g,\phi_m\rangle\phi_m(\cdot)\Big\|_2\\
&\le\epsilon+\Big\|\sum_{m=1}^{+\infty}\left(1-b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\right)^{n-1}\langle g,\phi_m\rangle\phi_m(\cdot)\Big\|_2. \qquad (123)
\end{aligned}$$

From (49), we have

$$\begin{aligned}
&\Big\|\sum_{m=1}^{+\infty}\left(1-b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\right)^{n-1}\langle g,\phi_m\rangle\phi_m(\cdot)\Big\|_2\\
&\quad=\Big\|\sum_{m=1}^{+\infty}\left(1-b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\right)^{n-1}\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|\xi_m^{-\tau}\xi_m^{\tau}\langle f,\phi_m\rangle\phi_m(\cdot)\Big\|_2\\
&\quad\le\sup_{m\ge1}\mathcal{I}(m)\mathcal{E}, \qquad (124)
\end{aligned}$$

where $\mathcal{I}(m):=\left(1-b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\right)^{n-1}\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|\xi_m^{-\tau}$, then

$$\mathcal{I}(m)\le\varphi_1T\Big(1-b\Big|\frac{\mathscr{B}_0}{\xi_m}\Big|^2\Big)^n\xi_m^{-\tau-1}. \qquad (125)$$

Let $\mathcal{G}(r)=\left(1-b\big|\frac{\mathscr{B}_0}{r}\big|^2\right)^{n-1}r^{-(\tau+1)}$, $r:=\xi_m$. After these calculation steps, we get

$$\mathcal{G}'(r)=\left\{\frac{r^{-\tau}\left(1-\frac{b\mathscr{B}_0^2}{r^2}\right)^n\left(b\mathscr{B}_0^2(2n+\tau-1)-r^2(\tau+1)\right)}{\left(r^2-b\mathscr{B}_0^2\right)^2}\right\}. \qquad (126)$$

To solve the equation $\mathcal{G}'(r_0)=0$, we can find that r_0 as follows:

$$r_0=\Big(\frac{b\mathscr{B}_0^2(2n+\tau-1)}{\tau+1}\Big)^{\frac12}, \qquad (127)$$

then

$$\begin{aligned}
\mathcal{I}(m)\le\varphi_1T\mathcal{G}(r_0)&=\varphi_1T\Big(1-\frac{\tau+1}{2n+\tau-1}\Big)^{n-1}\Big(\frac{b\mathscr{B}_0^2(2n+\tau-1)}{\tau+1}\Big)^{-\frac{\tau+1}{2}}\\
&\le\varphi_1T\Big(\frac{\tau+1}{b\mathscr{B}_0^22}\Big)^{\frac{\tau+1}{2}}n(\epsilon)^{-\frac{\tau+1}{2}}. \qquad (128)
\end{aligned}$$

Hence, we get

$$(\delta-1)\epsilon\le\varphi_1T\Big(\frac{\tau+1}{b\mathscr{B}_0^22}\Big)^{\frac{\tau+1}{2}}n(\epsilon)^{-\frac{\tau+1}{2}}\mathcal{E}. \qquad (129)$$

Combining (123) and (129), we obtain

$$n(\epsilon)\le\left(\frac{\tau+1}{2b\mathscr{B}_0^2}\right)\left(\frac{\varphi_1T}{\delta-1}\right)^{\frac{2}{\tau+1}}\left(\frac{\mathcal{E}}{\epsilon}\right)^{\frac{2}{\tau+1}}. \qquad (130)$$

The proof of this lemma is completed. □

Theorem 5.2. *Let $f(x)$ given by* (10) *be the exact solution of problem* (1), *and let $f_z^n(x)$ given by* (104). *Suppose the assumption* (4) *is satisfied, then we have*

$$\big\|f_z^{n,\epsilon} - f\big\|_2 \le \textit{ is of order } \epsilon^{\frac{\tau}{\tau+1}}, \tag{131}$$

where $\mathcal{C}_3 = b^{\frac{1}{2}} \left(\frac{\tau+1}{2b\mathscr{B}_0^2}\right)^{\frac{1}{2}} \left(\frac{\varphi_1 T}{\delta-1}\right)^{\frac{1}{\tau+1}} \mathcal{E}^{\frac{1}{\tau+1}} \mathcal{Q}(\varphi_0, \varphi_1, f) + |\varphi_0 T|^{-\frac{\tau}{\tau+1}} \mathcal{E}^{\frac{1}{\tau+1}} \big(\delta+1\big)^{\frac{1}{\tau+1}}$ *is positive constant.*

Proof. It is easy to see that

$$\big\|f_z^{m,\epsilon} - f\big\|_2 \le \big\|f_z^{n,\epsilon} - f_z^n\big\|_2 + \big\|f_z^n - f\big\|_2. \tag{132}$$

First of all, we have

$$\big\|f_z^{n,\epsilon} - f_z^n\big\|_2 \le b^{\frac{1}{2}} \left(\frac{\tau+1}{2b\mathscr{B}_0^2}\right)^{\frac{1}{2}} \left(\frac{\varphi_1 T}{\delta-1}\right)^{\frac{1}{\tau+1}} \mathcal{E}^{\frac{1}{\tau+1}}\ \epsilon^{\frac{\tau}{\tau+1}} \mathcal{Q}(\varphi_0, \varphi_1, f). \tag{133}$$

Afterthat, it yields

$$\begin{aligned}
\Big\|\mathcal{K}\big(f_z^n - f\big)\Big\|_2 &= \Big\| \sum_{m=1}^{+\infty} \Big[1 - \Big[1 - \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n\Big]\Big]^z \langle g, \phi_m\rangle \phi_m(\cdot)\Big\|_2 \\
&= \Big\| \sum_{m=1}^{+\infty} \Big[1 - \Big[1 - \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n\Big]\Big]^z \langle g - g^\epsilon, \phi_m\rangle \phi_m(\cdot) \\
&\qquad + \sum_{m=1}^{+\infty} \Big(1 - \Big[1 - \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n\Big]^z\Big) \langle g^\epsilon, \phi_m\rangle \phi_m(\cdot)\Big\|_2 \\
&\le \Bigg\| \sum_{m=1}^{+\infty} \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n \langle g - g^\epsilon, \phi_m\rangle \phi_m(\cdot)\Bigg\|_2 \\
&\qquad + \Big\| \sum_{m=1}^{+\infty} \Big(1 - \Big[1 - \Big(1 - a\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n\Big]^z\Big) \langle g, \phi_m\rangle \phi_m(\cdot)\Big\|_2 \\
&\le (\delta+1)\epsilon.
\end{aligned} \tag{134}$$

Due to (49), we have

$$\begin{aligned}
\big\|f_z^n - f\big\|_{\mathcal{H}^\tau(\Omega)} &= \Big\| \sum_{m=1}^{+\infty} \Big(1 - \Big[1 - \Big(1 - b\big|\mathbf{Q}_m^{\gamma,\kappa}(T,\varphi)\big|^2\Big)^n\Big]^z\Big) \xi_m^\tau \langle f, \phi_m\rangle \phi_m(\cdot)\Big\|_2 \\
&\le \Big\| \sum_{m=1}^{+\infty} \xi^\tau \langle f, \phi_m\rangle \phi_m(\cdot)\Big\|_2 \le \mathcal{E}.
\end{aligned} \tag{135}$$

According to Theorem 3.3 and estimation formula (135), we obtain

$$\big\|f_z^n - f\big\|_2 \le |\varphi_0 T|^{-\frac{\tau}{\tau+1}} \mathcal{E}^{\frac{1}{\tau+1}} (\delta+1)^{\frac{1}{\tau+1}} \epsilon^{\frac{\tau}{\tau+1}}. \tag{136}$$

Combining (132), (135) and (136), we obtain

$$\begin{aligned}
\big\|f_z^{n,\epsilon} - f\big\|_2 &\le b^{\frac{1}{2}} \left(\frac{\tau+1}{2b\mathscr{B}_0^2}\right)^{\frac{1}{2}} \left(\frac{\varphi_1 T}{\delta-1}\right)^{\frac{1}{\tau+1}} \mathcal{E}^{\frac{1}{\tau+1}}\ \epsilon^{\frac{\tau}{\tau+1}} \mathcal{Q}(\varphi_0, \varphi_1, f) \\
&\qquad + |\varphi_0 T|^{-\frac{\tau}{\tau+1}} \mathcal{E}^{\frac{1}{\tau+1}} \big(\delta+1\big)^{\frac{1}{\tau+1}} \epsilon^{\frac{\tau}{\tau+1}}.
\end{aligned} \tag{137}$$

The Theorem 5.2 is proved. ∎

6. NUMERICAL EXAMPLES

This section presents numerical results demonstrating the efficacy of our proposed schemes. We consider the following diffusion equation associated with fractional derivative:

$$\begin{cases} \mathbb{D}_t^{\gamma,\kappa} u(x,t) - \Delta u(x,t) = \varphi(t) f(x), & \text{in } \Omega \times (0,T], \\ u(x,t) = 0 & \text{in } \partial\Omega \times (0,T], \\ u(x,0) = 0, & \text{in } \Omega. \end{cases} \tag{138}$$

with the following nonlocal condition

$$\int_0^T u(x,t)dt = g(x), \qquad \text{in } \Omega. \tag{139}$$

where $\kappa > 0$ and $\gamma \in (0,1)$. Here $\Omega \in \mathcal{R}^d, (d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. Here $\mathbb{D}_t^{\gamma,\kappa}$ is called tempered Caputo of order α which is defined by (see p. 430 [1])

$$\mathbb{D}_t^{\gamma,\kappa} z(t) = \frac{e^{-\kappa t}}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} \frac{d}{ds}\big(e^{\kappa s} z(s)\big) ds, \tag{140}$$

where Γ is the Gamma function and we suppose $\gamma = 0.5, \kappa = 1, \Omega = (0,\pi), T = 1$

To obtain the exact value of $g(x)$, we solve the equation above using the finite difference method. The split time and space variables are $\Delta t = T/N$ and $\Delta x = \pi/M$. Thus, $t_n = n\Delta t$ $(n = 0,1,2,\ldots,N), x_i = i\Delta x (i = 0,1,2,\ldots,M)$, and the approximate value of the unknown function is denoted by $u_i^n \approx u(x_i, t_n)$. Direct calculation yields the eigenvalues $\xi_n = n^2$ and $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(n\pi x)$ for $n = 1, 2, \cdots$.
By solving the above discrete format and taking $U^N = g$ with Python version 3.12 software with some libraries as follows **numpy**, **sympy** and **matplotlib**, the value g can be obtained. The discrete form of the time fractional derivative is as follows [30].
Next, we generate the data with error by adding random perturbation, i.e.,

$$g^\varepsilon = g + \varepsilon(2\mathbf{rand}(\text{size}(g)) - 1),$$

where the function $\text{randn}(\cdot)$ produces a list of random numbers with a mean of 0 and a variance of 1, and ε represents the relative error level. The error level is given as follows

$$\varepsilon = \left\| g^\delta - g \right\| = \left(\frac{1}{M+1} \sum_{i=1}^{M+1} (g_i - g_i^\varepsilon)^2 \right)^{1/2}.$$

To verify the accuracy of the numerical results, the relative errors are given by the following definition

$$err = \frac{\sqrt{\sum \left(f - f_\mu^\delta\right)^2}}{\sqrt{\sum f^2}}.$$

In reality, acquiring the prior bound E is challenging, and the choice of prior regularization parameters relies on the prior bound condition. Consequently, we exclusively present numerical outcomes based on the posterior regularization parameter selection criteria. We set the initial value $\varphi(t) = \frac{10}{\sqrt{\pi}} t^{1/2} e^{-t/2}, g(x) = \left(-6e^{-1/2} + 4\right) \sin(2x), M = 100, N = 100$ and the noise levels $\varepsilon = 0.05, 0.01, 0.001$.
In numerical examples, we consider the smooth function

$$f(x) = \sin(2x), x \in [0, \pi].$$

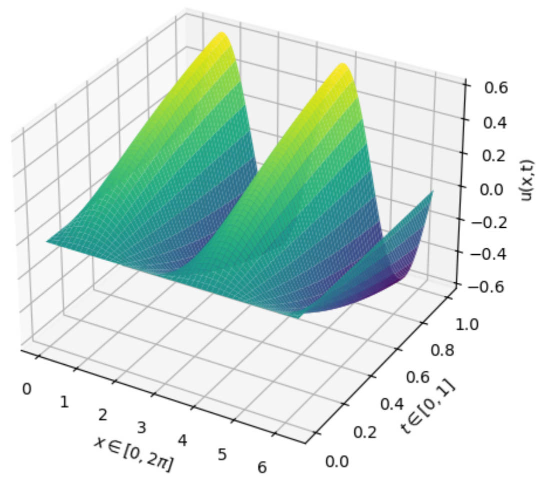
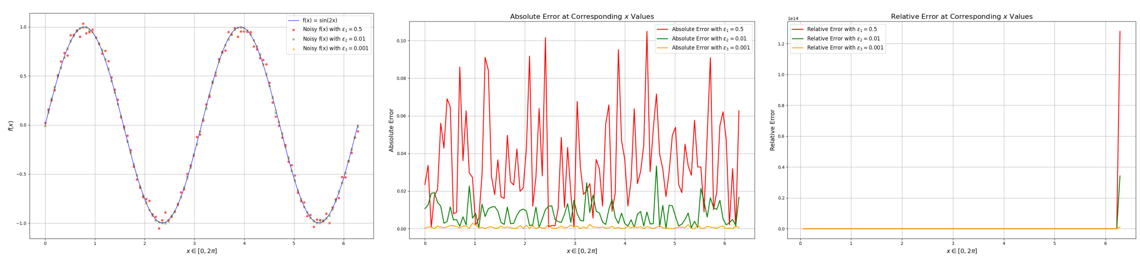


FIGURE 1. The exact solution $u(x,t) = t\exp(-0.5t)\sin(2x)$ for $(x,t) \in (0,2\pi) \times (0,1)$



(a) The source function and noise (b) Absolute error (c) Relative error

FIGURE 2. The source function and errors for $\gamma = 0.5$, respectively

ε	**0.05**	**0.01**	**0.001**
$\gamma = 0.1$	0.0395	0.0158	0.0009
$\gamma = 0.3$	0.1462	0.0452	0.0143
$\gamma = 0.5$	0.0559	0.0125	0.0015
$\gamma = 0.9$	0.1315	0.0432	0.0172

TABLE 1. The relative errors between the exact solution and the regularization solutions.

Figure 2(a) depicts the exact $f(x)$ and the regularization approximation solution $f^\varepsilon(x)$ with $\gamma = 0.1, 0.3, 0.5, 0.7$. Figure 1 depicts the solution in 3D graph for $(x,t) \in (0,2\pi) \times (0,1)$. Table 1 shows that the smaller ε, the lower the relative error between the exact and regularization solutions.

Competing interests. The author declares no competing interests.

References

[1] M.A. Zaky, Existence, Uniqueness and Numerical Analysis of Solutions of Tempered Fractional Boundary Value Problems, Appl. Numer. Math. 145 (2019), 429–457. https://doi.org/10.1016/j.apnum.2019.05.008.

[2] H.D. Binh, N.D. Huy, L.V.C. Hoan, N.H. Can, On Continuity Results for Some Differential Equations with Caputo-Fabrizio Derivative, Hacettepe J. Math. Stat. 55 (2025), 1062–1086. https://doi.org/10.15672/hujms.1670526.

[3] V.N. Minh, N.H. Can, Dependence of Solutions on the Order of the Caputo-Hadamard Derivative and Illustrative Examples, Fract. Calc. Appl. Anal. 28 (2025), 2991–3021. https://doi.org/10.1007/s13540-025-00461-x.

[4] N.A. Triet, D.H.Q. Nam, N.H. Can, L.D. Long, New Regularization and Error Estimate on Terminal Value Problem for Elliptic Equations, Discrete Contin. Dyn. Syst. Ser. S 17 (2024), 1028–1048. https://doi.org/10.3934/dcdss.2023110.

[5] H.D. Binh, N.D. Huy, N.A. Tuan, N.H. Can, On Nonlinear Sobolev Equation with the Caputo Fractional Operator and Exponential Nonlinearity, Math. Methods Appl. Sci. 47 (2024), 1492–1513. https://doi.org/10.1002/mma.9624.

[6] N.H. Can, L.T.M. Duc, N.H. Luc, N.A. Tuan, On the Initial Value Problem for Parabolic Equations with Memory Terms of Fractional Type, Filomat 38 (2024), 7895–7921. https://doi.org/10.2298/fil2422895c.

[7] T. Abdeljawad, S. Banerjee, G.C. Wu, Discrete Tempered Fractional Calculus for New Chaotic Systems with Short Memory and Image Encryption, Optik 218 (2020), 163698. https://doi.org/10.1016/j.ijleo.2019.163698.

[8] F. Sabzikar, M.M. Meerschaert, J. Chen, Tempered Fractional Calculus, J. Comput. Phys. 293 (2015), 14–28. https://doi.org/10.1016/j.jcp.2014.04.024.

[9] C. Li, W. Deng, High Order Schemes for the Tempered Fractional Diffusion Equations, Adv. Comput. Math. 42 (2016), 543–572. https://doi.org/10.1007/s10444-015-9434-z.

[10] X. Wu, W. Deng, E. Barkai, Tempered Fractional Feynman–Kac Equation: Theory and Examples, Phys. Rev. E 93 (2016), 032151. https://doi.org/10.1103/physreve.93.032151.

[11] B. Lu, Y. Zhang, D.M. Reeves, H. Sun, C. Zheng, Application of Tempered-Stable Time Fractional-Derivative Model to Upscale Subdiffusion for Pollutant Transport in Field-Scale Discrete Fracture Networks, Mathematics 6 (2018), 5. https://doi.org/10.3390/math6010005.

[12] A. Kirsch, An Introduction to the Mathematical Theory of Inverse Problems, 3rd ed., Springer International Publishing, 2021. https://doi.org/10.1007/978-3-030-63343-1.

[13] C. Yu, S. Wei, Y. Zhang, Y. Zheng, Z. Yu, et al., Quantifying Colloid Fate and Transport through Dense Vegetation and Soil Systems Using a Particle-Plugging Tempered Fractional-Derivative Model, J. Contam. Hydrol. 224 (2019), 103484. https://doi.org/10.1016/j.jconhyd.2019.04.007.

[14] A. Hanyga, Wave Propagation in Media with Singular Memory, Math. Comput. Model. 34 (2001), 1399–1421. https://doi.org/10.1016/s0895-7177(01)00137-6.

[15] M.M. Meerschaert, F. Sabzikar, M.S. Phanikumar, A. Zeleke, Tempered Fractional Time Series Model for Turbulence in Geophysical Flows, J. Stat. Mech. Theory Exp. 2014 (2014), P09023. https://doi.org/10.1088/1742-5468/2014/09/p09023.

[16] M.M. Meerschaert, Y. Zhang, B. Baeumer, Tempered Anomalous Diffusion in Heterogeneous Systems, Geophys. Res. Lett. 35 (2008), L17403. https://doi.org/10.1029/2008gl034899.

[17] M. Chen, W. Deng, A Second-Order Accurate Numerical Method for the Space–Time Tempered Fractional Diffusion-Wave Equation, Appl. Math. Lett. 68 (2017), 87–93. https://doi.org/10.1016/j.aml.2016.12.010.

[18] J. Deng, L. Zhao, Y. Wu, Fast Predictor-Corrector Approach for the Tempered Fractional Differential Equations, Numer. Algorithms 74 (2017), 717–754. https://doi.org/10.1007/s11075-016-0169-9.

[19] C. Li, W. Deng, L. Zhao, Well-Posedness and Numerical Algorithm for the Tempered Fractional Differential Equations, Discrete Contin. Dyn. Syst. Ser. B 24 (2019), 1989–2015. https://doi.org/10.3934/dcdsb.2019026.

[20] M.L. Morgado, M. Rebelo, Well-Posedness and Numerical Approximation of Tempered Fractional Terminal Value Problems, Fract. Calc. Appl. Anal. 20 (2017), 1239–1262. https://doi.org/10.1515/fca-2017-0065.

[21] M.A. Zaky, Recovery of High Order Accuracy in Jacobi Spectral Collocation Methods for Fractional Terminal Value Problems with Non-Smooth Solutions, J. Comput. Appl. Math. 357 (2019), 103–122. https://doi.org/10.1016/j.cam.2019.01.046.

[22] K. Sakamoto, M. Yamamoto, Initial Value/Boundary Value Problems for Fractional Diffusion-Wave Equations and Applications to Some Inverse Problems, J. Math. Anal. Appl. 382 (2011), 426–447. https://doi.org/10.1016/j.jmaa.2011.04.058.

[23] F. Geng, Y. Lin, Application of the Variational Iteration Method to Inverse Heat Source Problems, Comput. Math. Appl. 58 (2009), 2098–2102. https://doi.org/10.1016/j.camwa.2009.03.002.

[24] A. Shidfar, A. Babaei, A. Molabahrami, Solving the Inverse Problem of Identifying an Unknown Source Term in a Parabolic Equation, Comput. Math. Appl. 60 (2010), 1209–1213. https://doi.org/10.1016/j.camwa.2010.06.002.

[25] Z. Zhang, T. Wei, Identifying an Unknown Source in Time-Fractional Diffusion Equation by a Truncation Method, Appl. Math. Comput. 219 (2013), 5972–5983. https://doi.org/10.1016/j.amc.2012.12.024.

[26] J. Wang, Y. Zhou, T. Wei, Two Regularization Methods to Identify a Space-Dependent Source for the Time-Fractional Diffusion Equation, Appl. Numer. Math. 68 (2013), 39–57. https://doi.org/10.1016/j.apnum.2013.01.001.

[27] R. Gorenflo, Y. Luchko, M. Yamamoto, Time-Fractional Diffusion Equation in the Fractional Sobolev Spaces, Fract. Calc. Appl. Anal. 18 (2015), 799–820. https://doi.org/10.1515/fca-2015-0048.

[28] M.K. Beshtokov, Boundary Value Problems for a Pseudoparabolic Equation with the Caputo Fractional Derivative, Differ. Equ. 55 (2019), 884–893. https://doi.org/10.1134/s0012266119070024.

[29] Y. Wang, J. Xu, P.E. Kloeden, Asymptotic Behavior of Stochastic Lattice Systems with a Caputo Fractional Time Derivative, Nonlinear Anal. Theory Methods Appl. 135 (2016), 205–222. https://doi.org/10.1016/j.na.2016.01.020.

[30] Z. Sun, The Method of Order Reduction and Its Application to the Numerical Solutions of Partial Differential Equations, Science Press, Beijing, China, (2009).