ON INTUITIONISTIC FUZZY PRIMARY IDEAL OF A RING

P. K. SHARMA

ABSTRACT. The purpose of this paper is to introduce and investigate primary ideal and P-primary ideal in the intuitionistic fuzzy environment and lay down the foundation for the primary decomposition theorem in the intuitionistic fuzzy setting. Also a suitable characterization of intuitionistic fuzzy P-primary ideal will be discussed.

1. INTRODUCTION

The decomposition of an ideal into primary ideals is a traditional pillar of ideal theory. It provides the algebraic foundation for decomposing an algebraic variety into its irreducible components. From another point of view primary decomposition provides a generalization of the factorization of an integer as a product of prime-powers. A prime ideal in a ring R is in some sense a generalization of a prime number. Also, primary ideal is some sort of generalization of prime ideal. An ideal Q in a ring R is called primary if $Q \neq R$ and if

$$xy \in Q \Rightarrow \text{ either } x \in Q \text{ or } y^n \in Q \text{ for some } n \in \mathbb{N}.$$

In other words, Q is primary ideal $\Leftrightarrow R/Q \neq 0$ and every non-zero divisors in R/Q is nilpotent.

After the foundation of the theory fuzzy sets by Zadeh [17]. Mathematician started fuzzifying the algebraic concepts. Rosenfeld [13] was the first one to introduce the notion of fuzzy subgroup of a group. The concepts of fuzzy subring and ideal were introduced and studied by Liu in [9]. The notion like fuzzy (prime, primary, semi-prime, nil radical etc.) ideals were studied by Swamy at al. in [16] and Malik et al. in [10]. A detailed study of different algebraic structures in fuzzy setting can be found in [11].

One of the prominent generalizations of fuzzy sets theory is the theory of intuitionistic fuzzy sets introduced by Atanassov [1], [2] and [3]. Biswas introduced the notion of intuitionistic fuzzy subgroup of a group in [6]. The concepts of intuitionistic fuzzy subring and ideal were introduced and studied by Hur and others in [7]. The notion like intuitionistic fuzzy (prime, primary, semi-prime, nil etc.) ideals were studied in [4], [8], [12] and [15].

In this paper, we evaluate primary ideals and P-primary ideals in the intuitionistic fuzzy environment and laid down the foundation for the primary decomposition theorem in the intuitionistic fuzzy setting.

POST-GRADUATE DEPARTMENT OF MATHEMATICS, D.A.V. COLLEGE, JALANDHAR, PUNJAB, INDIA

E-mail address: pksharma@davjalandhar.com.

2. PRELIMINARIES

Throughout this paper R is a commutative ring with identity.

Definition 2.1. ([2,3]) An intuitionistic fuzzy set (IFS) A in X can be represented as an object of the form $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Remark 2.2. ([2,3])
(i) When $\mu_A(x) + \nu_A(x) = 1, \forall x \in X$. Then A is called a fuzzy set.
(ii) An IFS $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ is shortly denoted by $A(x) = (\mu_A(x), \nu_A(x)), \forall x \in X$. We denote by $IFS(X)$ the set of all IFSs of X.

If $A, B \in IFS(X)$, then $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$ and $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. For any subset Y of X, the intuitionistic fuzzy characteristic function χ_Y is an intuitionistic fuzzy set of X, defined as $\chi_Y(x) = (1,0), \forall x \in Y$ and $\chi_Y(x) = (0,1), \forall x \in X \backslash Y$. Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Then the crisp set $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta\}$ is called the (α, β)-level cut subset of A. Also the IFS $x_{(\alpha,\beta)}$ of X defined as $x_{(\alpha,\beta)}(y) = (\alpha, \beta)$, if $y = x$, otherwise $(0,1)$ is called the intuitionistic fuzzy point (IFP) in X with support x. By $x_{(\alpha,\beta)} \in A$ we mean $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. Further, if $f : X \to Y$ is a mapping and A, B be respectively IFS of X and Y. Then the image $f(A)$ is an IFS of Y is defined as $\mu_{f(A)}(y) = Sup\{\mu_A(x) : f(x) = y\}, \nu_{f(A)}(y) = Inf\{\nu_A(x) : f(x) = y\}$, for all $y \in Y$ and the inverse image $f^{-1}(B)$ is an IFS of X is defined as $\mu_{f^{-1}(B)}(x) = \mu_B(f(x)), \nu_{f^{-1}(B)}(x) = \nu_B(f(x))$, for all $x \in X$, i.e., $f^{-1}(B)(x) = B(f(x))$, for all $x \in X$. Also, the IFS A of X is said to be f-invariant if for any $x, y \in X$, whenever $f(x) = f(y)$ imply $A(x) = A(y)$.

Definition 2.3. ([4,7]) Let $A \in IFS(R)$. Then A is called an intuitionistic fuzzy ideal (IFI) of ring R if for all $x, y \in R$, the followings are satisfied
(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;
(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$;
(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;
(iv) $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Note that $\mu_A(0) \geq \mu_A(x) \geq \mu_A(1), \mu_A(0) \leq \mu_A(x) \leq \nu_A(1), \forall x \in R$. The set of all intuitionistic fuzzy ideals of R is denoted by $IFI(R)$.

Definition 2.4. ([5]) Let $A, B \in IFI(R)$. Then the intuitionistic fuzzy product AB and intuitionistic intrinsic product $A * B$ of A and B are defined as: For all $x \in R$

$$(\mu_{AB}(x), \nu_{AB}(x)) = \begin{cases} (Sup_{x=yz}(\mu_A(y) \wedge \mu_B(z)), Inf_{x=yz}(\nu_A(y) \vee \nu_B(z)), & \text{if } x = yz \\ (0,1), & \text{otherwise} \end{cases}$$

and

$$\mu_{A*B}(x) = \begin{cases} Sup[Inf_{i=1}^n\{\mu_A(a_i) \wedge \mu_B(b_i)\}] & \text{if } x = \Sigma_{i=1}^n a_i b_i, a_i, b_i \in R, n \in \mathbb{N} \\ 0, & \text{if } x \text{ is not expressible as } x = \Sigma_{i=1}^n a_i b_i \end{cases} \text{ and }$$

$$\nu_{A*B}(x) = \begin{cases} Inf[Sup_{i=1}^n\{\nu_A(a_i) \vee \nu_B(b_i)\}] & \text{if } x = \Sigma_{i=1}^n a_i b_i, a_i, b_i \in R, n \in \mathbb{N} \\ 1, & \text{if } x \text{ is not expressible as } x = \Sigma_{i=1}^n a_i b_i, \end{cases}$$

where as usual supremum and infimum of an empty set are taken to be 0 and 1 respectively. Note that $AB \subseteq A * B$.

Remark 2.5. ([7]) Let R be a commutative ring. Then for any $x_{(p,q)}, y_{(t,s)} \in IFP(R)$
(i) $x_{(p,q)} + y_{(t,s)} = (x + y)_{(p \wedge t, q \vee s)}$;
(ii) $x_{(p,q)} y_{(t,s)} = (xy)_{(p \wedge t, q \vee s)}$.

Theorem 2.6. ([5]) *Let $A \in IFS(R)$. Then A is an intuitionistic fuzzy ideal if and only if $A_{(\alpha, \beta)}$ is an ideal of R, for all $\alpha \leq \mu_A(0), \beta \geq \nu_A(0)$ with $\alpha + \beta \leq 1$. In particular, if A is an IFI of R, then $A_* = \{x \in R : \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)\}$ is always an ideal of R.*

Definition 2.7. ([4,7]) Let Q be a non-constant IFI of a ring R. Then Q is said to be an intuitionistic fuzzy prime ideal of R, if for any two IFIs A, B of R such that $AB \subseteq Q$ implies that either $A \subseteq Q$ or $B \subseteq Q$.

Theorem 2.8. ([4]) *Let Q be an IFI of a ring R. Then for any $x_{(p,q)}, y_{(t,s)} \in IFP(R)$ the following are equivalent:*
(i) Q is an intuitionistic fuzzy prime ideal of R
(ii) $x_{(p,q)} y_{(t,s)} \subseteq Q$ implies $x_{(p,q)} \subseteq Q$ or $y_{(t,s)} \subseteq Q$.

Theorem 2.9. ([7]) *If Q is an intuitionistic fuzzy prime ideal of a ring R, then the following conditions hold:*
(i) $Q(0) = (1, 0)$,
(ii) Q_* is a prime ideal of R,
(iii) $Img(Q) = \{(1, 0), (t, s)\}$, where $t, s \in [0, 1)$ such that $t + s \leq 1$.

Definition 2.10. ([8]) Let A be an IFI of a ring R. The intuitionistic nil radical of A is an IFS denoted by $\sqrt{A}$ defined by

$$\mu_{\sqrt{A}}(x) = \vee\{\mu_A(x^n) : n \in \mathbb{N}\} \text{ and } \nu_{\sqrt{A}}(x) = \wedge\{\nu_A(x^n) : n \in \mathbb{N}\}$$

for all $x \in R$.

Theorem 2.11. ([8]) *For every IFIs A and B of ring R, we have*
(i) $A \subseteq \sqrt{A}$;
(ii) $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$;
(iii) $\sqrt{\sqrt{A}} = \sqrt{A}$.

Theorem 2.12. ([8]) *Let A, B be two IFIs of a ring R. Then*
$$\sqrt{AB} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}.$$

Theorem 2.13. ([8]) *Let $f : R \to R_1$ be a ring homomorphism. If A, B are intuitionistic fuzzy ideals of R and R_1 respectively, then*
(i) $\sqrt{f^{-1}(B)} = f^{-1}(\sqrt{B})$;
(ii) $\sqrt{f(A)} = f(\sqrt{A})$, provided A is f-invariant, that is, $f(x) = f(y)$ implies $A(x) = A(y)$, and f is onto.

3. INTUITIONISTIC FUZZY PRIMARY IDEAL

In this section, we give a complete characterization of an intuitionistic fuzzy primary ideal of a commutative ring R with unity.

Definition 3.1. Let Q be a non-constant IFI of a ring R. Then Q is said to be an intuitionistic fuzzy primary ideal of R, if for any two IFIs A, B of R such that $AB \subseteq Q$ implies that either $A \subseteq Q$ or $B \subseteq \sqrt{Q}$.

From the definition, it is clear that every intuitionistic fuzzy prime ideal of R is an intuitionistic fuzzy primary ideal of R. But converse need not be true (see Example (3.8)).

Theorem 3.2. *Let Q be an IFI of a ring R. Then for any $x_{(p,q)}, y_{(t,s)} \in IFP(R)$ the following are equivalent:*
(i) Q is an intuitionistic fuzzy primary ideal of R.
(ii) $x_{(p,q)} y_{(t,s)} \subseteq Q$ implies $x_{(p,q)} \subseteq Q$ or $y_{(t,s)} \subseteq \sqrt{Q}$.

Proof. $(i) \Rightarrow (ii)$ Suppose that Q is an intuitionistic fuzzy primary ideal of R.
Let $x_{(p,q)}, y_{(t,s)} \in IFP(R)$ such that $x_{(p,q)} y_{(t,s)} \subseteq Q$. Then
$\mu_Q(xy) \geq \mu_{x_{(p,q)} y_{(t,s)}}(xy) = \mu_{(xy)_{(p \wedge t, q \vee s)}}(xy) = p \wedge t$ and
$\nu_Q(xy) \leq \nu_{x_{(p,q)} y_{(t,s)}}(xy) = \nu_{(xy)_{(p \wedge t, q \vee s)}}(xy) = q \vee s$

$$\Rightarrow \mu_Q(xy) \geq p \wedge t, \nu_Q(xy) \leq q \vee s \tag{1}$$

Let us define two IFSs A, B as

$$A(z) = \begin{cases} (p,q), & \text{if } z \in \langle x \rangle \\ (0,1), & \text{if otherwise} \end{cases} ; \quad B(z) = \begin{cases} (t,s), & \text{if } z \in \langle y \rangle \\ (0,1), & \text{if otherwise.} \end{cases}$$

Clearly A, B are IFIs of R such that $x_{(p,q)} \subseteq A$ and $y_{(t,s)} \subseteq B$.

Now $\mu_{AB}(z) = Sup_{z=uv}\{\mu_A(u) \wedge \mu_B(v)\} = p \wedge t$, when $u \in \langle x \rangle, v \in \langle y \rangle$ and
$\nu_{AB}(z) = Inf_{z=uv}\{\nu_A(u) \vee \nu_B(v)\} = q \vee s$, when $u \in \langle x \rangle, v \in \langle y \rangle$.
Hence $\mu_{AB}(z) = p \wedge t \leq \mu_Q(z), \nu_{AB}(z) = q \vee s \geq \nu_Q(z)$, from (1) when $z = uv$, where $u \in \langle x \rangle, v \in \langle y \rangle$,
otherwise $\mu_{AB}(z) = 0, \nu_{AB}(z) = 1$. That is $AB \subseteq Q$.
As Q is an intuitionistic fuzzy primary ideal of R. Therefore, $A \subseteq Q$ or $B \subseteq \sqrt{Q}$.
Then $x_{(p,q)} \subseteq A \subseteq Q$ or $y_{(t,s)} \subseteq B \subseteq \sqrt{Q}$ implies that $x_{(p,q)} \subseteq Q$ or $y_{(t,s)} \subseteq \sqrt{Q}$.

$(ii) \Rightarrow (i)$ Let $A, B \in IFI(R)$ such that $AB \subseteq Q$. Suppose that A is not subset of Q. Then there exists
$x \in R$ such that $\mu_A(x) > \mu_Q(x), \nu_A(x) < \mu_Q(x)$.
Let $\mu_A(x) = p, \nu_A(x) = q$. Let $y \in R$ and $\mu_B(y) = t, \nu_B(y) = s$. If $z = xy$, then
$\mu_Q(z) = \mu_Q(xy) \geq \mu_{AB}(xy) \geq \mu_A(x) \wedge \mu_B(y) = p \wedge t = \mu_{x_{(p,q)} y_{(t,s)}}(xy) = \mu_{x_{(p,q)} y_{(t,s)}}(z)$ implies
$\mu_Q(z) \geq \mu_{x_{(p,q)} y_{(t,s)}}(z)$. Similarly, we have $\nu_Q(z) \leq \nu_{x_{(p,q)} y_{(t,s)}}(z)$.

Also, if $\mu_{x_{(p,q)} y_{(t,s)}}(0) = 0, \nu_{x_{(p,q)} y_{(t,s)}}(0) = 1$. Then $\mu_Q(z) \geq \mu_{x_{(p,q)} y_{(t,s)}}(z)$ and $\nu_Q(z) \leq \nu_{x_{(p,q)} y_{(t,s)}}(z)$, for
all $z \in R$. Hence $x_{(p,q)} y_{(t,s)} \subseteq Q$. By (ii) either $x_{(p,q)} \subseteq Q$ or $y_{(t,s)} \subseteq \sqrt{Q}$. That is either $\mu_Q(x) \geq p, \nu_Q(x) \leq q$
or $\mu_{\sqrt{Q}}(y) \geq t, \nu_{\sqrt{Q}}(y) \leq s$. Since $p \geq \mu_Q(x), q \leq \nu_Q(x)$. So $\mu_B(y) = t \leq \mu_{\sqrt{Q}}(y), \nu_B(y) = s \geq \nu_{\sqrt{Q}}(y)$. So
$B \subseteq \sqrt{Q}$. Thus Q is an intuitionistic fuzzy primary ideal of R. $\qquad \square$

The following theorem, which relates intuitionistic fuzzy primary ideal to primary ideal of the ring, will
be needed in the proof of Theorem (3.7).

Theorem 3.3. *Let A be an intuitionistic fuzzy primary ideal of R. Then $A_{(\alpha,\beta)}$ is a primary ideal of R, where*
$\alpha, \beta \in [0,1]$ *such that $\alpha + \beta \leq 1$ and $\alpha \leq \mu_A(0), \beta \geq \nu_A(0)$.*

Proof. Let $A_{(\alpha,\beta)}$ be an intuitionistic fuzzy primary ideal of R. Let $ab \in A_{(\alpha,\beta)}$ for some $a, b \in R$. Then
$\mu_A(ab) \geq \alpha$ and $\nu_A(ab) \leq \beta \Rightarrow (ab)_{(\alpha,\beta)} = a_{(\alpha,\beta)} b_{(\alpha,\beta)} \subseteq A$, since A is an intuitionistic fuzzy primary ideal
of R, either $a_{(\alpha,\beta)} \subseteq A$ or $b_{(\alpha,\beta)} \subseteq \sqrt{A}$, i.e., $b^n_{(\alpha,\beta)} \subseteq A$, for some $n \in \mathbb{N}$.

If $a_{(\alpha,\beta)} \subseteq A$, then $\mu_A(a) \geq \alpha$ and $\nu_A(a) \leq \beta \Rightarrow a \in A_{(\alpha,\beta)}$.

If $b^n_{(\alpha,\beta)} \subseteq A$, then $\mu_A(b^n) \geq \alpha$ and $\nu_A(b^n) \leq \beta \Rightarrow b^n \in A_{(\alpha,\beta)}$, i.e., $b \in \sqrt{A_{(\alpha,\beta)}}$.

Hence $A_{(\alpha,\beta)}$ is a primary ideal of R.   □

Remark 3.4. Let A be an intuitionistic fuzzy primary ideal of R. Then
$$A_* = \{x \in R : \mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0)\}$$
is a primary ideal of R.

Proof. Clear from Theorem (3.3) as $A_{(\alpha,\beta)} = A_*$, when $\alpha = \mu_A(0)$ and $\beta = \nu_A(0)$.   □

Remark 3.5. The converse of the Theorem (3.3) is not always true, see the following example:

Example 3.6. Let $R = \mathbb{Z}$, be the ring of integers. Define $A \in IFS(R)$ as follow

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x \in 4\mathbb{Z} - \{0\} \\ 0.4, & \text{if } x \in 4\mathbb{Z} - 2\mathbb{Z} \\ 0, & \text{if otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.3, & \text{if } x \in 4\mathbb{Z} - \{0\} \\ 0.5, & \text{if } x \in 4\mathbb{Z} - 2\mathbb{Z} \\ 1, & \text{if otherwise} \end{cases}$$

Clearly, $A \in IFI(R)$. By some manipulation, we can see that, for all $\alpha, \beta \in (0, 1]$, $A_{(\alpha,\beta)}$ is a primary ideal of R. But it can be easily checked that A is not an intuitionistic fuzzy primary ideal of R. For, if we take $a = 5, b = 4$, then $(5)_{(1/3,1/2)}(4)_{(2/3,1/3)} = (20)_{(1/3,1/2)} \subseteq A$, but $(4)_{(2/3,1/3)} \notin A$ and $(5^n)_{(1/3,1/2)} \notin A, \forall n \in \mathbb{N}$ (also, $(5)_{(1/3,1/2)} \notin A$ and $(4^n)_{(2/3,1/3)} \notin A, \forall n \in \mathbb{N}$).

The following theorem characterized, intuitionistic fuzzy primary ideal completely.

Theorem 3.7. (a) *Let J be a primary ideal of commutative ring R with unity 1_R and $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta \leq 1$. If A is an IFS of R defined by*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in J \\ \alpha, & \text{if otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in J \\ \beta, & \text{otherwise.} \end{cases}$$

for all $x \in R$. Then A is an intuitionistic fuzzy primary ideal of R.
(b) Conversely, any intuitionistic fuzzy primary ideal can be obtained as in (a).

Proof. **(a)** Since J is a primary ideal of R, $J \neq R$, so A is non-constant intuitionistic fuzzy ideal of R. We show that A is an intuitionistic fuzzy primary ideal of R.

Suppose $a_{(s,t)}, b_{(p,q)} \in IFP(R)$ are such that $a_{(s,t)}b_{(p,q)} \subseteq A$ and $b_{(p,q)}$ is not a subset of A. We show that $a_{(s,t)} \subseteq \sqrt{A}$. Now $b_{(p,q)}$ is not a subset of A, then $\mu_A(b) < p, \nu_A(b) > q$.

Suppose that $\mu_A(b) = \alpha < p$ and $\nu_A(b) = \beta > q$, hence $b \notin J$.
Since $(ab)_{(s \wedge p, t \vee q)} = a_{(s,t)}b_{(p,q)} \subseteq A$, then $\mu_A(ab) \geq s \wedge p, \nu_A(ab) \leq t \vee q$.

If $\mu_A(ab) = 1$ and $\nu_A(ab) = 0$, then $ab \in J$. Since $b \notin J$ and J is a primary ideal of R, we have $a^n \in J$, for some $n \in \mathbb{N}$. Hence $\mu_A(a^n) = 1$ and $\nu_A(a^n) = 0$. Thus $\mu_A(a^n) = 1 \geq s$ and $\nu_A(a^n) = 0 \leq t$ implies that $a^n_{(s,t)} \subseteq A$, i.e., $a_{(s,t)} \subseteq \sqrt{A}$.

If $\mu_A(ab) = \alpha$ and $\nu_A(ab) = \beta$, then $\alpha \geq s \wedge p$ and $\alpha \leq t \vee q$. But $\alpha < p, \beta > q$ implies that $s \leq \alpha, t \geq \beta$.

Thus $\mu_A(a^n) \geq \mu_A(a) \geq \mu_A(ab) = \alpha \geq s$ and $\nu_A(a^n) \leq \nu_A(a) \leq \nu_A(ab) = \beta \leq t$ This implies that $a^n_{(s,t)} \subseteq A$, i.e., $a_{(s,t)} \subseteq \sqrt{A}$. Hence A is an intuitionistic fuzzy primary ideal of R.

(b) Let A be an intuitionistic fuzzy primary ideal of R. We show that A is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in J \\ \alpha, & \text{if otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in J \\ \beta, & \text{otherwise.} \end{cases}$$

for all $x \in R$, where $\alpha, \beta \in (0,1]$ such that $\alpha + \beta \leq 1$.

Claim (1) $A_* = \{x \in R : \mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0)\}$ is a primary ideal of R.

Since A is a non-constant intuitionistic fuzzy primary ideal of R, so $A_* \neq R$. For all $a, b \in R$, if $ab \in A_*$ implies $\mu_A(ab) = \mu_A(0)$ and $\nu_A(ab) = \nu_A(0)$ so that $(ab)_{(\mu_A(0),\nu_A(0))} = a_{(\mu_A(0),\nu_A(0))}b_{(\mu_A(0),\nu_A(0))} \subseteq A$, then $a_{(\mu_A(0),\nu_A(0))} \subseteq A$ or $b^n_{(\mu_A(0),\nu_A(0))} \subseteq A$, for some $n \in \mathbb{N}$.

Case(i) If $a_{(\mu_A(0),\nu_A(0))} \subseteq A$, then $\mu_A(a) \geq \mu_A(0)$ and $\nu_A(a) \leq \nu_A(0)$ but $\mu_A(0) \geq \mu_A(a)$ and $\nu_A(0) \leq \nu_A(a)$ for all $a \in R$. Hence $\mu_A(a) = \mu_A(0)$ and $\nu_A(a) = \nu_A(0)$ so $a \in A_*$.

Case(ii) If $b^n_{(\mu_A(0),\nu_A(0))} \subseteq A$, then $\mu_A(b^n) \geq \mu_A(0)$ and $\nu_A(b^n) \leq \nu_A(0)$ but $\mu_A(0) \geq \mu_A(b^n)$ and $\nu_A(0) \leq \nu_A(b^n)$ for all $b^n \in R$. Hence $\mu_A(b^n) = \mu_A(0)$ and $\nu_A(b^n) = \nu_A(0)$ so $b^n \in A_*$, i.e., $b \in \sqrt{A_*}$. Thus A_* is a primary ideal of R. Now, take $A_* = J$.

Claim (2) We claim $\mu_A(0) = 1, \nu_A(0) = 0$. Suppose that $\mu_A(0) < 1, \nu_A(0) > 0$. Since A is non-constant intuitionistic fuzzy primary ideal of R, therefore there exists an element $a \in R$ such that $\mu_A(a) < \mu_A(0), \nu_A(a) > \nu_A(0)$.

Define IFSs C, D of R as $C = \chi_{A_*}$, $D(x) = A(0), \forall x \in R$. It is easy to verify that C, D are IFIs of R. Since $\mu_C(0) = 1 > \mu_A(0), \nu_C(0) = 0 < \nu_A(0)$ and $\mu_D(a) = \mu_A(0) > \mu_A(a), \nu_D(a) = \nu_A(0) < \nu_A(a)$. So C is not a subset of A and D is not a subset of A and so C is not a subset of $\sqrt{A}$ and D is not a subset of $\sqrt{A}$. Thus CD is not a subset of A, which is not true, since for all $x, y \in R$, we have $\mu_A(xy) \geq \mu_C(x) \wedge \mu_D(y)$ and $\nu_A(xy) \leq \nu_C(x) \vee \nu_D(y)$. Thus $\mu_A(0) = 1, \nu_A(0) = 0$.

Claim (3) A has two values.

Since A_* is a primary ideal of R, $A_* \neq R$. Then there exists $z \in R \backslash A_*$. We will show that $\mu_A(y) = \mu_A(z) < \mu_A(0)$ and $\nu_A(y) = \nu_A(z) > \nu_A(0)$, for all $y \in R$ such that $y \notin A_*$. Now $z \notin A_* \Rightarrow \mu_A(z) < 1 = \mu_A(0)$ and $\nu_A(z) > 0 = \nu_A(0)$ so $z_{(1,0)}$ is not a subset of A and $z_{(\mu_A(z),\nu_A(z))} = z_{(1,0)}1_{(\mu_A(z),\nu_A(z))} \subseteq A$. Thus $1^n_{(\mu_A(z),\nu_A(z))} \subseteq A$, i.e., $\mu_A(1_R) \geq \mu_A(z)$ and $\nu_A(1_R) \leq \nu_A(z)$. Since $x = 1_R.x$, for all $x \in R$, we have $\mu_A(x) = \mu_A(x.1_R) \geq \mu_A(x) \wedge \mu_A(1_R) \geq \mu_A(x) \wedge \mu_A(z)$, i.e., $\mu_A(z) \leq \mu_A(x)$. Similarly we have $\nu_A(z) \geq \nu_A(x)$.

Take $x = y$, we have $\mu_A(z) \leq \mu_A(y)$ and $\nu_A(z) \geq \nu_A(y)$. Similarly, $\mu_A(y) \leq \mu_A(z)$ and $\nu_A(y) \geq \nu_A(z)$. Hence $\mu_A(z) = \mu_A(y)$ and $\nu_A(z) = \nu_A(y)$.

Hence, every intuitionistic fuzzy primary ideal of R, is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in J \\ \alpha, & \text{if otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in J \\ \beta, & \text{otherwise}. \end{cases}$$

for all $x \in R$, where $J = A_*$ is a primary ideal of R.   □

Theorem 3.8. *If J be a non-trivial ideal in a ring R, then χ_J is an intuitionistic fuzzy primary ideal of R if, and only if, J is a primary ideal of R.*

Proof. This follows immediately from Theorem (3.7).   □

Theorem (3.8) is particularly useful in deciding whether, an intuitionistic fuzzy ideal, is primary or not. The following example illustrate this.

Example 3.9. Let $R = \mathbb{Z}$, be the ring of integers. Then

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in <p^k> \\ 0.25, & \text{if otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in <p^k> \\ 0.75, & \text{otherwise}. \end{cases}$$

where p is a prime integer and $k > 1$. Then, by Theorem (3.8), A is an intuitionistic fuzzy primary ideal of $\mathbb{Z}$, since $<p^k>$ is a primary ideal of $\mathbb{Z}$. Notice that A is not an intuitionistic fuzzy prime ideal of R.

In the following example, we show that, if Q_1, Q_2 are two intuitionistic fuzzy primary ideals of a ring R, then $Q_1 \cap Q_2$ need not be an intuitionistic fuzzy primary ideal of R.

Example 3.10. Let $R = \mathbb{Z}$, be the ring of integers. Take $I = 2\mathbb{Z}, J = 3\mathbb{Z}$. Clearly, I and J are primary (in fact prime) ideals in R. Define $Q_1 = \chi_I, Q_2 = \chi_J$. Then, by Theorem (3.8), Q_1 and Q_2 are intuitionistic fuzzy primary ideals of R. Also, $Q_1 \cap Q_2 = \chi_{I \cap J} = \chi_{6\mathbb{Z}}$, which is not an intuitioniatic fuzzy primary ideal of R, as $6\mathbb{Z}$ is not a primary ideal in $\mathbb{Z}$.

Definition 3.11. An intuitionistic fuzzy primary ideal Q of ring R with $\sqrt{Q} = P$, is called an intuitionistic fuzzy P-primary ideals of ring R.

Theorem 3.12. *Let $Q_1, Q_2, ..., Q_n$ be intuitionistic fuzzy P-primary ideals of ring R with $P = \sqrt{Q_i}, \forall i = 1, 2, ..., n$, an intitionistic fuzzy prime ideal of R. Then $Q = \bigcap_{i=1}^{n} Q_i$ is an intuitionistic fuzzy P-primary ideal of R.*

Proof. Let $x_{(p,q)}, y_{(t,s)} \in IFP(R)$ be such that $x_{(p,q)} y_{(t,s)} \subseteq Q = \bigcap_{i=1}^{n} Q_i$ and $x_{(p,q)} \notin Q$. Then $x_{(p,q)} \notin Q_j$, for some $j \in \{1, 2, ..., n\}$ also $x_{(p,q)} y_{(t,s)} \subseteq Q_j, \forall j \in \{1, 2, ..., n\}$. Since each Q_j is an intuitionistic fuzzy P-primary ideals of R, we have

$$y_{(t,s)} \in \sqrt{Q_j} = P = \bigcap_{i=1}^{n} \sqrt{Q_i} = \sqrt{\bigcap_{i=1}^{n} Q_i} = \sqrt{Q}.$$

Hence Q is an intuitionistic fuzzy P-primary ideals of R.   □

In the next theorems we show that, both the image and inverse image of an intuitionistic fuzzy primary (P-primary) ideal under a ring epimorphism are again intuitionistic fuzzy primary (P-primary) ideal.

Theorem 3.13. *Let f be an ring epimorphism from R to R_1. If A is an intuitionistic fuzzy primary ideal of R such that $\chi_{kerf} \subseteq A$, then $f(A)$ is an intuitionistic fuzzy primary ideal of R_1.*

Proof. Now, it is easy to see that $f(A)$ is an intuitionistic fuzzy ideal of R_1.

We show that $f(A)$ is an intuitionistic fuzzy primary ideal of R_1. Since A is an intuitionistic fuzzy primary ideal of R, so A is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in J \\ \alpha, & \text{if otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in J \\ \beta, & \text{otherwise.} \end{cases}$$

for all $x \in R$, where $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta \le 1$ and $J = A_*$ is a primary ideal of R.

We first claim that, if A_* is a primary ideal of R and $\chi_{kerf} \subseteq A$, then $f(A_*)$ is primary ideal of R_1.

Let $x \in \chi_{kerf}$. Then $\mu_{\chi_{kerf}}(x) = 1 \le \mu_A(x)$ and $\nu_{\chi_{kerf}}(x) = 0 \ge \nu_A(x)$ implies that $\mu_A(x) = \mu_A(0)$ and $\nu_A(x) = \nu_A(0) \Rightarrow x \in A_*$. Thus, $kerf \subseteq A_*$.

For all $a_1, b_1 \in R_1$, since f is epimorphism, therefore there exists $a, b \in R$ such that $a_1 = f(a)$ and $b_1 = f(b)$. Now, $ab \in A_*$ and A_* is a primary ideal of R, so either $a \in A_*$ or $b^n \in A_*$, for some $n \in \mathbb{N}$.

If $a \in A_*$, then $a_1 = f(a) \in f(A_*)$ and if $b^n \in A_*$, then $b_1^n = f(b^n) = (f(b))^n \in f(A_*)$.

Thus $f(A_*)$ is a primary ideal of R_1. So by Theorem (3.7), for all $y \in R_1$,

$$\mu_{f(A)}(y) = \begin{cases} 1, & \text{if } y \in f(A_*) \\ \alpha, & \text{if otherwise} \end{cases} \quad ; \quad \nu_{f(A)}(y) = \begin{cases} 0, & \text{if } y \in f(A_*) \\ \beta, & \text{otherwise.} \end{cases}$$

Hence $f(A)$ is an intuitionistic fuzzy primary ideal of R_1. $\qquad \square$

Theorem 3.14. *Let f be an ring epimorphism from R to R_1. If A is an intuitionistic fuzzy P-primary ideal of R such that $\chi_{kerf} \subseteq A$, then $f(A)$ is an intuitionistic fuzzy $f(P)$-primary ideal of R_1.*

Proof. This follows from Theorem (3.13) and Theorem (2.13)(ii) $\qquad \square$

Theorem 3.15. *Let f be a ring epimorphism from R to R_1. If B is an intuitionistic fuzzy primary ideal of R_1, then $f^{-1}(B)$ is an intuitionistic fuzzy primary ideal of R.*

Proof. Let B be an intuitionistic fuzzy primary ideal of R_1. Then

$$\mu_B(y) = \begin{cases} 1, & \text{if } y \in B_* \\ \alpha, & \text{if otherwise} \end{cases} \quad ; \quad \nu_B(y) = \begin{cases} 0, & \text{if } y \in B_* \\ \alpha', & \text{otherwise.} \end{cases}$$

for all $y \in R_1$ and B_* is a primary ideal of R_1.

We first show that $f^{-1}(B_*)$ is a primary ideal of R.

For all $a, b \in R$, if $ab \in f^{-1}(B_*) \Rightarrow f(ab) \in B_*$, i.e., $f(a)f(b) \in B_*$. As B_* is primary ideal of R_1. Therefore, either $f(a) \in B_*$ or $(f(b))^n = f(b^n) \in B_*$, for some $n \in \mathbb{N}$.

If $f(a) \in B_*$, then $a \in f^{-1}(B_*)$ and if $f(b^n) \in B_*$, then $b^n \in f^{-1}(B_*)$. Hence

$$\mu_{f^{-1}(B)}(x) = \begin{cases} 1, & \text{if } x \in f^{-1}(B_*) \\ \alpha, & \text{if otherwise} \end{cases} \quad ; \quad \nu_{f^{-1}(B)}(x) = \begin{cases} 0, & \text{if } x \in f^{-1}(B_*) \\ \beta, & \text{otherwise.} \end{cases}$$

Hence $f^{-1}(B)$ is an intuitionistic fuzzy primary ideal of R. $\qquad \square$

Theorem 3.16. *Let f be a ring epimorphism from R to R_1. If B is an intuitionistic fuzzy P-primary ideal of R_1, then $f^{-1}(B)$ is an intuitionistic fuzzy $f^{-1}(P)$-primary ideal of R.*

Proof. This follows from Theorem (3.15) and Theorem (2.13)(i)   □

4. INTUITIONISTIC FUZZY PRIMARY DECOMPOSITION

In this section, we study the decomposability of an intuitionistic fuzzy ideal in a Noetherian ring, in terms of intuitionistic fuzzy primary ideals, such that the set of their respective intuitionistic fuzzy radical ideals, are independent of the particular decomposition.

To begin this section, we first recall from [14] the definition of residual quotient $(A : B)$ of an intuitionistic fuzzy ideal A by an intuitionistic fuzzy set B in a ring R.

Definition 4.1. For any IFI A of a ring R and for any IFS B of R, the intuitionistic fuzzy residual quotient of A by B is denoted by $(A : B)$ and is defined as

$$(A : B) = \bigcup \{x_{(p,q)} \in IFP(R) : x_{(p,q)}B \subseteq A\}$$

For any IFP $x_{(p,q)}$ of the ring R, we use a streamlined notation $(A : x_{(p,q)})$ for $(A : \langle x_{(p,q)} \rangle)$, where $\langle x_{(p,q)} \rangle = \bigcap \{C : C$ is an IFI of R such that $x_{(p,q)} \subseteq C\}$, be an IFI generated by $x_{(p,q)}$. There is no difficulty in seeing that $(A : x_{(p,q)})$ is an IFI of R and $A \subseteq (A : x_{(p,q)})$.

Theorem 4.2. *Let Q be an intuitioistic fuzzy P-primary ideal of ring R, where $P = \sqrt{Q}$. If $x_{(p,q)} \in IFP(R)$ be any intuitionistic fuzzy point of R. Then*
(i) If $x_{(p,q)} \in Q$, then $(Q : x_{(p,q)}) = \chi_R$;
(ii) If $x_{(p,q)} \notin Q$, then $(Q : x_{(p,q)})$ is an intuitionistic fuzzy P-primary ideal and $\sqrt{(Q : x_{(p,q)})} = P$;
(iii) If $x_{(p,q)} \notin \sqrt{Q}$, then $(Q : x_{(p,q)}) = Q$.

Proof. Let $x_{(p,q)} \in IFP(R)$, Q be an intuitionistic fuzzy primary ideal of R such that $P = \sqrt{Q}$.

(i) If $x_{(p,q)} \in Q$, then $(Q : x_{(p,q)}) = \bigcup \{y_{(t,s)} \in IFP(R) : y_{(t,s)}x_{(p,q)} \subseteq Q\}$.
Now $(Q : x_{(p,q)}) \subseteq \chi_R$ always. For other inclusion.
If $y_{(t,s)} \in \chi_R$, then $y_{(t,s)}x_{(p,q)} = (yx)_{(t \wedge p, s \vee q)} \subseteq Q$. This implies $y_{(t,s)} \in (Q : x_{(p,q)})$. Thus $\chi_R \subseteq (Q : x_{(p,q)})$.
Hence $(Q : x_{(p,q)}) = \chi_R$.

(ii) Obviously $Q \subseteq (Q : x_{(p,q)})$. Let $y_{(t,s)} \in (Q : x_{(p,q)})$. So $y_{(t,s)}x_{(p,q)} \subseteq Q$. Since $x_{(p,q)} \notin Q$ imply that $y_{(t,s)} \in \sqrt{Q} = P$. This means that $Q \subseteq (Q : x_{(p,q)}) \subseteq P$ and so $\sqrt{Q} \subseteq \sqrt{(Q : x_{(p,q)})} \subseteq \sqrt{P} = P$. This imply $\sqrt{(Q : x_{(p,q)})} = P$.

Now, we show that $(Q : x_{(p,q)})$ is an intuitionistic fuzzy primary ideal of R.
Assume that $a_{(u_1,v_1)}b_{(u_2,v_2)} \in (Q : x_{(p,q)})$ and $b_{(u_2,v_2)} \notin \sqrt{(Q : x_{(p,q)})}$, then
$a_{(u_1,v_1)}b_{(u_2,v_2)}x_{(p,q)} \in Q$, i.e., $(a_{(u_1,v_1)}x_{(p,q)})b_{(u_2,v_2)} \in Q$ and Q is intuitionistic fuzzy P-primary ideal of R,
This imply $a_{(u_1,v_1)}x_{(p,q)} \in Q$ or $b_{(u_2,v_2)} \in \sqrt{Q} = P = \sqrt{(Q : x_{(p,q)})}$. This imply $a_{(u_1,v_1)}x_{(p,q)} \in Q$. Thus $a_{(u_1,v_1)} \in (Q : x_{(p,q)})$. Hence $(Q : x_{(p,q)})$ is an intuitionistic fuzzy primary ideal of R.

(iii) Since $Q \supseteq x_{(p,q)} \cap Q \supseteq x_{(p,q)}Q$, i.e., $x_{(p,q)}Q \subseteq Q$. Therefore, by the properties of IF residual quotient, we have $Q \subseteq (Q : x_{(p,q)})$. Further, $x_{(p,q)}(Q : x_{(p,q)}) \subseteq Q$. As Q is an intuitionistic fuzzy primary ideal of R and $x_{(p,q)} \notin \sqrt{Q}$ implies that $(Q : x_{(p,q)}) \subseteq Q$. Hence $(Q : x_{(p,q)}) = Q$.   □

Theorem 4.3. *If $Q_1, Q_2, \ldots\ldots, Q_n$ are IFIs of ring R and $x_{(p,q)} \in IFP(R)$, then*
$$(\bigcap_{i=1}^n Q_i : x_{(p,q)}) = \bigcap_{i=1}^n (Q_i : x_{(p,q)}).$$

Proof. Now $y_{(t,s)} \in (\bigcap_{i=1}^n Q_i : x_{(p,q)}) \Leftrightarrow y_{(t,s)} x_{(p,q)} \subseteq \bigcap_{i=1}^n Q_i$

$\Leftrightarrow y_{(t,s)} x_{(p,q)} \subseteq Q_i, \forall i = 1, 2, \ldots., n$

$\Leftrightarrow y_{(t,s)} \in (Q_i : x_{(p,q)}), \forall i = 1, 2, \ldots., n$

$\Leftrightarrow y_{(t,s)} \in \bigcap_{i=1}^n (Q_i : x_{(p,q)})$.

Hence $(\bigcap_{i=1}^n Q_i : x_{(p,q)}) = \bigcap_{i=1}^n (Q_i : x_{(p,q)})$. □

Definition 4.4. A primary decomposition of an intuitionistic fuzzy ideal A, is an expression $A = \bigcap_{i=1}^n Q_i$, of A, as a finite intersection of intuitionistic fuzzy primary ideals Q_i of a ring R.

Definition 4.5. An intuitionistic fuzzy primary decomposition of an intuitionistic fuzzy ideal $A = \bigcap_{i=1}^n Q_i$ of ring R, is said to be minimal if:

    (1) all intuitionistic fuzzy primary ideal Q_i have distinct $\sqrt{Q_i}$;

    (2) $\bigcap_{j \neq i=1}^n Q_j$ is not a subset of Q_i.

Remark 4.6. If the intuitionistic fuzzy primary decomposition $A = \bigcap_{i=1}^n Q_i$ is not minimal, that is if $\sqrt{Q_j} = \sqrt{Q_k} = P$ for $j \neq k$, then we may achieve (1) of definition (4.5) by replacing Q_j and Q_k by $Q' = Q_j \cap Q_k$ which is an intuitionistic fuzzy P-primary ideal of R, by Theorem (3.12). Repeating this process, we get will arrive at an intuitionistic fuzzy primary decomposition in which all $\sqrt{Q_i}$ are distinct. If $\bigcap_{j \neq i=1}^n Q_j \subseteq Q_i$, we may simply omit Q_i. Repeating this process, we will achieve (2) of definition (4.5).

Lemma 4.7. *Let $A_1, A_2, \ldots., A_n$ be IFIs of ring R and let P be an intuitionistic fuzzy prime ideal of R. Then*

    (1) *If $\bigcap_{i=1}^n A_i \subseteq P$, then $A_i \subseteq P$ for some i;*

    (2) *If $\bigcap_{i=1}^n A_i = P$, then $A_i = P$ for some i.*

Proof. **(1)** Suppose A_i is not a subset of P for all i. Then $\exists^s$, $(x_i)_{(p_i,q_i)} \in A_i$ such that $(x_i)_{(p_i,q_i)} \notin P$ for $1 \leq i \leq n$. Therefore $(x_1)_{(p_1,q_1)}(x_2)_{(p_2,q_2)}\ldots\ldots(x_n)_{(p_n,q_n)} \subseteq A_1 A_2 \ldots\ldots A_n \subseteq \bigcap_{i=1}^n A_i \subseteq P$. But, since P is an intuitionistic fuzzy prime ideal and $A_1 A_2 \ldots. A_n \subseteq P$, then $A_i \subseteq P$ for some i.

    (2) If $P = \bigcap_{i=1}^n A_i$, then $P \subseteq A_i$ for some i, and from part (1), $A_i \subseteq P$ for some i. Hence $P = A_i$, for some i. □

Definition 4.8. An intuitionistic fuzzy prime ideal P in a ring R is called an intuitionistic fuzzy associated prime ideal of an IFI A, if $P = \sqrt{(A : x_{(p,q)})}$ for some $x_{(p,q)} \in IFP(R)$.

Moreover, for an IFI A of a ring R. We define $IF - ASS(A)$ to be the set of all intuitionistic fuzzy prime ideals associated with the IFI A, i.e.,
$$IF - ASS(A) = \{\sqrt{(A : x_{(p,q)})} : \sqrt{(A : x_{(p,q)})} \text{ is an IFPI of } R, x_{(p,q)} \in IFP(R)\}.$$

Theorem 4.9. *Let A be an IFI of a Noetherian ring R. Suppose that $A = \bigcap_{i=1}^n Q_i$, is a minimal intuitionistic fuzzy primary decomposition of A, and let $P_i = \sqrt{Q_i}, 1 \leq i \leq n$. Then the set $IF - ASS(A) = \{P_i, i = 1, 2, \ldots, n\}$ is independent of the particular decomposition.*

Proof. Let $A = \bigcap_{i=1}^n Q_i$ with $P_i = \sqrt{Q_i}, 1 \leq i \leq n$ be the minimal intuitionistic fuzzy primary decomposition of A. Consider any $x_{(p,q)} \in IFP(R)$, we have
$(A : x_{(p,q)}) = (\bigcap_{i=1}^n Q_i : x_{(p,q)}) = \bigcap_{i=1}^n (Q_i : x_{(p,q)})$.

    Hence $\sqrt{(A : x_{(p,q)})} = \bigcap_{i=1}^n \sqrt{(Q_i : x_{(p,q)})}$.

Also, by Theorem (4.2), if $x_{(p,q)} \in Q_j$ then $\sqrt{(Q_j : x_{(p,q)})} = \chi_R$ and if, $x_{(p,q)} \notin Q_j$,

then $\sqrt{(Q_j : x_{(p,q)})} = P_j$, be an intuitionistic fuzzy prime ideal of R. So

$$\sqrt{(A : x_{(p,q)})} = \bigcap_{i=1}^{n} \sqrt{(Q_i : x_{(p,q)})} = \bigcap_{x_{(p,q)} \notin Q_j} P_j.$$

Now, suppose that $P \in IF - ASS(A)$, then $P = \sqrt{(A : x_{(p,q)})}$ be an intuitionistic fuzzy prime ideal of R, for some $x_{(p,q)} \in IFP(R)$. Since $\sqrt{(A : x_{(p,q)})} = \bigcap_{x_{(p,q)} \notin Q_j} P_j$, then by Lemma (4.7)(2) we have $\sqrt{(A : x_{(p,q)})} = P_j$ for some j. So, $P \in \{P_i, i = 1, 2, ..., n\}$. Therefore, $IF - ASS(A) \subseteq \{P_i, i = 1, 2, ..., n\}$.

Conversely, as the decomposition is minimal so $\bigcap_{j \neq i=1}^{n} Q_j$ is not a subset of Q_i. Then for each $i \in \{1, 2, ..., n\}$, there exists $(x_i)_{(p_i, q_i)} \in \bigcap_{j \neq i=1}^{n} Q_j$ and $(x_i)_{(p_i, q_i)} \notin Q_i$, we have

$$\sqrt{(A : (x_i)_{(p_i, q_i)})} = \bigcap_{j=1}^{n} \sqrt{(Q_j : (x_j)_{(p_j, q_j)})} = P_i$$

(Since all other's $\sqrt{(Q_j : (x_j)_{(p_j, q_j)})} = \chi_R$, for $j \neq i$ by Theorem (4.2)).
So, $P_i \in IF - ASS(A)$. Therefore, $\{P_i, i = 1, 2, ..., n\} \subseteq IF - ASS(A)$.
Thus, $IF - ASS(A) = \{P_i, i = 1, 2, ..., n\}$. Hence $IF - ASS(A)$ is independent of the particular decomposition. $\qquad \square$

Example 4.10. Let $R = Z_{p_1^{n_1}} \times Z_{p_2^{n_2}} \times ..... \times Z_{p_k^{n_k}}$ be a commutative ring of order $n = p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$, where p_i are distinct primes. Let $R = \langle x_1, x_2, ..., x_k \rangle$ such that $o(x_i) = p_i^{n_i}$, for $1 \leq i \leq k$. Let $U_0 = \langle 0 \rangle$, $U_1 = \langle x_1 \rangle$, $U_2 = \langle x_1, x_2 \rangle, ..., U_k = \langle x_1, x_2, ..., x_k \rangle = R$ be the chain of ideals of R such that $U_0 \subset U_1 \subset .... \subset U_{k-1} \subset U_k$.
Let A be any intuitionistic fuzzy ideal of R defined by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in U_0 \\ \alpha_1 & \text{if } x \in U_1 \backslash U_0 \\ \alpha_2 & \text{if } x \in U_2 \backslash U_1 \\ .... \\ \alpha_k & \text{if } x \in U_k \backslash U_{k-1} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0 & \text{if } x \in U_0 \\ \beta_1 & \text{if } x \in U_1 \backslash U_0 \\ \beta_2 & \text{if } x \in U_2 \backslash U_1 \\ .... \\ \beta_k & \text{if } x \in U_k \backslash U_{k-1}. \end{cases}$$

where $1 = \alpha_0 \geq \alpha_1 \geq ..... \geq \alpha_k$ and $0 = \beta_0 \leq \beta_1 \leq .... \leq \beta_k$ and the pair (α_i, β_i) are called double pins and the set $\wedge(A) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), ..., (\alpha_k, \beta_k)\}$ is called the set of double pinned flags for the IFI A of R.

Define IFSs A_i on R as follows:

$$\mu_{A_i}(x) = \begin{cases} 1, & \text{if } x \in R_i \\ \alpha_{i+1}, & \text{if otherwise} \end{cases} \quad ; \quad \nu_{A_i}(x) = \begin{cases} 0, & \text{if } x \in R_i \\ \beta_{i+1}, & \text{otherwise.} \end{cases}$$

where $\alpha_i, \beta_i \in (0, 1)$ such that $\alpha_i + \beta_i \leq 1$, for $1 \leq i \leq k$ and $\alpha_{k+1} = \alpha_1, \beta_{k+1} = \beta_1$ and $R_i = Z_{p_1^{n_1}} \times ..... \times Z_{p_{i-1}^{n_{i-1}}} \times \langle 0 \rangle \times Z_{p_{i+1}^{n_{i+1}}} \times .... \times Z_{p_k^{n_k}}$ is a primary ideal of R. Clearly, A_i are intuitionistic fuzzy primary ideal of R. It can be easily checked that $A = \cap_{i=1}^{n} A_i$ is an intuitionistic fuzzy primary decomposition of A.

Example 4.11. Consider $R = \prod_{i=1}^{\infty} \mathbf{Z_2}$, a direct product of infinitely many copies of the field $\mathbf{Z_2} = \{\bar{0}, \bar{1}\}$ be a boolean ring. Then R is a ring, which is not a Noetherian ring, as the strictly ascending chain of ideals $\mathbf{0} \subset \mathbf{Z_2} \times \mathbf{0} \subset \mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{0} \subset ......$ is not stationary.
For every $t_i, s_i \in [0, 1)$ such that $t_i + s_i \leq 1$, define $A_i \in IFS(R)$ as

$$\mu_{A_i}(x) = \begin{cases} 1, & \text{if } x = \prod_{i=1}^{\infty} \bar{0} \\ t_i, & \text{if otherwise} \end{cases} \quad ; \quad \nu_{A_i}(x) = \begin{cases} 0, & \text{if } x = \prod_{i=1}^{\infty} \bar{0} \\ s_i, & \text{if otherwise.} \end{cases}$$

for all $x \in R$. Then by Theorem (2.9), A_i is an intuitionistic fuzzy prime ideal and hence primary ideal of R.

Consider the IFI A of R defined by $A(x) = (0, 1), \forall x \in R$. Then A has no intuitionistic fuzzy primary decomposition in R, i.e., $A \neq \bigcap_{i=1}^{n} A_i$, for any $n \in \mathbb{N}$.

5. CONCLUSION

In this paper, we explored the fundamental ideas of intuitionistic fuzzy primary and P-primary ideal of a commutative ring R. We proved that an intuitionistic fuzzy primary ideal is a two valued intuitionistic fuzzy set with base set as a primary ideal (the base set of an intuitionistic fuzzy ideal A is defined as the set $\{x \in R | \mu_A(x) = \mu_A(0); \nu_A(x) = \nu_A(0)\}$ and vice versa. We also investigated the behaviour of intuitionistic fuzzy primary ideal under ring homomorphism. The structure of intuitionistic fuzzy P-primary ideal of a commutative R has been fully explored. Many properties of intuitionistic fuzzy primary ideals have been studied in terms of residual quotients. We have also laid down the foundation of the most important property in ring theory: decomposition of an ideal in terms of primary ideals in the intuitionistic fuzzy environment.

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