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ON THE EXISTENCE RESULTS OF FIRST AND SECOND-ORDER DIFFERENTIAL INCLUSIONS INVOLVING THE ϕ-LAPLACIAN IN $\mathbb{R}^n$

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ABSTRACT. In this paper, we will establish the existence of solutions to systems of first-order and second-order differential inclusions, with nonlinear differential operators, satisfying some boundary value conditions. We consider the case where the right-hand side is a lower semi-continuous set-valued map.

1. INTRODUCTION

In this paper, we will establish the existing results for the following systems of first-order and second-order differential inclusions:

$$\begin{cases} (\phi(x(t)))' \in F(t,x(t)) \text{ a.e. on } [0,T], \\ \int_0^T \phi(x(s))ds = a, \end{cases} \tag{1.1}$$

$$\begin{cases} (\phi(x(t)))' \in F(t,x(t)) \text{ a.e. on } [0,T], \\ x(\eta) = a, \end{cases} \tag{1.2}$$

$$\begin{cases} (\phi(x(t)))' \in F(t,x(t)) \text{ a.e. on } [0,T], \\ x(0) = x(T), \end{cases} \tag{1.3}$$

$$\begin{cases} (\phi(x'(t)))' \in F(t,x(t)) \text{ a.e. on } [0,T], \\ x(\tau) = a,\ x'(\eta) = b, \end{cases} \tag{1.4}$$

$$\begin{cases} (\phi(x'(t)))' \in F(t,x(t)) \text{ a.e. on } [0,T], \\ x(0) = a,\ x(\eta) = x(T), \end{cases} \tag{1.5}$$

and

$$\begin{cases} (\phi(x'(t)))' \in F(t,x(t)) \text{ a.e. on } [0,T], \\ x(0) = x'(\eta),\ x(\tau) = x(T), \end{cases} \tag{1.6}$$

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where $F : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued map which is lower semi-continuous with respect to the second argument, $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an homeomorphism, $T \in \mathbb{R}^{+*}$, $\eta, \tau \in [0,T]$ and $a, b \in \mathbb{R}^n$.

For boundary value problems with lower semi-continuous set-valued maps $F(t,.)$, in [1,2], the authors have proved the existence of solutions for Problems (1.2), with $\eta = 0$, (1.3), and (1.4), with $\tau = \eta = 0$ or $\tau = \eta = T$, in the case where $n = 1$ and ϕ, defined on $]-\alpha,\alpha[$ or $\mathbb{R}$, is an increasing homeomorphism for (1.2) and (1.3) and is uniquely an homeomorphism for (1.4). They have applied the method of upper and lower solutions, combined with the topological degree. In addition, in [3], the authors have established the existence of solutions for Problems (1.4), with $a = b$ and $\tau = 0$, (1.5), with $a = 0$, and (1.6), in the case where $n = 1$ and $\phi :]-\alpha,\alpha[\to \mathbb{R}$, with $0 < \alpha < +\infty$, is an increasing homeomorphism such that $\phi(0) = 0$. They have used the topological degree without the method of upper and lower solutions.

For boundary value problems for differential inclusions with ϕ-Laplacian, recently, Goli et al [4] have studied the following problem

$$\begin{cases} (b(x(t))\phi(x'(t)))' \in A(x(t)) + F(t, x(t), x'(t)) \text{ a.e. on } [0,T], \\ b(x(0))\phi(x'(0)) = g_1(x(0), x(T), x'(0), x'(T)), \\ b(x(T))\phi(x'(T)) = g_2(x(0), x(T), x'(0), x'(T)), \end{cases}$$

where $\phi :]-\alpha,\alpha[\longrightarrow \mathbb{R}$, with $0 < \alpha < +\infty$, is an increasing homeomorphism such that $\phi(0) = 0$, $A : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is a maximal monotone map, $F : [0,T] \times \mathbb{R}^2 \longrightarrow 2^{\mathbb{R}}$ is a L^P-Caratheodory multifunction, $p > 1$, with nonempty, convex and closed values, $b : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly positive continuous function and $g_1, g_2 : \mathbb{R}^4 \longrightarrow \mathbb{R}$ are continuous functions.

For boundary value problems for differential equations with ϕ-Laplacian, we refer the reader to the recent paper of Santos [15] and the references therein.

In this paper, we use different methods to give the existence of solutions for Problems (1.1)-(1.6). We suppose that the set-valued map F is measurable and lower semi-continuous with respect to the second argument and has compact values. The function ϕ is not necessary increasing or satisfies the condition $\phi(0) = 0$. It is an homeomorphism or verifies some monotonicity conditions which ensure that it is an homeomorphism. The main results of this work extend the previous cited above [1–3] and others in the literature related to this kind of problems.

2. Preliminaries

Let E be a Banach space equipped with the norm $\|.\|$, $I = [0,T]$, J a subset of $\mathbb{R}$, $\mathcal{C}(J,E)$ is the space of continuous functions endowed with the usual uniform norm that we denote $\|.\|_\infty$ and $\mathcal{C}^1(J,E)$ denotes the Banach space of continuously differentiable functions on J. The set $L^1([0,T],\mathbb{R}^n)$ refers to the Banach space of Lebesgue integrale functions from $[0,T]$ into $\mathbb{R}^n$. We say that a subset A of $I \times \mathbb{R}^n$ is $\mathcal{L} \otimes \mathcal{B}-$measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $J \times D$ where J is Lebesgue measurable in I and D is Borel measurable in $\mathbb{R}^n$. A multi-valued map is said to be measurable if its graph is measurable.

Now, we give some definitions of the concepts which will be used in the sequel.

Definition 2.1. Let E be a separable Banach space, X a nonempty closed subset of E and $F : X \to 2^E$ a multi-valued map with nonempty closed values. We say that F is lower semi-continuous if the set $\{x \in X : F(x) \cap C \neq \emptyset\}$ is open for any open set C in E.

Definition 2.2. A subset B of $L^1(I,\mathbb{R}^n)$ is decomposable if for all $u,\ v \in B$ and $J \subset I$ measurable, the function $u\mathcal{X}_J + v\mathcal{X}_{I\setminus J} \in B$, where $\mathcal{X}$ denotes the characteristic function.

Let $F : I \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multi-valued map with nonempty compact values. Assign to F the multi-valued operator $\mathcal{F} : \mathcal{C}(I, \mathbb{R}^n) \to 2^{L^1(I,\mathbb{R}^n)}$, defined by

$$\mathcal{F}(x) = \Big\{ y \in L^1(I, \mathbb{R}^n) : y(t) \in F(t, x(t)) \ for\ a.e.\ t \in I \Big\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F. We say that F is the lower semi-continuous type if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

For a lower semi-continuous set-valued operator which has nonempty closed and decomposable values, we have the following result.

Lemma 2.3. *[5] Let E be a separable metric space and $\Gamma : E \to 2^{L^1(I,\mathbb{R}^n)}$ be a multi-valued operator which is lower semi-continuous and has nonempty closed and decomposable values. Then Γ has a continuous selection, i.e there exists a continuous function $g : E \to L^1(I, \mathbb{R}^n)$ such that $g(y) \in \Gamma(y)$ for every $y \in E$.*

Definition 2.4. Let X and Y be topological spaces. A function $f : X \to Y$ is said compact if $f(X)$ is relatively compact.

The notion of the fixed point index is widely used in the sequel. Here we give its definition and some related results. For more detail, we refer the reader to [12, 14].

Definition 2.5. Let U be an open subset of $\mathbb{R}^n$ and $f : \overline{U} \to \mathbb{R}^n$ be a continuous compact function. An index is an application which associates (f, U) the integer $index(f, U)$ with the following properties:

(1) (Existence) if $index(f, U) \neq 0$, then f has a fixed point in U,
(2) (Normalisation) if $f(u) = u_0 \in U$ for all $u \in U$, then $index(f, U) = 1$,
(3) (Additivity) if $Fix(f) \subset U_1 \cup U_2$ where U_1 and U_2 are disjoint open subsets of U, then $index(f, U) = index(f, U_1) + index(f, U_2)$,
(4) (Homotopy) if $H : [0, 1] \times U \to \mathbb{R}^n$ is a continuous compact function such that $\cup_{\lambda\in[0,1]} Fix(H(\lambda, .))$ is compact, then $index(H(\lambda, .), U) = index(H(0, .), U)$ for all $\lambda \in [0, 1]$,
(5) (Excision) if $V \subset U$ is open and $Fix(f) \subset V$, then $index(f, U) = index(f, V)$.

Remark 2.6. If $f(x) \neq x$ for all $x \in \partial U$, then $Fix(f) = \Big\{ x \in U : f(x) = x \Big\}$ is compact.

In this paper, we will use the following assumptions.

$(H1)$ $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an homeomorphism,
$(H2)$ $F : I \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued map with nonempty compact values satisfying
  (1) $(t, x) \mapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B} - measurable$,
  (2) $x \mapsto F(t, x)$ is lower semi-continuous for almost all $t \in I$,
$(H3)$ there exists $m \in L^1(I, \mathbb{R}^+)$ such that

$$\|F(t, x)\| := \sup\{\|y\| : \ y \in F(t, x)\} \leq m(t), \quad \forall t \in I,\ \forall x \in \mathbb{R}^n.$$

In the sequel, we will use the following important Lemma.

Lemma 2.7. *[11] If conditions $(H2)$ and $(H3)$ are satisfied, then F is of the lower semi-continuous type.*

3. First order boundary problems

Set

$$W^{1,1}_{\phi}(J, \mathbb{R}^n) = \left\{ u \in \mathcal{C}(J, \mathbb{R}^n) : \phi(u) \text{ is absolutely continuous } \right\}.$$

A function $x : [0, T] \to \mathbb{R}^n$ is considered as a solution of (1.1) (resp. (1.2) and (1.3)) if it belongs to $W^{1,1}_{\phi}([0, T], \mathbb{R}^n)$ and satisfies the conditions of (1.1) (resp. (1.2) and (1.3)).

3.1. **First order with integral condition and with nonlocal condition.** The aim of this section is to establish the existence of solutions for the following problem of first order differential inclusions $(\phi(x(t)))' \in F(t, x(t))$ for almost every $t \in [0, T]$, with the condition $\int_0^T \phi(x(s))ds = a$ and with the condition $x(\eta) = a$.

Theorem 3.1. *Assume that the conditions* $(H1)$, $(H2)$ *and* $(H3)$ *are hold. Then Problem (1.1) (resp. (1.2)) has at least one solution.*

Proof. By Lemma 2.7, F is of the lower semi-continuous type. Then, by Lemma 2.3, there exists a continuous function $f : \mathcal{C}([0, T], \mathbb{R}^n) \to L^1([0, T], \mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \mathcal{C}([0, T], \mathbb{R}^n)$, where $\mathcal{F}$ is the Niemytzki operator associated with F. First, consider the family of problems defined, for $\lambda \in [0, 1]$, by

$$\begin{cases} (\phi(x(t)))' = \lambda f(x)(t) \text{ a.e. on } [0, T], \\ \int_0^T \phi(x(s))ds = a, \end{cases} \tag{3.1}$$

and

$$\begin{cases} (\phi(x(t)))' = \lambda f(x)(t) \text{ a.e. on } [0, T], \\ x(\eta) = a, \end{cases} \tag{3.2}$$

and consider the operators $\mathcal{Q}_1, \mathcal{Q}_2 : [0, 1] \times \mathcal{C}([0, T], \mathbb{R}^n) \to \mathcal{C}([0, T], \mathbb{R}^n)$ defined by

$$\mathcal{Q}_1(\lambda, x)(t) = \phi^{-1}\left(\frac{a}{T} + \lambda \int_0^T \left(\frac{s - T}{T}\right) f(x)(s)ds + \lambda N_f(x)(t)\right)$$

and

$$\mathcal{Q}_2(\lambda, x)(t) = \phi^{-1}\left(\phi(a) - \lambda\Big[N_f(x)(\eta) - N_f(x)(t)\Big]\right),$$

for all $(\lambda, x) \in [0, 1] \times \mathcal{C}([0, T], \mathbb{R}^n)$ and all $t \in [0, T]$, where

$$N_f : \mathcal{C}([0, T], \mathbb{R}^n) \to \mathcal{C}([0, T], \mathbb{R}^n)$$

is defined, for all $x \in \mathcal{C}([0, T], \mathbb{R}^n)$ by

$$N_f(x)(t) = \int_0^t f(x)(s)ds, \quad \forall t \in [0, T].$$

Remark that, since N_f is continuous and compact and ϕ is an homeomorphism, the operator $\mathcal{Q}_i$, $i \in \{1, 2\}$, is also continuous and compact.

Claim 3.2. *If x is a fixed point of $\mathcal{Q}_1(\lambda, .)$, x is a solution of (3.1). Moreover, there exists $R_1 > 0$ such that $\|x\|_\infty \le R_1$.*

Proof. If x is a fixed point of $\mathcal{Q}_1(\lambda, .)$, we have

$$\phi(x(t)) = \frac{a}{T} + \lambda \int_0^T \left(\frac{s - T}{T}\right) f(x)(s)ds + \lambda N_f(x)(t), \quad \forall t \in [0, T].$$

By integration between 0 and T, we obtain

$$\begin{aligned} \int_0^T \phi(x(s))ds &= a + \lambda \int_0^T (s - T) f(x)(s)ds + \lambda \int_0^T N_f(x)(s)ds \\ &= a + \lambda \int_0^T (s - T) f(x)(s)ds + \lambda\left[T N_f(x)(T) - \int_0^T s f(x)(s)ds\right] \\ &= a. \end{aligned}$$

In addition, by derivating $\phi(x(t))$, we get $(\phi(x(t)))' = \lambda f(x)(t)$ for almost every $t \in [0, T]$. Hence x is a solution of (3.1). On the other hand, for all $t \in [0, T]$, we have $\|f(x)(t)\| \le m(t)$. Then, for all $t \in [0, T]$

$$\begin{aligned}\left\|\frac{a}{T} + \lambda \int_0^T \Big(\frac{s-T}{T}\Big) f(x)(s)ds + \lambda N_f(x)(t)\right\| &\le \frac{\|a\|}{T} + \int_0^T m(\tau)d\tau + \int_0^T m(\tau)d\tau \\ &= \frac{\|a\|}{T} + 2\int_0^T m(\tau)d\tau := r_1.\end{aligned}$$

Since ϕ is an homeomorphism, there exists $R_1 > 0$ such that

$$\left\|\phi^{-1}\left(\frac{a}{T} + \lambda \int_0^T \Big(\frac{s-T}{T}\Big) f(x)(s)ds + \lambda N_f(x)(t)\right)\right\| \le R_1, \quad \forall t \in [0, T].$$

Thus, for all $t \in [0, T]$, $\|x(t)\| \le R_1$. Consequently $\|x\|_\infty \le R_1$. □

Claim 3.3. *If x is a fixed point of $\mathcal{Q}_2(\lambda, .)$, x is a solution of (3.2). Moreover, there exists $R_2 > 0$ such that $\|x\|_\infty \le R_2$.*

Proof. If x is a fixed point of $\mathcal{Q}_2(\lambda, .)$, one has

$$x(t) = \phi^{-1}\left(\phi(a) - \lambda\Big[N_f(x)(\eta) - N_f(x)(t)\Big]\right), \quad \forall t \in [0, T].$$

It is clair that $x(\eta) = \phi^{-1}(\phi(a)) = a$ and $(\phi(x(t)))' = \lambda f(x)(t)$ for almost all $t \in [0, T]$. Hence x is a solution of (3.2). In addition, for all $t \in [0, T]$, we have

$$\begin{aligned}\left\|\phi(a) - \lambda\Big[N_f(x)(\eta) - N_f(x)(t)\Big]\right\| &= \left\|\phi(a) - \lambda \int_t^\eta f(x)(s)ds\right\| \\ &\le \|\phi(a)\| + \int_0^T m(s)ds := r_2\end{aligned}$$

Then, as above, there exists $R_2 > 0$ such that $\|x\|_\infty \le R_2$. □

Now, fix $\alpha_i > R_i$, $i \in \{1, 2\}$, and set $\mathcal{O}_i = \Big\{x \in \mathcal{C}([0, T], \mathbb{R}^n) : \|x\|_\infty < \alpha_i\Big\}$. Remark that, for all $i \in \{1, 2\}$, $x \ne \mathcal{Q}_i(\lambda, x)$ for all $(\lambda, x) \in [0, 1] \times \partial\mathcal{O}_i$,

$$\mathcal{Q}_1(0, x) = \phi^{-1}(\frac{a}{T}) \in \mathcal{O}_1, \quad \forall x \in \mathcal{O}_1 \text{ and } \mathcal{Q}_2(0, x) = a \in \mathcal{O}_2, \quad \forall x \in \mathcal{O}_2.$$

Then, by normalisation property, we conclude that $index(\mathcal{Q}_i(0, .), \mathcal{O}_i) = 1$ for all $i \in \{1, 2\}$. By homotopy property, we get, for all $i \in \{1, 2\}$,

$$index(\mathcal{Q}_i(\lambda, .), \mathcal{O}_i) = index(\mathcal{Q}_i(0, .), \mathcal{O}_i) = 1, \quad \forall \lambda \in [0, 1].$$

Consequently, for all $\lambda \in [0, 1]$, $\mathcal{Q}_i(\lambda, .)$, for all $i \in \{1, 2\}$, has a fixed point. In particular, for $\lambda = 1$, Problem (1.1) (resp. (1.2)) has at least one solution. □

3.2. **Periodic problem.** We set

$$\mathcal{C}_P(I, \mathbb{R}^n) = \Big\{u \in \mathcal{C}(I, \mathbb{R}^n) : \ u(0) = u(T)\Big\}, \ \mathcal{C}_0(I, \mathbb{R}^n) = \Big\{u \in \mathcal{C}(I, \mathbb{R}^n) : \ u(0) = 0\Big\}$$

and

$$W^{1,1}(J, F) = \Big\{u \in \mathcal{C}(J, F) : u \text{ is absolutely continuous } \Big\}$$

where F is a space.

In the sequel, we need the following lemma.

Lemma 3.4. *[10] Let $u \in W^{1,1}(I, \mathbb{R}^n)$ and set $A = \{t \in I / \|u(t)\| > 0\}$. Then*

$$\|u\|_\infty \in W^{1,1}(A, \mathbb{R}) \text{ and } \|u(t)\|' = \frac{\langle u(t), u'(t)\rangle}{\|u(t)\|} \text{ a.e. } t \in A.$$

In the next, the following maximum principle type result will play a key role.

Lemma 3.5. *[9] Let $\omega \in W^{1,1}(I, \mathbb{R})$ such that, for almost all $t \in \{t \in I : \omega(t) > 0\}$, $\omega'(t) < 0$. If $\omega(0) \leq \omega(T)$, then $\omega(t) \leq 0$ for all $t \in I$.*

We recall here the Schauder's fixed point theorem.

Theorem 3.6. *[6] (**Schauder's fixed point theorem**) Let E be a Banach space and C be a nonempty closed convex set in E. Let $f : C \to C$ be a continuous map such that $f(C) \subset K$, where K is a compact subset of C. Then f has a fixed point in K.*

Now, we give the definition of solution-tube of (1.3).

Definition 3.7. Let $v \in W^{1,1}_\phi(I, \mathbb{R}^n)$ and $M \in W^{1,1}(I, [0, +\infty[)$. We say that (v, M) is a solution-tube of (1.3) if

(1) for every $t \in I$ and any $x \in \mathbb{R}^n$ such that $\|\phi(x) - \phi(v(t))\| = M(t)$, there exists $z \in F(t, x)$ such that $\langle \phi(x) - \phi(v(t)), z - (\phi(v(t)))'\rangle \leq M(t)M'(t)$,
(2) $(\phi(v(t)))' \in F(t, v(t))$ and $M'(t) = 0$ for a.e. $\{t \in I : M(t) = 0\}$,
(3) $\|\phi(v(0)) - \phi(v(T))\| \leq M(0) - M(T)$.

We note that the method of solution-tube has been successfully applied to study the existence of solutions for periodic value problem. We refer the reader to [8] and the references therein.

In this section, we will establish the existence of solutions to Problem (1.3).

Theorem 3.8. *Assume that the assumptions $(H1)$, $(H2)$ and $(H3)$ are satisfied. If Problem (1.3) has a solution-tube $(v, M) \in W^{1,1}_\phi(I, \mathbb{R}^n) \times W^{1,1}(I, [0, +\infty[)$, then it has at least one solution $x \in W^{1,1}_\phi(I, \mathbb{R}^n)$ such that*

$$\|\phi(x(t)) - \phi(v(t))\| \leq M(t), \quad \forall t \in I.$$

Proof. Consider the multi-valued mapping $\overline{F} : I \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ defined by

$$\overline{F}(t, x) = \phi(\overline{x}) + F(t, \overline{x}) \cap G(t, x),\ for\ all\ (t, x) \in I \times \mathbb{R}^n,$$

where

$$G(t,x) = \begin{cases} (\phi(v(t)))' \quad \text{if } M(t) = 0, \\ \mathbb{R}^n \text{ if } \|\phi(x) - \phi(v(t))\| \leq M(t) \text{ and } M(t) \neq 0, \\ \left\{ z \in \mathbb{R}^n : \langle \phi(x) - \phi(v(t)), z - (\phi(v(t)))'\rangle \leq M'(t)\|\phi(x) - \phi(v(t))\| \right\}, \text{ otherwise }, \end{cases}$$

and $\overline{x}$ is defined as follows

$$\phi(\overline{x}) = \begin{cases} \phi(x), \text{ if } \quad \|\phi(x) - \phi(v(t))\| \leq M(t), \\ \phi(v(t)) + \frac{M(t)}{\|\phi(x) - \phi(v(t))\|}(\phi(x) - \phi(v(t))), \text{ otherwise }. \end{cases}$$

Claim 3.9. *For all $t \in I$, $x \mapsto G(t, x)$ is lower semi-continuous.*

Proof. It is clair that $G(t,.)$ is lower semi-continuous on $\mathbb{R}^n$ if $M(t)=0$ and at all $x\in\mathbb{R}^n$ such that $\|\phi(x)-\phi(v(t))\|\le M(t)$ if $M(t)\neq 0$. Now, let $t\in I$ such that $M(t)\neq 0$ and $x\in\mathbb{R}^n$ such that $\|\phi(x)-\phi(v(t))\|>M(t)$. Let $(x_k)_k$ a sequence of $\mathbb{R}^n$ which converges to x and $z\in G(t,x)$. We will prove that there exists a sequence $(z_k)_k$ of $\mathbb{R}^n$ which converges to z and such that $z_k\in G(t,x_k)$ for all k. By continuity of $\|\phi(.)-\phi(v(t))\|$, for k large enough, we get $0<M(t)<\|\phi(x_k)-\phi(v(t))\|$. If

$$\langle \phi(x)-\phi(v(t)), z-(\phi(v(t)))'\rangle < M'(t)\|\phi(x)-\phi(v(t))\|,$$

by continuity of $\langle \phi(.)-\phi(v(t)), z-(\phi(v(t)))'\rangle$, for k large enough, we obtain

$$\langle \phi(x_k)-\phi(v(t)), z-(\phi(v(t)))'\rangle < M'(t)\left\|\phi(x_k)-\phi(v(t))\right\|.$$

Then, in this case, we take $z_k=z\in G(t,x_k)$ for k large enough. Next, if

$$\langle \phi(x)-\phi(v(t)), z-(\phi(v(t)))'\rangle = M'(t)\|\phi(x)-\phi(v(t))\|,$$

set

$$\begin{gathered}\alpha_k=\langle \phi(x_k)-\phi(v(t)), z-(\phi(v(t)))'\rangle - M'(t)\|\phi(x)-\phi(v(t))\|,\\ \beta_k=M'(t)\left\|\phi(x_k)-\phi(v(t))\right\|-M'(t)\|\phi(x)-\phi(v(t))\|,\\ \gamma_k=\max\left\{0,\frac{\alpha_k-\beta_k}{\|\phi(x_k)-\phi(v(t))\|}\right\} \text{ and } z_k=z-\gamma_k\frac{\phi(x_k)-\phi(v(t))}{\|\phi(x_k)-\phi(v(t))\|}.\end{gathered}$$

Since $(\alpha_k)_k$ and $(\beta_k)_k$ converge to 0, $(z_k)_k$ converges to z. On the other hand, we have

$$\begin{aligned}&\langle \phi(x_k)-\phi(v(t)), z_k-(\phi(v(t)))'\rangle\\ &=\langle \phi(x_k)-\phi(v(t)), z-(\phi(v(t)))'\rangle-\gamma_k\frac{\|\phi(x_k)-\phi(v(t))\|^2}{\|\phi(x_k)-\phi(v(t))\|}\\ &=M'(t)\|\phi(x)-\phi(v(t))\|+\alpha_k-\gamma_k\left\|\phi(x_k)-\phi(v(t))\right\|\\ &=M'(t)\left\|\phi(x_k)-\phi(v(t))\right\|-\beta_k+\alpha_k-\max\{0,\alpha_k-\beta_k\}\\ &\le M'(t)\left\|\phi(x_k)-\phi(v(t))\right\|\end{aligned}$$

Thus $z_k\in G(t,x_k)$. So, we conclude that $G(t,.)$ is lower semi-continuous on x. □

Claim 3.10. *$\overline{F}$ satisfies all assumptions of F.*

Proof. We have

$$\overline{F}(t,x) \quad = \quad \begin{cases}\phi(v(t))+(\phi(v(t)))' \quad \text{if } M(t)=0,\\ \phi(x)+F(t,x) \text{ if } \|\phi(x)-\phi(v(t))\|\le M(t) \text{ and } M(t)\neq 0,\\ \phi(\bar{x})+F(t,\bar{x})\cap G(t,x), \text{ otherwise },\end{cases}$$

Let $t\in I$ such that $M(t)\neq 0$ and $x\in\mathbb{R}^n$ such that $\|\phi(x)-\phi(v(t))\|>M(t)$. Since $\|\phi(\bar{x})-\phi(v(t))\|=M(t)$, by definition of solution-tube, there exists $z\in F(t,\bar{x})$ such that $\langle \phi(\bar{x})-\phi(v(t)), z-(\phi(v(t)))'\rangle\le M(t)M'(t)$. Then,

$$\left\langle \frac{M(t)}{\|\phi(x)-\phi(v(t))\|}(\phi(x)-\phi(v(t))), z-(\phi(v(t)))'\right\rangle \le M(t)M'(t).$$

Thus,

$$\langle \phi(x)-\phi(v(t)), z-(\phi(v(t)))'\rangle \le M'(t)\|\phi(x)-\phi(v(t))\|.$$

So $z\in G(t,x)$. Hence $\overline{F}$ has nonempty values. Now, since G has closed values and, for all $t\in I$, $G(t,.)$ is lower semi-continuous, the set-valued map $\overline{F}$ has compact values and, for all $t\in I$, $\overline{F}(t,.)$ is lower semi-continuous. The other hypotheses are obvious. □

Now, by the last claim and by Lemma 2.7, $\overline{F}$ is the lower semi-continuous type. So, there exists a continuous function $g : \mathcal{C}(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ such that, for all $y \in \mathcal{C}(I, \mathbb{R}^n)$, $g(y) \in \mathcal{F}(y)$ where $\mathcal{F}$ is the Niemytzki operator associated with $\overline{F}$.

Let us consider the operators $L : \mathcal{C}_P(I, \mathbb{R}^n) \to \mathcal{C}_0(I, \mathbb{R}^n)$ defined by

$$L(u)(t) = u(t) - u(0) + \int_0^t u(s)ds, \quad \forall u \in \mathcal{C}_P(I, \mathbb{R}^n), \quad \forall t \in I,$$

$\Phi : \mathcal{C}_P(I, \mathbb{R}^n) \to \mathcal{C}_P(I, \mathbb{R}^n)$ defined by $\Phi(u)(t) = \phi^{-1}(u(t))$ for all $u \in \mathcal{C}_P(I, \mathbb{R}^n)$ and all $t \in I$, and $\mathcal{K} : \mathcal{C}_P(I, \mathbb{R}^n) \to \mathcal{C}_P(I, \mathbb{R}^n)$ defined by $\mathcal{K}(u) = \Phi \circ L^{-1} \circ N_g(u)$ for all $u \in \mathcal{C}_P(I, \mathbb{R}^n)$.

Claim 3.11. *L is continuous and invertible.*

Proof. Let $(x_k)_k \subset \mathcal{C}_P(I, \mathbb{R}^n)$ which converges to x in $\mathcal{C}_P(I, \mathbb{R}^n)$. Then, for all $k \in \mathbb{N}$ and all $t \in I$, we have

$$\begin{aligned} \|L(x_k)(t) - L(x)(t)\| &\leq \|x_k(t) - x(t)\| + \|x_k(0) - x(0)\| + \int_0^t \|x_k(s) - x(s)\|ds \\ &\leq \|x_k(t) - x(t)\| + \|x_k(0) - x(0)\| + T \max_{s \in I} \|x_k(s) - x(s)\| \end{aligned}$$

The right hand side converges to 0. So the operator L is continuous. Now, let $y \in \mathcal{C}_0(I, \mathbb{R}^n)$. The function x defined by

$$x(t) = Ae^{-t} - \int_0^t e^{s-t} y(s)ds + y(t), \quad \forall t \in I,$$

where $A = \frac{1}{e^T - 1}\left[e^T y(T) - \int_0^T e^s y(s)ds\right]$, is the unique function in $\mathcal{C}_P(I, \mathbb{R}^n)$ satisfying $L(x)(t) = y(t)$ for all $t \in I$. Then L is invertible. □

Now, since the operator $\mathcal{K}$ is continuous and compact, by the Schauder fixed point theorem, $\mathcal{K}$ has a fixed point which is a solution of the following problem

$$\begin{cases} (\phi(x(t)))' + \phi(x(t)) \in \overline{F}(t, x(t)) \text{ a.e. on } I, \\ x(0) = x(T). \end{cases} \tag{3.3}$$

Claim 3.12. *Every solution x of (3.3) verifies $\|\phi(x(t)) - \phi(v(t))\| \leq M(t)$ for all $t \in I$.*

Proof. Let x be a solution of (3.3) and put $\omega(t) = \|\phi(x(t)) - \phi(v(t))\| - M(t)$.

- If $t \in \left\{t \in I : \ M(t) = 0 \text{ and } \omega(t) > 0\right\}$, we have

  $$\overline{F}(t, x(t)) = \{\phi(v(t)) + (\phi(v(t)))'\}.$$

  Since x is a solution of (3.3), we get

  $$(\phi(x(t)))' - (\phi(v(t)))' = -(\phi(x(t)) - \phi(v(t))).$$

  So

  $$\begin{aligned} \omega'(t) &= \frac{\langle \phi(x(t)) - \phi(v(t)), \phi'(x(t)) - \phi'(v(t)) \rangle}{\|\phi(x(t)) - \phi(v(t))\|} \\ &= \frac{-\|\phi(x(t)) - \phi(v(t))\|^2}{\|\phi(x(t)) - \phi(v(t))\|} \\ &< 0. \end{aligned}$$

- If $t \in \left\{t \in I : \ M(t) > 0 \text{ and } \omega(t) > 0\right\}$, since x is a solution of (3.3), there exists $z \in F(t, \overline{x}(t)) \cap G(t, x(t))$ such that

  $$\phi'(x(t)) + \phi(x(t)) = \phi(\overline{x}(t)) + z.$$

Then

$$\begin{aligned}
\omega'(t) &= \frac{\langle \phi(x(t)) - \phi(v(t)), \phi'(x(t)) - \phi'(v(t))\rangle}{\|\phi(x(t)) - \phi(v(t))\|} - M'(t) \\
&= \frac{\langle \phi(x(t)) - \phi(v(t)), \phi(\overline{x}(t)) - \phi(x(t)) + z - \phi'(v(t))\rangle}{\|\phi(x(t)) - \phi(v(t))\|} - M'(t) \\
&= \frac{\langle \phi(x(t)) - \phi(v(t)), \phi(\overline{x}(t)) - \phi(x(t))\rangle}{\|\phi(x(t)) - \phi(v(t))\|} + \frac{\langle \phi(x(t)) - \phi(v(t)), z - \phi'(v(t))\rangle}{\|\phi(x(t)) - \phi(v(t))\|} - M'(t) \\
&\leq \frac{\langle \phi(x(t)) - \phi(v(t)), \frac{M(t)}{\|\phi(x(t)) - \phi(v(t))\|}(\phi(x(t)) - \phi(v(t))) + \phi(v(t)) - \phi(x(t))\rangle}{\|\phi(x(t)) - \phi(v(t))\|} \\
&= \left(\frac{M(t)}{\|\phi(x(t)) - \phi(v(t))\|} - 1\right) \frac{\|\phi(x(t)) - \phi(v(t))\|^2}{\|\phi(x(t)) - \phi(v(t))\|} \\
&= M(t) - \|\phi(x(t)) - \phi(v(t))\| \\
&< 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\omega(0) &= \|\phi(x(0)) - \phi(v(0))\| - M(0) \\
&= \|\phi(x(T)) - \phi(v(0))\| - M(0) \\
&= \|\phi(x(T)) - \phi(v(T)) + \phi(v(T)) - \phi(v(0))\| - M(0) \\
&\leq \|\phi(x(T)) - \phi(v(T))\| + \|\phi(v(T)) - \phi(v(0))\| - M(0) \\
&\leq \|\phi(x(T)) - \phi(v(T))\| - M(T) \\
&= \omega(T).
\end{aligned}$$

So, by Lemma 3.5, $\omega(t) \leq 0$ for all $t \in I$ which implies that

$$\|\phi(x(t)) - \phi(v(t))\| \leq M(t), \quad \forall t \in I.$$

□

Now, any solution x of (3.3) is a solution of (1.3) because

$$\|\phi(x(t)) - \phi(v(t))\| \leq M(t) \text{ and } \overline{F}(t, x(t)) = \phi(x(t)) + F(t, x(t)), \quad \forall t \in I.$$

□

In the scalar case where $n = 1$, a function $\alpha \in W^{1,1}_{\phi}([0,T], \mathbb{R})$ is called a lower solution of (1.3), if $(\phi(\alpha(t)))' \in F(t, \alpha(t))$, for almost all $t \in [0,T]$, and $\alpha(0) \leq \alpha(T)$. Similarly a function $\beta \in W^{1,1}_{\phi}([0,T], \mathbb{R})$ is called an upper solution of (1.3), if $(\phi(\beta(t)))' \in F(t, \beta(t))$, for almost all $t \in [0,T]$, and $\beta(0) \geq \beta(T)$.

For $n = 1$, we have the following result.

Corollary 3.13. *Assume that the assumptions $(H2)$ and $(H3)$ are hold and ϕ is an increasing homeomorphism. If there exist $\alpha \in W^{1,1}_{\phi}(I, \mathbb{R})$ and $\beta \in W^{1,1}_{\phi}(I, \mathbb{R})$ are respectively lower and upper solutions of (1.3) with $\alpha(t) \leq \beta(t)$ for all $t \in I$. Then, the periodic problem (1.3) has at least one solution $x \in W^{1,1}_{\phi}(I, \mathbb{R})$ such that $\alpha(t) \leq x(t) \leq \beta(t)$ for every $t \in I$.*

Proof. Let $\phi(v(t)) = \frac{\phi(\beta(t))+\phi(\alpha(t))}{2}$ and $M(t) = \frac{\phi(\beta(t))-\phi(\alpha(t))}{2}$ for all $t \in I$. It is clair that $v \in W^{1,1}_{\phi}(I, \mathbb{R})$ and $M \in W^{1,1}(I, [0, +\infty[)$. On the other hand, let $t \in I$ and $x \in \mathbb{R}$ such that $|\phi(x) - \phi(v(t))| = M(t)$. If $\phi(x) - \phi(v(t)) = M(t)$, $x = \beta(t)$ and hence $(\phi(x) - \phi(v(t)))(z - (\phi(v(t)))') = M(t)M'(t)$, for $z = (\phi(\beta(t)))' \in F(t, x)$. If $\phi(x) - \phi(v(t)) = -M(t)$, $x = \alpha(t)$ so

$$(\phi(x) - \phi(v(t)))(z - (\phi(v(t)))') = M(t)M'(t),$$

for $z = (\phi(\alpha(t)))' \in F(t,x)$. Next, let $t \in I$ such that $M(t) = 0$. We have $\alpha(t) = \beta(t)$ and $\phi(v(t)) = \phi(\alpha(t))$. Hence $(\phi(v(t)))' \in F(t, v(t))$ and $M'(t) = 0$. In addition, one has

$$\Big|\phi(v(0)) - \phi(v(T))\Big| = \frac{\phi(\beta(0)) - \phi(\beta(T)) - \phi(\alpha(0)) + \phi(\alpha(T))}{2} = M(0) - M(T).$$

We conclude that (v, M) is a solution-tube of (1.3). Consequently, since all assumptions of Theorem 3.8 are hold, the periodic problem (1.3) has a solution $x \in W^{1,1}_{\phi}(I,\mathbb{R})$ such that $\big|\phi(x(t)) - \phi(v(t))\big| \leq M(t)$ for all $t \in I$. Since ϕ is an increasing homeomorphism, we deduce that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in I$. □

In the rest of this section, we consider Problem (1.3) without the condition of the existence of solution-tube.

Theorem 3.14. *If the assumptions $(H2)$ and $(H3)$ are hold and $\phi : B_r \to \mathbb{R}^n$, where $r > 0$ and $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$, is an homeomorphism, the periodic problem (1.3) has at least one solution $x \in W^{1,1}_{\phi}(I,\mathbb{R})$.*

Proof. As above, there exists a continuous function $f : \mathcal{C}([0,T],\mathbb{R}^n) \to L^1([0,T],\mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \mathcal{C}([0,T],\mathbb{R}^n)$. Consider the family of problems defined, for $\lambda \in [0,1]$, by

$$\begin{cases} (\phi(x(t)))' = \lambda f(x)(t) \text{ a.e. on } [0,T], \\ \phi(x(0)) = \lambda\phi(x(T)), \end{cases} \tag{3.4}$$

and consider the operator $\mathcal{H} : [0,1] \times \mathcal{C}([0,T],\mathbb{R}^n) \to \mathcal{C}([0,T],\mathbb{R}^n)$ defined by

$$\mathcal{H}(\lambda, x)(t) = \phi^{-1}\Bigg(\lambda\phi(x(T)) + \lambda N_f(x)(t)\Bigg)$$

for all $(\lambda, x) \in [0,1] \times \mathcal{C}([0,T],\mathbb{R}^n)$ and all $t \in [0,T]$. The operator $\mathcal{H}$ is continuous and compact. We note that if x is a fixed point of $\mathcal{H}(\lambda,.)$, x is a solution of (3.4) and $\|x\|_\infty < r$. Now, fix $R > r$ and set $\mathcal{U} = \Big\{x \in \mathcal{C}([0,T],\mathbb{R}^n) : \|x\|_\infty < R\Big\}$. We have $x \neq \mathcal{H}(\lambda, x)$, for all $(\lambda, x) \in [0,1] \times \partial\mathcal{U}$, and $\mathcal{H}(0,x) = \phi^{-1}(0) \in \mathcal{U}$ for all $x \in \mathcal{U}$. If we apply the normalisation property and the homotopy property, we get $index(\mathcal{H}(\lambda,.),\mathcal{U}) = index(\mathcal{H}(0,.),\mathcal{U}) = 1$ for all $\lambda \in [0,1]$. Consequently, for all $\lambda \in [0,1]$, $\mathcal{H}(\lambda,.)$ has a fixed point. In particular, for $\lambda = 1$, Problem (1.3) has at least one solution. □

4. Second order boundary value problems

Set

$$W^{2,1}_{\phi}(I,\mathbb{R}^n) = \Bigg\{u \in \mathcal{C}^1(I,\mathbb{R}^n) : \phi(u') \text{ is absolutely continuous}\Bigg\}.$$

Definition 4.1. A function $x : [0,T] \to \mathbb{R}^n$ is said a solution of (1.4) (resp. (1.5) and (1.6)) if it belongs to $W^{2,1}_{\phi}([0,T],\mathbb{R}^n)$ and satisfies the conditions of (1.4) (resp. (1.5) and (1.6)).

4.1. **Two points boundary value problems.** In this section, we shall prove the following result.

Theorem 4.2. *Assume that the conditions $(H1)$, $(H2)$ and $(H3)$ are hold. Then Problem (1.4) has at least one solution.*

Proof. As in the last sections, we consider the problems defined, for $\lambda \in [0,1]$, by

$$\begin{cases} (\phi(x'(t)))' = \lambda f(x)(t) \; a.e. \; on \; [0,T] \\ x(\tau) = a, \; x'(\eta) = b, \end{cases} \tag{4.1}$$

where $f:\mathcal{C}([0,T],\mathbb{R}^n)\to L^1([0,T],\mathbb{R}^n)$ is continuous and such that $f(y)\in\mathcal{F}(y)$ for all $y\in\mathcal{C}([0,T],\mathbb{R}^n)$. We define the operator

$$\mathcal{M}:[0,1]\times\mathcal{C}^1([0,T],\mathbb{R}^n)\to\mathcal{C}^1([0,T],\mathbb{R}^n),$$

by

$$\mathcal{M}(\lambda,x)(t)=a+\int_\tau^t\phi^{-1}\Big(\phi(b)+\lambda\big(N_f(x)(s))-N_f(x)(\eta)\big)\Big)ds,$$

for all $(\lambda,x)\in[0,1]\times\mathcal{C}^1([0,T],\mathbb{R}^n)$ and all $t\in[0,T]$. The operator $\mathcal{M}$ is continuous and compact. It is easy to verify that any fixed point of $\mathcal{M}(\lambda,.)$ is a solution of (4.1). Moreover, for all $\lambda\in[0,T]$, for all fixed point x of $\mathcal{M}(\lambda,.)$ and for all $t\in[0,T]$, we have

$$\begin{aligned}\Big\|\phi(b)+\lambda\big(N_f(x)(t)-N_f(x)(\eta)\big)\Big\| &\le\|\phi(b)\|+2\lambda\int_0^T\|f(x)(s)\|ds\\ &\le\|\phi(b)\|+2\int_0^T m(s)ds=r_1.\end{aligned}$$

Then there exists $r_2>0$ such that

$$\left\|\phi^{-1}\Big(\phi(b)+\lambda\big(N_f(x)(t)-N_f(x)(\eta)\big)\Big)\right\|\le r_2,$$

for all $\lambda\in[0,T]$, for all fixed point x of $\mathcal{M}(\lambda,.)$ and for all $t\in[0,T]$. Thus

$$\begin{aligned}\|x(t)\|= \quad &\le\|a\|+2\int_0^T\left\|\phi^{-1}\Big(\phi(b)+\lambda\big(N_f(x)(s)-N_f(x)(\eta)\big)\Big)\right\|ds\\ &\le\|a\|+2Tr_2\end{aligned}$$

for all $\lambda\in[0,T]$, for all fixed point x of $\mathcal{M}(\lambda,.)$ and for all $t\in[0,T]$. Consequently, there exists $R>\|a\|+2Tr_2$ such that $\|x\|_\infty<R$ for all $\lambda\in[0,T]$ and for all fixed point x of $\mathcal{M}(\lambda,.)$. Now, set $\mathcal{U}=\{x\in\mathcal{C}^1([0,T],\mathbb{R}^n):\|x\|_\infty<R\}$. Observe that $x\neq\mathcal{M}(\lambda,x)$ for all $(\lambda,x)\in[0,1]\times\partial\mathcal{U}$ and $\mathcal{M}(0,x)=u\in\mathcal{U}$ for all $x\in\mathcal{U}$, where $u(t)=a+b(t-\tau)$, $\forall t\in[0,T]$. Then, by normalisation property and homotopy property, we get

$$index(\mathcal{M}(\lambda,.),\mathcal{U})=index(\mathcal{M}(0,.),\mathcal{U})=1,\quad\forall\lambda\in[0,1].$$

Hence, for all $\lambda\in[0,1]$, $\mathcal{M}(\lambda,.)$ has a fixed point which is a solution of (4.1). So, if we take $\lambda=1$, we get the existence of solution of Problem (1.4). □

4.2. **Three and four points boundary value problems.** In this section, we will use the following assumption.

$(H1^*)$ $\phi:\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function which satisfies the following two conditions:
 (1) for any $x_1,x_2\in\mathbb{R}^n, x_1\neq x_2$, $\langle\phi(x_1)-\phi(x_2),x_1-x_2\rangle>0$,
 (2) there exists a function $\alpha:\mathbb{R}^+\to\mathbb{R}^+$ such that $\lim_{s\to+\infty}\alpha(s)=+\infty$ and

$$\langle\phi(x),x\rangle\ge\alpha(\|x\|)\|x\|,\ \text{for all } x\in\mathbb{R}^n.$$

Remark 4.3. Under the condition $(H1^*)$, ϕ is an homeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$ (see [7]).

Now, for fixed $h\in C([0,T],\mathbb{R}^n)$ and $c\in\mathbb{R}^n$, let us define

$$L_h(c)=\int_\rho^T\phi^{-1}(c+h(t))dt,$$

where $\rho\in\{\tau,\eta\}\subset[0,T[$. The following lemma plays a fundamental role in the rest of this paper.

Lemma 4.4. *[13] If ϕ satisfies $(H1^*)$, then the function L_h has the following properties:*

(1) *for any fixed $h \in C([0,T],\mathbb{R}^n)$, the equation $L_h(c) = 0$ has a unique solution $\tilde{c}(h)$,*
(2) *the function $\tilde{c} : C([0,T],\mathbb{R}^n) \to \mathbb{R}^n$ is continuous and sends bounded sets into bounded sets.*

We note that Lemma 4.4, in [13], has been proved for $\rho = 0$. The same proof is valid for $\rho \in]0,T[$ without any modification.

We shall prove the following main results.

Theorem 4.5. *Assume that the conditions $(H1^*)$, $(H2)$ and $(H3)$ are hold and $\tau, \eta \in [0,T[$. Then, Problem (1.5) (resp. (1.6)) has at least one solution.*

Proof. Consider the following modified problems

$$\begin{cases} (\phi(x'(t)))' = \lambda f(x)(t) \text{ a.e. on } [0,T], \\ x(0) = a,\ x(\eta) = x(T), \end{cases} \tag{4.2}$$

and

$$\begin{cases} (\phi(x'(t)))' = \lambda f(x)(t) \text{ a.e. on } [0,T], \\ x(0) = x'(\eta),\ x(\tau) = x(T). \end{cases} \tag{4.3}$$

Let us define the operators $\mathcal{A},\ \mathcal{K} : [0,1] \times \mathcal{C}^1([0,T],\mathbb{R}^n) \to \mathcal{C}^1([0,T],\mathbb{R}^n)$ by

$$\mathcal{A}(\lambda,x)(t) = a + \int_0^t \phi^{-1}\Big(\tilde{c}_1(\lambda N_f(x)) + \lambda N_f(x)(s)\Big)ds,$$

and

$$\mathcal{K}(\lambda,x)(t) = \phi^{-1}\Big(\tilde{c}_2(\lambda N_f(x)) + \lambda N_f(x)(\eta)\Big) + \int_0^t \phi^{-1}\Big(\tilde{c}_2(\lambda N_f(x)) + \lambda N_f(x)(s)\Big)ds,$$

for all $(\lambda,x) \in [0,1] \times \mathcal{C}^1([0,T],\mathbb{R}^n)$ and all $t \in [0,T]$, with $\tilde{c}_1(\lambda N_f(x))$ and $\tilde{c}_2(\lambda N_f(x))$ are uniquely determined by Lemma 4.4 and satisfies the following

$$\int_\eta^T \phi^{-1}(\tilde{c}_1(\lambda N_f(x)) + \lambda N_f(x)(s))ds = 0 \text{ and } \int_\tau^T \phi^{-1}(\tilde{c}_2(\lambda N_f(x)) + \lambda N_f(x)(s))ds = 0.$$

Note that, by taking $\lambda = 0$ in the above equalities, we get $\tilde{c}_1(0) = \tilde{c}_2(0) = \phi(0)$. It follows from the assumptions $(H1^*)$, $H2$ and $H3$, using Lemma 4.4, that the operators $\mathcal{A}$ and $\mathcal{K}$ are continuous and compact. Now, we claim that any fixed point of $\mathcal{A}(\lambda,.)$ is a solution of (4.2) and any fixed point of $\mathcal{K}(\lambda,.)$ is a solution of (4.3). Indeed, if x is a fixed point of $\mathcal{A}(\lambda,.)$, we have

$$x(t) = a + \int_0^t \phi^{-1}\Big(\tilde{c}_1(\lambda N_f(x)) + \lambda N_f(x)(s)\Big)ds, \quad \forall t \in [0,T].$$

Then, $(\phi(x'(t)))' = \lambda f(x)(t)$ for almost all $t \in [0,T]$, $x(0) = a$ and $x(\eta) = x(T)$ because

$$\int_\eta^T \phi^{-1}(\tilde{c}_1(\lambda N_f(x)) + \lambda N_f(x)(s))ds = 0.$$

If x is a fixed point of $\mathcal{K}(\lambda,.)$, we have

$$x(t) = \phi^{-1}\Big(\tilde{c}_2(\lambda N_f(x)) + \lambda N_f(x)(\eta)\Big) + \int_0^t \phi^{-1}\Big(\tilde{c}_2(\lambda N_f(x)) + \lambda N_f(x)(s)\Big)ds, \quad \forall t \in [0,T].$$

Then, $x'(t) = \phi^{-1}\Big(\tilde{c}_2(\lambda N_f(x)) + \lambda N_f(x)(t)\Big)$ and $(\phi(x'(t)))' = \lambda f(x)(t)$ for almost all $t \in [0,T]$. So $x(0) = x'(\eta)$ and, since

$$\int_\tau^T \phi^{-1}(\tilde{c}_2(\lambda N_f(x)) + \lambda N_f(x)(s))ds = 0,$$

$x(\tau) = x(T)$. On the other hand, for all $t \in [0,T]$ and all $x \in \mathcal{C}^1([0,T],\mathbb{R}^n)$, we have $\|f(x)(t)\| \leq m(t)$. Then

$$\|\lambda N_f(x)\|_\infty \leq \int_0^T m(s)ds, \quad \forall x \in \mathcal{C}^1([0,T],\mathbb{R}^n), \forall \lambda \in [0,1].$$

So, by Lemma 4.4, there exists $r_1 > 0$ and $r_2 > 0$ such that

$$\|\tilde{c}_1(\lambda N_f(x))\| \leq r_1 \text{ and } \|\tilde{c}_2(\lambda N_f(x))\| \leq r_2, \quad \forall x \in \mathcal{C}^1([0,T],\mathbb{R}^n), \forall \lambda \in [0,1].$$

Hence, for all $t \in [0,T]$, for all $x \in \mathcal{C}^1([0,T],\mathbb{R}^n)$ and all $\lambda \in [0,1]$,

$$\Big\|\tilde{c}_1(\lambda N_f(x)) + \lambda N_f(x)(t)\Big\| \leq r_1 + \int_0^T m(s)ds := r_3$$

and

$$\Big\|\tilde{c}_2(\lambda N_f(x)) + \lambda N_f(x)(t)\Big\| \leq r_2 + \int_0^T m(s)ds := r_4.$$

Since ϕ is an homeomorphism, there exists $r_5 > 0$ and $r_6 > 0$ such that

$$\Big\|\phi^{-1}\Big(\tilde{c}_1(\lambda N_f(x)) + \lambda N_f(x)(t)\Big)\Big\| \leq r_5$$

and

$$\Big\|\phi^{-1}\Big(\tilde{c}_2(\lambda N_f(x)) + \lambda N_f(x)(t)\Big)\Big\| \leq r_6,$$

for all $t \in [0,T]$, for all $x \in \mathcal{C}^1([0,T],\mathbb{R}^n)$ and all $\lambda \in [0,1]$. Thus

$$\begin{aligned}\|\mathcal{A}(\lambda, x(t))\| &\leq \|a\| + \int_0^t \Big\|\phi^{-1}\Big(\tilde{c}_1(\lambda N_f(x)) + \lambda N_f(x)(t)\Big)\Big\|ds \\ &\leq \|a\| + r_5 T := R_1\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{K}(\lambda, x(t))\| \leq& \Big\|\phi^{-1}\Big(\tilde{c}(\lambda N_f(x)) + \lambda N_f(x)(\eta)\Big)\Big\| + \int_0^t \Big\|\phi^{-1}\Big(\tilde{c}(\lambda N_f(x)) + \lambda N_f(x)(t)\Big)\Big\|ds \\ \leq& \, r_6(1+T) := R_2,\end{aligned}$$

for all $t \in [0,T]$, for all $x \in \mathcal{C}^1([0,T],\mathbb{R}^n)$ and all $\lambda \in [0,1]$. Now, for $i \in \{1,2\}$, set $\mathcal{O}_i = \Big\{x \in \mathcal{C}^1([0,T],\mathbb{R}^n) : \|x\|_\infty < R_i + 1\Big\}$. Remark that $x \neq \mathcal{A}(\lambda, x)$ for all $(\lambda, x) \in [0,1] \times \partial\mathcal{O}_1$ and $\mathcal{A}(0,x) = a \in \mathcal{O}_1$ for all $x \in \mathcal{O}_1$. Also $x \neq \mathcal{K}(\lambda, x)$ for all $(\lambda, x) \in [0,1] \times \partial\mathcal{O}_2$ and $\mathcal{K}(0,x) = 0 \in \mathcal{O}_2$ for all $x \in \mathcal{O}_2$. Then, by normalisation property and homotopy property, we conclude that

$$index(\mathcal{A}(\lambda,.), \mathcal{O}_1) = index(\mathcal{K}(\lambda,.), \mathcal{O}_2) = 1, \quad \forall \lambda \in [0,1].$$

Consequently, for all $\lambda \in [0,1]$, $\mathcal{A}(\lambda,.)$ and $\mathcal{K}(\lambda,.)$ have a fixed point. In particular, for $\lambda = 1$, Problems (1.5) and (1.6) have at least one solution. □

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