

NEW SRQP METHOD FOR SOLVING NONLINEAR COMPLEMENTARITY PROBLEMS

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ABSTRACT. Extending the framework developed in [5], this study introduces a refined variant of the square root quadratic proximal (SRQP) method for the resolution of nonlinear complementarity problems. Unlike the conjugate gradient scheme adopted in [5], which constructs search directions through a combination of current and preceding iterates, the proposed approach incorporates a simplified correction mechanism that avoids such combinations. Under the assumption that the mapping F is pseudomonotone, the global convergence of the method is rigorously established. Furthermore, preliminary numerical experiments are reported to assess the efficiency and robustness of the proposed algorithm.

1. INTRODUCTION

NCP seeks to identify a vector $x \in \mathbb{R}^n$ satisfying

$$(1.1) \quad x \geq 0, \quad F(x) \geq 0 \quad \text{and} \quad x^T F(x) = 0,$$

F represents a nonlinear function from $\mathbb{R}^n$ onto itself. This study considers $F(x)$ to be continuous and pseudomonotone with respect to $\mathbb{R}_+^n$. Additionally, it is assumed that the solution set for (1.1), represented by Ω^* , is not empty.

One common method for solving the NCP is to find the zero point of $O(x) = F(x) + C_{R_+^n}(x)$, i.e., find $x^* \in \mathbb{R}_+^n$ such that $0 \in O(x^*)$, where $C_{R_+^n}(\cdot)$ is the normal cone operator to $\mathbb{R}_+^n$ defined by

$$C_{R_+^n}(t) = \begin{cases} \{u \in \mathbb{R}^n : u^T(v - t) \leq 0, \quad \forall v \in \mathbb{R}_+^n\} & \text{if } t \in \mathbb{R}_+^n, \\ \emptyset & \text{otherwise.} \end{cases}$$

A well-established method for solving this problem is the proximal point algorithm, which initiates with an arbitrary starting $x^0 \in \mathbb{R}_+^n$ and $\beta_k \geq \beta > 0$, updates x^{k+1} for solving :

$$(1.2) \quad (\text{PPA}) \quad 0 \in \beta_k O(x) + \nabla_x q(x, x^k).$$

where

$$(1.3) \quad q(x, x^k) = \frac{1}{2} \|x - x^k\|^2$$

is a quadratic function of x .

Recently, numerous studies have focused on developing innovative interior point methods to address NCP. These methods share a common characteristic which enforce the new iterates $\{x^{k+1}\}$ to stay in the

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interior of R_{++}^n . Auslender, *et al.* [1] have proposed a new type of proximal interior algorithms via replacing the quadratic function (1.3) by $dist_\phi(x, x^k)$ which could be defined as

$$dist_\phi(x, y) = \sum_{j=1}^n y_j^2 \phi(y_j^{-1} x_j).$$

Let $\nu > \mu > 0$ two fixed parameters, and define

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu\varphi(t) & \text{if } t > 0 \\ +\infty & \text{otherwise,} \end{cases}$$

where $\varphi(t)$ is a φ -divergence function that respects these essential properties:

- 1) φ is twice continuously differentiable on the interior of R^n i.e. $(0, +\infty)$.
- 2) φ is strictly convex on its domain.
- 3) $\lim_{x \rightarrow 0^+} \frac{d\varphi(x)}{dx} = -\infty$.
- 4) $\varphi(1) = \frac{d\varphi(1)}{dx} = 0$ and $\frac{d^2\varphi(1)}{dx^2} > 0$.
- 5) There exists $\nu \in \left(\frac{1}{2} \frac{d^2\varphi(1)}{dx^2}, \frac{d^2\varphi(1)}{dx^2}\right)$ such that

$$\left(1 - \frac{1}{t}\right) \left(\frac{d^2\varphi(1)}{dx^2} + \nu(t-1)\right) \leq \frac{d\varphi(t)}{dx} \leq \frac{d^2\varphi(1)}{dx^2}(t-1) \quad \forall t > 0.$$

Auslender, *et al.* proposed a specialized logarithmic-quadratic proximal (LQP) using $\varphi_1(t) = t - \log(t) - 1$ in the definition of $\phi(t)$ (with $\nu = 2, \mu = 1$).

Later on, Noor and Bnouhachem [6], have proposed a new modified LQP method by using $\varphi_2(t) = t \log(t) - t + 1$ as a φ -divergence function (with $\nu = 1, \mu \in (0, 1)$)

Let $\nu = \frac{1}{2}$ and $\mu \in (0, \frac{1}{2})$, in our contribution, we used the φ function $\varphi_3(t) = (\sqrt{t} - 1)^2$, which satisfies all five required properties of a φ -divergence function, and we get

$$\phi(t) = \begin{cases} \frac{1}{4}(t-1)^2 + \mu(\sqrt{t}-1)^2 & \text{if } t > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Then the problem (1.2) becomes for given $x^k \in R_+^n$ and $\beta_k \geq \beta > 0$, the new iterate x^{k+1} is unique solution of the following set-valued equation:

$$(1.4) \quad (\text{SRQP}) \quad 0 \in \beta_k O(x) + \nabla_x dist_\phi(x, x^k),$$

where

$$(1.5) \quad dist_\phi(x, x^k) = \begin{cases} \frac{1}{4}\|x - x^k\|^2 + \mu \sum_{j=1}^n \left(x_j^k x_j - 2(x_j^k)^2 \sqrt{\frac{x_j}{x_j^k}} + (x_j^k)^2 \right) & \text{if } x \in R_{++}^n, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that

$$(1.6) \quad \begin{aligned} \nabla_x dist_\phi(x, x^k) &= \frac{1}{2}(x - x^k) + \mu \sum_{j=1}^n \left(x_j^k - \frac{(x_j^k)^2}{\sqrt{x_j^k}} \frac{1}{\sqrt{x_j}} \right) \\ &= \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}). \end{aligned}$$

Where $X_k = \text{diag}(\sqrt{x_1^k}^3, \dots, \sqrt{x_n^k}^3)$ and $\sqrt{x} = (\sqrt{x_1}, \dots, \sqrt{x_n})$.

Then the problem (1.4) is equivalent to the following systems of nonlinear equations

$$(1.7) \quad \beta_k F(x) + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = 0.$$

Solving the subproblem (1.7) exactly presents significant challenges in practice, often excluding practical applications. To mitigate this issue, it is advisable to pursue approximate solutions instead of exact ones. In this paper, we proposed a prediction-correction method to solve (1.7) approximately. Numerical results are provided to substantiate the efficacy of the proposed method.

2. PRELIMINARIES

We list some important results which will be required in our following analysis.

First, we denote $Pr_{R_+^n}(\cdot)$ as the projection under the Euclidean norm, i.e.,

$$Pr_{R_+^n}(z) = \min\{\|z - x\| \mid x \in R_+^n\}.$$

A basic property of the mapping of projection is

$$(2.1) \quad (y - Pr_{R_+^n}(y))^T (Pr_{R_+^n}(y) - x) \geq 0, \quad \forall y \in R^n, \quad \forall x \in R_+^n.$$

From (2.1), it is easy to verify that

$$(2.2) \quad \|Pr_{R_+^n}(v) - u\|^2 \leq \|v - u\|^2 - \|v - Pr_{R_+^n}(v)\|^2, \quad \forall v \in R^n, u \in R_+^n.$$

Definition 2.1. The operator $F : R^n \rightarrow R^n$ is said to be monotone, if

$$\forall u, v \in R^n, \quad (v - u)^T (F(v) - F(u)) \geq 0.$$

Definition 2.2. The operator $F : R^n \rightarrow R^n$ is said to be pseudomonotone, if

$$\forall u, v \in R^n, \quad (v - u)^T F(u) \geq 0 \Rightarrow (v - u)^T F(v) \geq 0.$$

3. THE PROPOSED METHOD AND CONVERGENCE RESULTS

At the k th iteration, SRQP finds the exact solution for the following system of equations:

$$(3.1) \quad \beta_k F(x) + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = 0.$$

We now present an SRQP-based interior prediction-correction method for solving NCP. For given $x^k > 0$ and $\beta_k > 0$, each iteration of the proposed method consists of two steps, the first step offers a predictor $\tilde{x}^k$ and the second step produces the new iterate x^{k+1} .

Prediction step: Find an approximate solution $\tilde{x}^k$ of (3.1), called predictor, such that

$$(3.2) \quad 0 \approx \beta_k F(x) + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = \xi^k$$

and ξ^k which satisfies

$$(3.3) \quad \|\xi^k\| \leq \eta \|x^k - \tilde{x}^k\|,$$

where

$$\eta < \frac{1}{2}, \quad \eta^2 + \mu^2 < 1.$$

Correction step: For $\alpha > 0$ and $0 < a_1 < 1$, the new iterate $x^{k+1}(\alpha)$ is defined by

$$(3.4) \quad x^{k+1}(\alpha) = a_1 x^k + (1 - a_1) Pr_{R_+^n} \left[x^k - \frac{\alpha \beta_k}{1 + \mu} F(\tilde{x}^k) \right].$$

How to choose a suitable step length $\alpha > 0$ will be discussed later.

Remark 3.1. (3.3) implies that

$$(3.5) \quad |(x^k - \tilde{x}^k)^T \xi^k| \leq \eta \|x^k - \tilde{x}^k\|^2,$$

where

$$\eta < \frac{1}{2}, \quad \eta^2 + \mu^2 < 1.$$

Remark 3.2. Note that in the special case $\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k))$, if F is Lipschitz continuous in R_+^n with Lipschitz constant $L > 0$, i.e.,

$$\|F(x^k) - F(\tilde{x}^k)\| \leq L \|x^k - \tilde{x}^k\|.$$

If we choose β_k satisfying $0 < \beta_k \leq \frac{\eta}{L}$, then the above inequalities (3.3) are satisfied.

The following Lemma plays a crucial role in the convergence analysis.

Lemma 3.3. For given $x^k > 0$ and $q \in R^n$, let x be the positive solution of the following equation:

$$(3.6) \quad q + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = 0,$$

where $X_k = \text{diag}(\sqrt{x_1^k}, \dots, \sqrt{x_n^k})$ and $\sqrt{x} = (\sqrt{x_1}, \dots, \sqrt{x_n})$,
then for any $y \geq 0$ we have

$$(3.7) \quad (x - y)^T(-q) \geq \frac{1+\mu}{4} (\|x - y\|^2 - \|x^k - y\|^2) + \frac{1-\mu}{4} \|x^k - x\|^2.$$

Proof. [5] □

Lemma 3.4. Let

$$(3.8) \quad \Theta(\alpha) = \|x^k - x^*\|^2 - \|x^{k+1}(\alpha) - x^*\|^2,$$

then we have

$$(3.9) \quad \Theta(\alpha) \geq (1 - a_1) \{ \|x^k - x_*^k\|^2 + 2\alpha(x_*^k - \tilde{x}^k)^T d^k - \frac{\alpha\mu}{1+\mu} \|x^k - \tilde{x}^k\|^2 \},$$

where

$$(3.10) \quad x_*^k := P_{R_+^n} \left[x^k - \frac{\alpha\beta_k}{1+\mu} F(\tilde{x}^k) \right] \quad \text{and} \quad d^k := \frac{1}{2}(x^k - \tilde{x}^k) + \frac{1}{1+\mu} \xi^k.$$

Proof. By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (3.6) and $y = x_*^k := P_{R_+^n} \left[x^k - \frac{\alpha\beta_k}{1+\mu} F(\tilde{x}^k) \right]$ in (3.7), it follows

$$(3.11) \quad (x_*^k - \tilde{x}^k)^T \left(\frac{1}{1+\mu} (\xi^k - \beta_k F(\tilde{x}^k)) \right) \leq \frac{1}{4} (\|x^k - x_*^k\|^2 - \|\tilde{x}^k - x_*^k\|^2) - \frac{1-\mu}{4(1+\mu)} \|x^k - \tilde{x}^k\|^2.$$

Recall

$$(3.12) \quad \frac{1}{2} (x_*^k - \tilde{x}^k)^T (x^k - \tilde{x}^k) = \frac{1}{4} (\|\tilde{x}^k - x_*^k\|^2 - \|x^k - x_*^k\|^2) + \frac{1}{4} \|x^k - \tilde{x}^k\|^2.$$

Adding (3.11) and (3.12) we then obtain

$$(x_*^k - \tilde{x}^k)^T \left\{ \frac{1}{2} (x^k - \tilde{x}^k) + \frac{1}{1+\mu} (\xi^k - \beta_k F(\tilde{x}^k)) \right\} \leq \frac{\mu}{2(1+\mu)} \|x^k - \tilde{x}^k\|^2,$$

which implies

$$(3.13) \quad 2\alpha(x_*^k - \tilde{x}^k)^T \left\{ \frac{1}{2} (x^k - \tilde{x}^k) + \frac{1}{1+\mu} (\xi^k - \beta_k F(\tilde{x}^k)) \right\} - \frac{\alpha\mu}{(1+\mu)} \|x^k - \tilde{x}^k\|^2 \leq 0.$$

Since $x^* \in \Omega^* \subset R_+^n$ and $x_*^k = P_{R_+^n} \left[x^k - \frac{\alpha\beta_k}{1+\mu} F(\tilde{x}^k) \right]$, it follows that

$$(3.14) \quad \|x_*^k - x^*\|^2 \leq \|x^k - \frac{\alpha\beta_k}{1+\mu} F(\tilde{x}^k) - x^*\|^2 - \|x^k - \frac{\alpha\beta_k}{1+\mu} F(\tilde{x}^k) - x_*^k\|^2.$$

From (3.4), we get

$$\begin{aligned} \|x^{k+1}(\alpha) - x^*\|^2 &= \|a_1(x^k - x^*) + (1 - a_1)(x_*^k - x^*)\|^2 \\ &= a_1^2 \|x^k - x^*\|^2 + (1 - a_1)^2 \|x_*^k - x^*\|^2 \\ &\quad + 2a_1(1 - a_1)(x^k - x^*)^T(x_*^k - x^*). \end{aligned}$$

Using the following identity

$$2(a + b)^T b = \|a + b\|^2 - \|a\|^2 + \|b\|^2$$

and (3.14), we obtain

$$\begin{aligned} \|x^{k+1}(\alpha) - x^*\|^2 &= a_1^2 \|x^k - x^*\|^2 + (1 - a_1)^2 \|x_*^k - x^*\|^2 \\ &\quad + a_1(1 - a_1)\{\|x^k - x^*\|^2 - \|x^k - x_*^k\|^2 + \|x_*^k - x^*\|^2\} \\ &= a_1 \|x^k - x^*\|^2 + (1 - a_1) \|x_*^k - x^*\|^2 \\ &\quad - a_1(1 - a_1) \|x^k - x_*^k\|^2 \\ &\leq a_1 \|x^k - x^*\|^2 + (1 - a_1) \|x^k - \frac{\alpha\beta_k}{1+\mu} F(\tilde{x}^k) - x^*\|^2 \\ &\quad - (1 - a_1) \|x^k - \frac{\alpha\beta_k}{1+\mu} F(\tilde{x}^k) - x_*^k\|^2 \\ &\quad - a_1(1 - a_1) \|x^k - x_*^k\|^2. \end{aligned}$$

Using the definition of $\Theta(\alpha)$, we get

$$\begin{aligned} \Theta(\alpha) &\geq (1 - a_1^2) \|x^k - x_*^k\|^2 + \frac{2(1 - a_1)\alpha\beta_k}{1 + \mu} (x_*^k - x^k)^T F(\tilde{x}^k) \\ (3.15) \quad &\quad + \frac{2(1 - a_1)\alpha\beta_k}{1 + \mu} (x^k - x^*)^T F(\tilde{x}^k). \end{aligned}$$

Since $\tilde{x}^k \in R_+^n$ and x^* is solution of NCP, using the pseudomonotonicity of F we have

$$(\tilde{x}^k - x^*)^T F(x^*) \geq 0 \Rightarrow (\tilde{x}^k - x^*)^T F(\tilde{x}^k) \geq 0$$

and consequently

$$(3.16) \quad (x^k - x^*)^T F(\tilde{x}^k) \geq (x^k - \tilde{x}^k)^T F(\tilde{x}^k).$$

Applying (3.16) to the last term in the right side of (3.15) and using $0 < a_1 < 1$, we obtain

$$\begin{aligned} \Theta(\alpha) &\geq (1 - a_1^2) \|x^k - x_*^k\|^2 + \frac{2(1 - a_1)\alpha\beta_k}{1 + \mu} (x_*^k - \tilde{x}^k)^T F(\tilde{x}^k) \\ (3.17) \quad &\geq (1 - a_1) \{\|x^k - x_*^k\|^2 + \frac{2\alpha\beta_k}{1 + \mu} (x_*^k - \tilde{x}^k)^T F(\tilde{x}^k)\}. \end{aligned}$$

Adding (3.13) and (3.17) and using the notation of d^k in (3.10), the lemma is proved. □

Theorem 3.5. Let $\Theta(\alpha)$ be defined in (3.8) and d^k be defined in (3.10), then for any $x^* \in \Omega^*$ and $\alpha > 0$, we have

$$(3.18) \quad \Theta(\alpha) \geq (1 - a_1)\Phi(\alpha),$$

where

$$(3.19) \quad \Phi(\alpha) = 2\alpha\psi_k - \alpha^2\|d^k\|^2$$

and

$$(3.20) \quad \psi_k = \frac{1}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k.$$

Proof. It follows from (3.9) that

$$\begin{aligned} \Theta(\alpha) &\geq (1-a_1)\{\|x^k - x_*^k\|^2 + 2\alpha(x_*^k - x^k)^T d^k + 2\alpha(x^k - \tilde{x}^k)^T d^k \\ &\quad - \frac{\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2\} \\ &= (1-a_1)\{\|x_*^k - x^k + \alpha d^k\|^2 - \alpha^2\|d^k\|^2 + 2\alpha(x^k - \tilde{x}^k)^T d^k \\ &\quad - \frac{\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2\} \\ &\geq (1-a_1)\{2\alpha(x^k - \tilde{x}^k)^T (\frac{1}{2}(x^k - \tilde{x}^k) + \frac{1}{1+\mu}\xi^k) \\ &\quad - \frac{\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 - \alpha^2\|d^k\|^2\} \\ &= (1-a_1)\left(2\alpha\left\{\frac{1}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k\right\} - \alpha^2\|d^k\|^2\right) \\ &= (1-a_1)\{2\alpha\psi_k - \alpha^2\|d^k\|^2\}, \end{aligned}$$

and (3.18) is proved. $\square$

$\Phi(\alpha)$ measures the progress obtained in the k th iteration. It is natural to choose a step length α_k which maximizes the progress. Note that $\Phi(\alpha_k)$ is a quadratic function of α and it reaches its maximum at

$$(3.21) \quad \alpha_k^* = \frac{\psi_k}{\|d^k\|^2}$$

and

$$(3.22) \quad \Phi(\alpha_k^*) = \alpha_k^* \psi_k.$$

In the next theorem we show that α_k^* and $\Phi(\alpha_k^*)$ are bounded away from zero, and it is one of the keys to prove the global convergence results.

Theorem 3.6. For given $x^k \in R_+^n$ and $\beta_k > 0$, let $\tilde{x}^k$ and ξ^k satisfied to the condition (3.3), then we have the following,

$$(3.23) \quad \alpha_k^* \geq \frac{1-2\eta}{4(1+\mu)}$$

and

$$(3.24) \quad \Phi(\alpha_k^*) \geq \frac{(1-2\eta)^2}{8(1+\mu)^2}\|x^k - \tilde{x}^k\|^2.$$

Proof. It follows from (3.20) and (3.5) that

$$(3.25) \quad \psi_k = \frac{1}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k \geq \left(\frac{1-2\eta}{2(1+\mu)}\right)\|x^k - \tilde{x}^k\|^2.$$

If $(x^k - \tilde{x}^k)^T \xi^k \leq 0$, since $\mu > 0$ it follows from (3.3) and (3.10) that

$$\|d^k\|^2 \leq \frac{1}{4}\|x^k - \tilde{x}^k\|^2 + \frac{1}{(1+\mu)^2}\|\xi^k\|^2$$

$$\begin{aligned}
(3.26) \quad & \leq \|x^k - \tilde{x}^k\|^2 + \|\xi^k\|^2 \\
& \leq 2\|x^k - \tilde{x}^k\|^2,
\end{aligned}$$

from (3.25) and (3.26), we obtain

$$\alpha_k^* = \frac{\varphi_k}{\|d^k\|^2} \geq \frac{1-2\eta}{4(1+\mu)}.$$

Otherwise, if $(x^k - \tilde{x}^k)^T \xi^k \geq 0$, it follows from $0 < \mu < 1$ and (3.3) that

$$\begin{aligned}
\psi_k &= \frac{1}{2(1+\mu)} \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu} (x^k - \tilde{x}^k)^T \xi^k \\
&\geq \frac{1}{1+\mu} \left\{ \frac{1}{4} \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu} (x^k - \tilde{x}^k)^T \xi^k \right. \\
&\quad \left. + \frac{1}{4} \|x^k - \tilde{x}^k\|^2 \right\} \\
&\geq \frac{1}{1+\mu} \left\{ \frac{1}{16} \|x^k - \tilde{x}^k\|^2 + \frac{1}{4(1+\mu)} (x^k - \tilde{x}^k)^T \xi^k \right. \\
&\quad \left. + \frac{1}{4(1+\mu)^2} \|\xi^k\|^2 \right\} \\
&= \frac{1}{4(1+\mu)} \|d^k\|^2
\end{aligned}$$

and thus

$$\alpha_k^* \geq \frac{1}{4(1+\mu)} \geq \frac{1-2\eta}{4(1+\mu)}.$$

Using (3.22), (3.23) and (3.25) directly we obtained (3.24). $\square$

For fast convergence, we take a relaxation factor $\gamma \in [1, 2)$ and the step-size α_k in (3.4) by $\alpha_k = \gamma \alpha_k^*$. Through simple manipulation we obtain

$$\begin{aligned}
(3.27) \quad \Phi(\gamma \alpha_k^*) &= 2\gamma \alpha_k^* \psi_k - (\gamma^2 \alpha_k^*) (\alpha_k^* \|d^k\|^2) \\
&= (2\gamma \alpha_k^* - \gamma^2 \alpha_k^*) \psi_k \\
&= \gamma(2 - \gamma) \Phi(\alpha_k^*)
\end{aligned}$$

It follows from Theorem 3.5 and Theorem 3.6 that there is a constant $c > 0$ such that

$$(3.28) \quad \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - c\|x^k - \tilde{x}^k\|^2 \quad \forall x^* \in \Omega^*.$$

The following result is very useful to prove the convergence of our method.

Lemma 3.7. [6] For given $x^k > 0$ and $\beta_k > 0$, let $\tilde{x}^k$ to be obtained by the prediction step (3.2), then for each $x \geq 0$ we have

$$(3.29) \quad (x - \tilde{x}^k)^T (\beta_k F(\tilde{x}^k) - \xi^k) \geq \frac{1}{2} (x^k - \tilde{x}^k)^T \{(1+\mu)x - (\mu x^k + \tilde{x}^k)\}.$$

Proof. By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (3.6) and $y = x$ in (3.7), it follows from (3.2) that

$$\begin{aligned}
(x - \tilde{x}^k)^T (\beta_k F(\tilde{x}^k) - \xi^k) &\geq \frac{1}{2} \left(\frac{1+\mu}{2} (\|\tilde{x}^k - x\|^2 - \|x^k - x\|^2) + \frac{1-\mu}{2} \|x^k - \tilde{x}^k\|^2 \right) \\
&= \frac{1}{2} ((1+\mu)x^T x^k - (1+\mu)x^T \tilde{x}^k - (1-\mu)(\tilde{x}^k)^T x^k) \\
&\quad - \frac{1}{2} (\mu \|x^k\|^2 + \|\tilde{x}^k\|^2) \\
&= \frac{1}{2} ((1+\mu)x^T (x^k - \tilde{x}^k) - (x^k - \tilde{x}^k)^T (\mu x^k + \tilde{x}^k)) \\
&= \frac{1}{2} (x^k - \tilde{x}^k)^T \{(1+\mu)x - (\mu x^k + \tilde{x}^k)\}.
\end{aligned}$$

Then the proof is completed. $\square$

Remark 3.8. [6] Setting $x = x^k$ in (3.29) and from (3.5) we obtain

$$\begin{aligned} (x^k - \tilde{x}^k)^T \beta_k F(\tilde{x}^k) &\geq \|\tilde{x}^k - x^k\|^2 + (x^k - \tilde{x}^k)^T \xi^k \\ (3.30) \qquad \qquad \qquad &\geq \left(\frac{1}{2} - \eta\right) \|x^k - \tilde{x}^k\|^2. \end{aligned}$$

Consequently, from the above inequality and (3.16), we have

$$(x^k - x^*)^T \beta_k F(\tilde{x}^k) \geq \left(\frac{1}{2} - \eta\right) \|x^k - \tilde{x}^k\|^2.$$

Hence, $-\beta_k F(\tilde{x}^k)$ is a descent direction of the distance function $\frac{1}{2} \|x - x^*\|^2$ at x^k .

Now, the convergence of the proposed method could be proved as follows:

Theorem 3.9. [6] If $\inf_{k=0}^{\infty} \beta_k = \beta > 0$, then the sequence $\{x^k\}$ generated by the proposed method converges to some x^∞ which is a solution of the NCP.

4. PRELIMINARY COMPUTATIONAL RESULTS

In the numerical experiments, it is necessary to compute an approximate solution $\tilde{x}^k$, in the special case $\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k))$, then (3.2) is equivalent to the following system of nonlinear equations

$$(4.1) \qquad \qquad \beta_k F(x^k) + \frac{1}{2}(\tilde{x}^k - x^k) + \mu(x^k - X_k(\sqrt{\tilde{x}^k})^{-1}) = 0,$$

hence

$$\frac{1}{2}\tilde{x}_j^k - \mu \frac{(\sqrt{x_j^k})^3}{\sqrt{\tilde{x}_j^k}} + \left(\beta_k F_j(x^k) - \frac{1}{2}x_j^k + \mu x_j^k \right) = 0, \quad j = 1, \dots, n.$$

Then

$$\frac{1}{2}\tilde{x}_j^k - \mu \frac{(\sqrt{x_j^k})^3}{\sqrt{\tilde{x}_j^k}} + \left(\beta_k F_j(x^k) - \frac{1}{2}x_j^k + \mu \frac{(\sqrt{x_j^k})^3}{\sqrt{x_j^k}} \right) = 0, \quad j = 1, \dots, n.$$

The recursion of classical Newton method for the above problem is

$$\tilde{x}_j^k := x_j^k - \frac{\beta_k}{1 + \mu} F_j(x^k)$$

The solution of (4.2) is $\tilde{x}^k > 0$, to avoid the non-positive value $\tilde{x}_j^k$ in the iteration process, we take

$$\tilde{x}_j^k := \max\left\{x_j^k - \frac{\beta_k}{1 + \mu} F_j(x^k), 0\right\}, \quad j = 1, \dots, n.$$

The detailed algorithm is as follows.

SRQP method

Step 0. Let $\beta_0 = 1, \eta := 0.45, \mu = 0.01 (\eta < \frac{1}{2} \text{ and } \eta^2 + \mu^2 < 1), 0 < a_1 < 1, \gamma = 1.95, \epsilon = 10^{-7}, k = 0$ and $x^0 > 0$.

Step 1. If $\| \min(x^k, F(x^k)) \|_\infty \leq \epsilon$, then stop. Otherwise, go to Step 2.

Step 2. (Prediction step)

$$\tilde{x}^k = P_{R_+^n} \left[x^k - \frac{\beta_k}{1 + \mu} F(x^k) \right], \quad \xi^k = \beta_k (F(\tilde{x}^k) - F(x^k)),$$

$$r = \frac{\|\xi^k\|}{\|x^k - \tilde{x}^k\|}.$$

While ($r > \eta$)

$$\beta_k = \frac{0.4}{r} * \beta_k, \quad \tilde{x}^k = P_{R_+^n} \left[x^k - \frac{\beta_k}{1 + \mu} F(x^k) \right],$$

$$\xi^k = \beta_k (F(\tilde{x}^k) - F(x^k)), \quad r = \frac{\|\xi^k\|}{\|x^k - \tilde{x}^k\|}.$$

end While

Step 3. (correction step)

$$\begin{aligned}\psi_k &= \frac{1}{2(1+\mu)} \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu} (x^k - \tilde{x}^k) \xi^k, \\ d^k &= \frac{1}{2} (x^k - \tilde{x}^k) + \frac{1}{1+\mu} \xi^k, \quad \alpha_k^* = \frac{\psi_k}{\|d^k\|^2}, \quad \alpha_k = \gamma \alpha_k^*, \\ x^{k+1}(\alpha) &= a_1 x^k + (1 - a_1) P_{R_+^n} \left[x^k - \frac{\alpha \beta_k}{1+\mu} F(\tilde{x}^k) \right],\end{aligned}$$

$$\text{Step 4. } \beta_{k+1} = \begin{cases} \frac{\beta_k * 0.33}{r} & \text{if } r \leq 0.15; \\ \beta_k & \text{otherwise.} \end{cases}$$

Step 5. $k:=k+1$; go to Step 1.

To test the proposed method, we consider the nonlinear complementarity problems:

$$(4.2) \quad x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0,$$

where

$$F(x) = D(x) + Mx + q,$$

$D(x)$ and $Mx + q$ are the nonlinear part and linear part of $F(x)$ respectively.

We form the linear part in the test problems similarly as in Harker and Pang [13]. In $D(x)$, the nonlinear is similar to the type of problems was tested in [17].

In all tests we take $a_1 = 0.01$. All iterations start with $x^1 = (1, \dots, 1)^T$, and stopped whenever

$$\|\min(x^k, F(x^k))\|_\infty \leq 10^{-7}.$$

All codes were written in Matlab, we compare the proposed method with those in [2], the test results for problem (4.2) are reported in Table 1. k is the number of iterations and l denotes the number of evaluations of mapping F .

TABLE 1. Numerical results for problem (4.2) with $q \in (-500, 500)$

n	The method in [2]			The proposed method		
	k	l	CPU(Sec.)	k	l	CPU(Sec.)
200	127	279	0.022	129	298	0.017
300	149	323	0.035	143	330	0.026
400	156	338	0.05	149	347	0.046
500	172	374	0.12	165	387	0.12
700	162	354	0.19	190	431	0.50
1000	158	341	0.71	169	385	0.86

5. CONCLUDING REMARKS

In this paper, we propose a new class of predictor-corrector algorithms for solving nonlinear complementarity problems using a novel square root quadratic proximal term. The numerical results demonstrate that our algorithm is efficient for the tested problems. Additionally, the computational results highlight its significant performance. This method requires only two projections to R_+^n and a few functional evaluations at each iteration.

Competing interests. The author declares no competing interests.

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