

## GLOBAL DYNAMICS OF A SEMILINEAR PARABOLIC EQUATION DRIVEN BY A CHAOTIC SYSTEM

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**ABSTRACT.** We study a closed interconnection between a semilinear parabolic equation on a bounded domain and a four-dimensional nonlinear oscillator. The finite-dimensional state modulates the parabolic evolution through four time-dependent coefficients, while the parabolic state acts back on the oscillator through a single bounded scalar observable, so that neither component evolves autonomously. Closing the loop changes the analysis in three ways: boundedness of the finite-dimensional component is no longer postulated but derived from a Lyapunov inequality that is required to survive bounded perturbations; the interconnection generates a genuine autonomous semiflow on the product space  $H_0^1(\Omega) \times \mathbb{R}^4$ , which removes any recourse to processes or skew-product constructions; and the one-way cascade reappears as the singular limit of vanishing feedback. We prove local and global well-posedness, exhibit a bounded absorbing set, and construct a compact global attractor in the product space, the asymptotic compactness being obtained from a uniform bound in a fractional power space. The same bound yields a uniform precompactness property for the family of attractors, from which we deduce that the attractors converge upper semicontinuously to the attractor of the uncoupled problem as the feedback intensity vanishes.

### 1. INTRODUCTION

Parabolic equations whose coefficients are produced by a chaotic ordinary differential equation are usually treated as *cascades*: one solves the finite-dimensional dynamics first and then inserts it into the partial differential equation as an exogenous signal. What one obtains is a non-autonomous evolution equation, and the standard way of restoring an autonomous structure is the skew-product formalism, in which the driving trajectory is adjoined to the phase space [2,3,8,14]. The construction is well understood, but it rests on one structural feature: the base dynamics is unaffected by the fibre.

The present paper removes that feature. We consider the interconnection

$$(1) \quad \text{oscillator} \rightleftharpoons \text{parabolic equation},$$

in which the parabolic state returns to the oscillator a single scalar quantity, namely the spatial average of a bounded nonlinear function of the solution. The feedback is deliberately minimal: one scalar, globally bounded and globally Lipschitz, entering one component of the vector field with an intensity  $\varepsilon \geq 0$ . Interconnections of ordinary differential equations with parabolic equations form a distinct mathematical class,

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and stability has to be assessed on the loop rather than on a drive–response pair [7, 9]. Coupled parabolic–ordinary systems are of course classical in another guise: partly dissipative reaction–diffusion systems, in which an ordinary differential equation is attached pointwise to a diffusive one, already possess a well-developed attractor theory [11]. Our interconnection differs from that setting in that the return channel is a single global functional of the parabolic state rather than a pointwise coupling.

Three consequences of closing the loop.

(i) *Boundedness becomes a conclusion.* In a cascade one may simply assume that the driving trajectory is bounded. Once the oscillator is perturbed by the parabolic state this is no longer legitimate, since the perturbed trajectory need not remain in the region where the unperturbed one lived. We therefore impose a Lyapunov inequality that is required to hold *uniformly over all admissible feedback values*, and we deduce from it an explicit bound on the finite-dimensional component. Lyapunov functionals are the natural device for estimates of this kind in concrete evolution models [15].

(ii) *The dynamics is autonomous on a product space.* Because the observable depends on the current parabolic state only, the pair  $z = (u, y)$  solves an autonomous evolution equation on  $\mathcal{X} = H_0^1(\Omega) \times \mathbb{R}^4$ . No process, no base flow and no skew-product extension are needed: the family  $\Phi(t)z_0 = (u(t), y(t))$  is a semiflow in the classical sense, so that the theory of dissipative semigroups applies verbatim [5, 12, 16].

(iii) *The cascade becomes a singular limit.* Letting the feedback intensity tend to zero returns the one-way problem, and one may ask whether the long-time dynamics does the same. We show that it does, but only in the upper semicontinuous sense: parts of the limiting attractor may fail to be reached by the perturbed ones. This question simply does not arise for a cascade, where  $\varepsilon$  is absent from the model.

The paper is organised so that the analytic tools come first and the proofs may then be read without interruption. Section 2 fixes the functional setting: the spaces in which the coupled state is measured, the diffusion semigroup and its smoothing properties, the fractional power spaces on which the compactness argument will rest, and the two forms of Gronwall’s inequality that the later estimates require. Section 3 introduces the interconnection itself, together with the feedback observable, the elementary properties of the parabolic nonlinearities and the two standing assumptions. The four main results are stated at the beginning of Section 4, which then proves the first two of them by establishing well-posedness and the dissipative estimates that produce the absorbing set, while Section 5 constructs the global attractor and examines the behaviour of the attractors as the feedback intensity vanishes. Finally, Section 6 reviews what closing the loop has cost and what it has gained, and collects the questions that remain open.

## 2. MATHEMATICAL FRAMEWORK

Before turning to the model itself we fix the functional setting, two features of which are dictated by the bidirectional structure of the problem: the phase space is a product of an infinite- and a finite-dimensional space, and the compactness of the dynamics will be measured in a fractional power space rather than in the domain of the diffusion operator.

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \leq 3$ , be a bounded domain with smooth boundary, and write  $\xi$  for the spatial variable. Set

$$H = L^2(\Omega), \quad V = H_0^1(\Omega),$$

with  $\|u\| = \|u\|_{L^2}$ ,  $\langle \cdot, \cdot \rangle$  the inner product of  $H$ , and  $\|u\|_V = \|\nabla u\|_{L^2}$ , the latter being an equivalent norm on  $V$  by the Poincaré inequality

$$(2) \quad \|u\| \leq c_\Omega \|u\|_V, \quad u \in V.$$

The phase space of the coupled problem is the Hilbert space

$$(3) \quad \mathcal{X} = V \times \mathbb{R}^4, \quad \|(u, y)\|_{\mathcal{X}}^2 = \|u\|_V^2 + |y|^2,$$

$|\cdot|$  being the Euclidean norm of  $\mathbb{R}^4$ . Choosing  $V$  rather than  $H$  for the parabolic factor is not a matter of convenience: the drift contains  $N_2(\nabla u)$ , so the vector field is only defined once the gradient is a function, and the  $\mathcal{X}$ -norm is exactly what Lemma 4.5 needs in order to be locally Lipschitz.

The diffusion is governed by

$$A = D\Delta, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad D > 0.$$

On a bounded smooth domain,  $-A$  is positive and self-adjoint on  $H$  with compact resolvent, and its spectrum consists of eigenvalues  $0 < D\lambda_1 \leq D\lambda_2 \leq \dots \rightarrow \infty$ , where  $\lambda_1 \geq c_\Omega^{-2}$  is the first eigenvalue of  $-\Delta$  under Dirichlet boundary conditions. Consequently  $A$  generates an analytic semigroup  $(e^{tA})_{t \geq 0}$  on  $H$ , and there is  $c_* > 0$  with

$$(4) \quad \|e^{tA}f\| \leq e^{-\delta t} \|f\|, \quad \|e^{tA}f\|_V \leq c_* t^{-1/2} e^{-\delta t} \|f\|, \quad f \in H, \quad t > 0,$$

where  $\delta = Dc_\Omega^{-2}$ ; see [6, Chapter 1] or [10, Chapter 2].

Since  $-A$  is positive and self-adjoint, its fractional powers  $(-A)^\alpha$ ,  $\alpha \in (0, 1)$ , are well defined by the spectral theorem, and we write

$$X^\alpha = D((-A)^\alpha), \quad \|u\|_{X^\alpha} = \|(-A)^\alpha u\|,$$

so that  $X^{1/2} = V$  and  $X^1 = D(A)$ . Analyticity yields the smoothing estimate

$$(5) \quad \|(-A)^\alpha e^{tA}\|_{\mathcal{L}(H)} \leq C_\alpha t^{-\alpha}, \quad t > 0, \quad 0 < \alpha < 1,$$

and the scale interpolates the usual Sobolev spaces,  $X^\alpha \hookrightarrow H^{2\alpha}(\Omega)$ . Since  $\Omega$  is bounded and smooth, the embedding  $H^{2\alpha}(\Omega) \hookrightarrow H^1(\Omega)$  is compact as soon as  $2\alpha > 1$ , whence

$$(6) \quad X^\alpha \Subset V, \quad \frac{1}{2} < \alpha < 1;$$

see [6, Chapter 1].

*Remark 2.1* (Why a fractional power and not  $D(A)$ ). The exponent range in (6) is what makes the compactness argument of Section 5 work, and the upper restriction  $\alpha < 1$  is essential. Compactness will be extracted from the variation-of-constants formula over a time interval of unit length, whose convolution term is estimated with (5) and therefore produces the integral

$$\int_0^1 r^{-\alpha} dr = \frac{1}{1-\alpha},$$

which is finite precisely because  $\alpha < 1$ . The same computation carried out in  $D(A)$  would require  $\alpha = 1$  and the divergent integral  $\int_0^1 r^{-1} dr$ : bounded forcing in  $H$  does not place the solution in the domain of the generator. Any  $\alpha \in (\frac{1}{2}, 1)$  resolves the difficulty, and by (6) it is already strong enough to give compactness in the phase space  $V$ .

Both the dissipative estimates and the vanishing-feedback limit rest on Gronwall-type arguments, but not on the same one. The first is classical.

**Lemma 2.2** (Differential form). *Let  $g \in C([s, \infty); \mathbb{R})$ ,  $k \geq 0$ , and let  $\psi \geq 0$  be absolutely continuous with  $\psi'(t) \leq g(t)\psi(t) + k$  for  $t \geq s$ . Then*

$$\psi(t) \leq \psi(s) \mathcal{E}(s, t) + k \int_s^t \mathcal{E}(\tau, t) d\tau, \quad \mathcal{E}(\sigma, t) = \exp\left(\int_\sigma^t g(\tau) d\tau\right), \quad t \geq s.$$

The second is the singular version, needed because the parabolic smoothing (4) generates a weakly singular kernel as soon as differences of solutions are measured in  $V$ .

**Lemma 2.3** (Singular form [6]). *Let  $T > 0$ ,  $0 \leq \gamma < 1$  and  $a, b \geq 0$ . If  $E : [0, T] \rightarrow [0, \infty)$  is bounded and measurable and satisfies*

$$E(t) \leq a + b \int_0^t (t-s)^{-\gamma} E(s) ds, \quad 0 \leq t \leq T,$$

*then  $\sup_{0 \leq t \leq T} E(t) \leq a \Theta$ , where  $\Theta = \Theta(b, \gamma, T)$  depends only on  $b, \gamma$  and  $T$ . In particular the bound is proportional to  $a$ .*

Throughout,  $z = (u, y)$  denotes the full state and  $z_0 = (u_0, y_0)$  the initial datum. For  $\varepsilon \geq 0$  we write  $\Phi_\varepsilon$  for the semiflow generated by the interconnection with feedback intensity  $\varepsilon$ , and  $\mathfrak{A}_\varepsilon$  for its global attractor; the subscript is omitted whenever  $\varepsilon$  is fixed. For  $B_1, B_2 \subset \mathcal{X}$ ,

$$\text{dist}_{\mathcal{X}}(B_1, B_2) = \sup_{z \in B_1} \inf_{w \in B_2} \|z - w\|_{\mathcal{X}}$$

is the Hausdorff semidistance. Constants denoted  $C, C_\alpha, C_T$  may change from line to line, but never depend on  $\varepsilon$  or on the initial datum unless this is indicated.

### 3. THE COUPLED MODEL

We now describe the system that will be analysed throughout the paper, a bidirectionally coupled PDE–ODE model in which a finite-dimensional nonlinear dynamics modulates the parabolic evolution, while a bounded spatial observable of the parabolic state acts as a feedback signal on the oscillator,

$$(7) \quad \begin{cases} \partial_t u = D\Delta u + \lambda \left( y_1(t)u + y_2(t)N_1(u) - y_3(t)N_2(\nabla u) + y_4(t) \sin u \right), \\ \dot{y} = F(y) + \varepsilon e_4 \mathcal{P}(u), \\ u|_{\partial\Omega} = 0, \quad u(\xi, 0) = u_0(\xi) \in V, \quad y(0) = y_0 \in \mathbb{R}^4, \end{cases}$$

where  $\lambda > 0$  is the modulation strength,  $\varepsilon \geq 0$  the feedback intensity,  $e_4 = (0, 0, 0, 1)^\top$ ,

$$(8) \quad N_1(s) = \frac{s^2}{1+s^2} \quad (s \in \mathbb{R}), \quad N_2(p) = \frac{|p|^2}{1+|p|^2} \quad (p \in \mathbb{R}^d),$$

and the feedback observable is

$$(9) \quad \mathcal{P}(u) = \frac{1}{|\Omega|} \int_{\Omega} \tanh(u(\xi)) d\xi.$$

The coupling is genuinely two-way, from  $y$  to  $u$  through the four coefficients  $y_i(t)$ , and from  $u$  to  $y$  through  $\mathcal{P}(u)$ .

*Remark 3.1* (Why the fourth component). The parabolic equation already contains the term  $y_4(t) \sin u$ . Feeding the observable into the fourth equation therefore closes the shortest possible loop,

$$u \longrightarrow y_4 \longrightarrow u,$$

and leaves the remaining three equations of the oscillator untouched. Nothing in the analysis uses the particular direction  $e_4$ , since every statement below holds verbatim for  $\dot{y} = F(y) + \varepsilon e e \mathcal{P}(u)$  with an arbitrary fixed  $e \in \mathbb{R}^4, |e| = 1$ .

The feedback enters the analysis only through two elementary inequalities. Since  $\tanh$  is bounded by 1 and globally 1-Lipschitz, the Cauchy–Schwarz inequality gives at once

$$(10) \quad |\mathcal{P}(u)| \leq 1, \quad |\mathcal{P}(u) - \mathcal{P}(w)| \leq |\Omega|^{-1/2} \|u - w\|, \quad u, w \in H.$$

The first inequality makes the perturbation of the oscillator uniformly small, and the second makes the coupled vector field locally Lipschitz on  $\mathcal{X}$ .

The two parabolic nonlinearities are globally Lipschitz, which is the only property of them that will be used.

**Lemma 3.2.** Let  $L_* = \frac{3\sqrt{3}}{8} = \frac{9}{8\sqrt{3}} < 1$ . Then

(a)  $0 \leq N_1 \leq 1$ ,  $0 \leq N_2 \leq 1$  and  $|\sin s| \leq 1$ , so that

$$\|N_1(u)\| + \|N_2(\nabla u)\| + \|\sin u\| \leq 3|\Omega|^{1/2}, \quad u \in V.$$

(b)  $N_1 : \mathbb{R} \rightarrow \mathbb{R}$  is globally Lipschitz with constant  $L_*$ , hence

$$\|N_1(u) - N_1(w)\| \leq L_* \|u - w\| \leq L_* c_\Omega \|u - w\|_V.$$

(c)  $N_2 : \mathbb{R}^d \rightarrow \mathbb{R}$  is globally Lipschitz with constant  $L_*$ , hence

$$\|N_2(\nabla u) - N_2(\nabla w)\| \leq L_* \|u - w\|_V, \quad u, w \in V.$$

(d)  $\|\sin u - \sin w\| \leq \|u - w\| \leq c_\Omega \|u - w\|_V$ .

*Proof.* Assertion (a) follows from  $0 \leq \tau/(1+\tau) < 1$  for  $\tau \geq 0$ , and (d) from the mean value theorem.

For (c), the function  $N_2$  is smooth on  $\mathbb{R}^d$  with

$$\nabla_p N_2(p) = \frac{2p}{(1+|p|^2)^2}.$$

Set  $r = |p|$  and  $f(r) = 2r(1+r^2)^{-2}$ . Then  $f'(r) = (2-6r^2)(1+r^2)^{-3}$ , so that  $f$  attains its maximum at  $r = 3^{-1/2}$  and

$$\sup_{p \in \mathbb{R}^d} |\nabla_p N_2(p)| = f(3^{-1/2}) = \frac{2 \cdot 3^{-1/2}}{(4/3)^2} = \frac{9}{8\sqrt{3}} = L_* < 1.$$

Since  $\mathbb{R}^d$  is convex, the mean value theorem gives  $|N_2(p) - N_2(q)| \leq L_* |p - q|$  for all  $p, q \in \mathbb{R}^d$ . Applying this pointwise with  $p = \nabla u(\xi)$  and  $q = \nabla w(\xi)$  and integrating over  $\Omega$  yields

$$\|N_2(\nabla u) - N_2(\nabla w)\| \leq L_* \|\nabla u - \nabla w\| = L_* \|u - w\|_V.$$

Assertion (b) is the same computation in dimension one, since  $N'_1(s) = 2s(1+s^2)^{-2}$ .  $\square$

*Remark 3.3.* Lemma 3.2 gives global Lipschitz bounds, and it does so without invoking any Sobolev embedding. This is worth emphasising, because no  $L^4$  bound on  $\nabla u$  is available for  $u \in V$ . The embedding  $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$  applies to  $u$ , not to  $\nabla u$ , and the boundedness of  $\nabla_p N_2$  is what makes any such embedding unnecessary.

It remains to state the two hypotheses under which all the results of the paper are established.

**(H0) Robust boundedness of the finite-dimensional component.** The vector field  $F : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is locally Lipschitz, and the finite-dimensional dynamics remains dissipative under bounded feedback, in the sense that there exist a coercive function  $W \in C^1(\mathbb{R}^4; \mathbb{R}_+)$ , that is,  $W(y) \rightarrow \infty$  as  $|y| \rightarrow \infty$ , and constants  $c_0, c_1 > 0$  such that

$$(11) \quad \nabla W(y) \cdot (F(y) + \varepsilon e_4 r) \leq c_0 - c_1 W(y), \quad y \in \mathbb{R}^4, |r| \leq 1.$$

**(H1) Uniform averaged dissipativity of the coupled trajectory.** There exist  $\mu > 0$  and  $C_a \geq 0$  such that, for every bounded set  $B \subset \mathcal{X}$ , there exists  $\tau_B \geq 0$  for which

$$(12) \quad \int_s^t y_1(\tau) d\tau \leq C_a - \mu(t-s), \quad t \geq s \geq \tau_B,$$

uniformly for all trajectories issued from  $B$ .

**Remark 3.4** (On the form of (H1)). Condition (12) is exactly the hypothesis used in the proof of Lemma 4.7, and it is strictly stronger than the requirement

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T y_1(\tau) d\tau \leq -\mu,$$

which controls only Cesàro averages started at the origin and gives no control of  $\int_s^t y_1$  that is uniform in  $s$  and in the initial datum. Such uniformity is indispensable if one wants a *single* absorbing set for the semiflow instead of a trajectory-dependent bound. Condition (12) holds, for instance, when the Cesàro limit is attained uniformly on the absorbing set of the oscillator, in particular when  $y$  evolves on a compact invariant set carrying an ergodic measure  $\nu$  with  $\int y_1 d\nu < -\mu$  and uniform convergence of the Birkhoff averages. Note also that  $y$  is here a component of the coupled state and not a prescribed external signal. For  $\varepsilon > 0$  the property is an assumption on the interconnection, it is not inherited automatically from the case  $\varepsilon = 0$ , and it persists as long as the feedback shifts the averages of  $y_1$  by less than  $\mu$ .

**Lemma 3.5** (Boundedness of the oscillator). *Assume (H0). Then every solution of the second equation in (7) satisfies*

$$(13) \quad W(y(t)) \leq W(y_0)e^{-c_1 t} + \frac{c_0}{c_1}(1 - e^{-c_1 t}), \quad t \geq 0.$$

*In particular  $y$  is global and, setting*

$$m_* = \sup \left\{ |y| : W(y) \leq \frac{2c_0}{c_1} \right\} < \infty,$$

*for every bounded set  $B \subset \mathbb{R}^4$  there is  $t_B \geq 0$  such that*

$$(14) \quad \sup_{1 \leq i \leq 4} |y_i(t)| \leq |y(t)| \leq m_*, \quad t \geq t_B, y_0 \in B.$$

*Proof.* By (10) we have  $|\mathcal{P}(u(t))| \leq 1$  for every  $t$ , so that (11) applies with  $r = \mathcal{P}(u(t))$  and gives  $\frac{d}{dt}W(y(t)) \leq c_0 - c_1 W(y(t))$  for as long as the solution exists. Lemma 2.2 yields (13). Coercivity of  $W$  then rules out blow-up, so that the solution is global and  $W(y(t)) \leq 2c_0/c_1$  for  $t \geq t_B = c_1^{-1} \log^+(c_1 c_0^{-1} \sup_{y_0 \in B} W(y_0))$ . The set  $\{W \leq 2c_0/c_1\}$  is bounded by coercivity, whence  $m_* < \infty$ .  $\square$

From now on  $t_*$  denotes a time after which both (14) and (12) hold.

**Remark 3.6** (The role of robustness in (H0)). In a one-way cascade one may simply postulate  $\sup_t |y_i(t)| \leq M$ . Here the bound (14) is a consequence of (11), and the reason for stating (H0) with the perturbation  $\varepsilon e_4 r$ ,  $|r| \leq 1$ , already included is precisely that the estimate has to be robust. It is required to hold for *every* admissible value of the feedback, not merely for the unperturbed field.

**Example 3.7** (Verification of (H0) under a robustness margin). Suppose the unperturbed field satisfies  $\nabla W(y) \cdot F(y) \leq \hat{c}_0 - \hat{c}_1 W(y)$  for some coercive  $W \in C^1(\mathbb{R}^4; \mathbb{R}_+)$ , and that the fourth partial derivative of  $W$  grows at most linearly in  $W$ , say  $|\partial_{y_4} W(y)| \leq \theta W(y) + \kappa_0$  with  $\theta, \kappa_0 \geq 0$ . Then, for  $|r| \leq 1$ ,

$$\nabla W(y) \cdot (F(y) + \varepsilon e_4 r) \leq (\hat{c}_0 + \varepsilon \kappa_0) - (\hat{c}_1 - \varepsilon \theta) W(y),$$

so that (11) holds with  $c_0 = \hat{c}_0 + \varepsilon \kappa_0$  and  $c_1 = \hat{c}_1 - \varepsilon \theta$  whenever

$$0 \leq \varepsilon < \frac{\hat{c}_1}{\theta} \quad (\theta > 0),$$

and for every  $\varepsilon \geq 0$  when  $\theta = 0$ . In the quadratic case  $W(y) = \frac{1}{2}|y|^2$  one has  $\partial_{y_4} W = y_4$ , so if  $y \cdot F(y) \leq a_0 - a_1 |y|^2$  with  $a_1 > 0$ , then (11) holds with  $c_0 = a_0 + \varepsilon/2$  and  $c_1 = 2a_1 - \varepsilon$ , provided  $0 \leq \varepsilon < 2a_1$ . A quantified smallness condition on the feedback intensity is therefore all that is needed once the unperturbed field is dissipative.

**Example 3.8** (Hyperchaotic oscillator with feedback). Let  $F = (F_1, F_2, F_3, F_4)$  be a four-dimensional oscillator exhibiting hyperchaotic behaviour in the sense of [13], whose fourth component carries the saturating nonlinearity

$$F_4(y) = y_3 + \frac{1}{2}e^{-y_4^2} + dy_4, \quad d < 0.$$

Fields of this type, including ones whose chaotic attractor is hidden, occur in secure-communication applications [1]. Inserting the observable into the fourth component only, the finite-dimensional part of (7) reads

$$\dot{y}_i = F_i(y) \quad (i = 1, 2, 3), \quad \dot{y}_4 = y_3 + \frac{1}{2}e^{-y_4^2} + dy_4 + \varepsilon \mathcal{P}(u),$$

the first three equations being left strictly unchanged. Their explicit form is immaterial for what follows, since only (H0) and (H1) are used.

*Remark 3.9* (Bounded chaos is not global dissipativity). The numerical observation of a bounded chaotic attractor does *not* establish (H0), and in particular does not justify the quadratic choice  $W(y) = \frac{1}{2}|y|^2$ . For a field of jerk type, in which  $\dot{y}_1 = y_2$  and  $\dot{y}_2 = y_3$ , the point  $y = (R, 0, 0, 0)$  gives  $y \cdot F(y) = 0$ , whereas the inequality  $y \cdot F(y) \leq a_0 - a_1|y|^2$  would force  $0 \leq a_0 - a_1R^2$ , which fails for large  $R$  when  $a_1 > 0$ . No estimate of that form can hold globally for such a field, so that (H0) must be verified with a genuinely anisotropic Lyapunov function  $W$ , or else the analysis must be restricted to a bounded positively invariant region. Global dissipativity in the sense of (11) and the existence of a bounded chaotic attractor for some initial data are distinct properties.

#### 4. MAIN RESULTS AND THE DISSIPATIVE ESTIMATES

We collect here the four statements that the rest of the paper establishes, in the order in which they will be proved.

**Theorem 4.1** (Local well-posedness). *For every  $(u_0, y_0) \in V \times \mathbb{R}^4$ , system (7) admits a unique maximal solution*

$$(u, y) \in C([0, T_{\max}); V \times \mathbb{R}^4),$$

*with*

$$u \in C^1((0, T_{\max}); H) \cap C((0, T_{\max}); D(A)), \quad y \in C^1([0, T_{\max}); \mathbb{R}^4),$$

*and the solution depends continuously on  $z_0$  in  $\mathcal{X}$ .*

**Theorem 4.2** (Global existence and dissipativity). *Assume (H0)–(H1). Then every solution of (7) is global, the family  $\Phi(t)z_0 = (u(t), y(t))$ ,  $t \geq 0$ , defines a continuous semiflow on  $\mathcal{X} = V \times \mathbb{R}^4$ , and  $\Phi$  possesses a bounded absorbing set in  $\mathcal{X}$ .*

**Theorem 4.3** (Global attractor). *Assume (H0)–(H1). Then the semiflow  $\Phi$  generated by (7) possesses a compact global attractor*

$$\mathfrak{A} \subset V \times \mathbb{R}^4,$$

*which is connected, invariant, and attracts every bounded subset of  $\mathcal{X}$ . Moreover  $\mathfrak{A}$  is bounded in  $X^\alpha \times \mathbb{R}^4$  for every  $\alpha \in (\frac{1}{2}, 1)$ .*

**Theorem 4.4** (Vanishing-feedback limit). *Assume (H0)–(H1) uniformly for  $\varepsilon \in [0, \varepsilon_0]$ , with constants  $c_0, c_1, \mu, C_a, t_*$  independent of  $\varepsilon$ . Then, for every  $\alpha \in (\frac{1}{2}, 1)$ ,*

$$\sup_{0 \leq \varepsilon \leq \varepsilon_0} \sup_{(u, y) \in \mathfrak{A}_\varepsilon} \left( \|u\|_{X^\alpha} + |y| \right) < \infty,$$

so that  $\overline{\bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \mathfrak{A}_\varepsilon}$  is compact in  $\mathcal{X}$ , and the attractors converge upper semicontinuously to the attractor  $\mathfrak{A}_0$  of the uncoupled problem,

$$\lim_{\varepsilon \rightarrow 0^+} \text{dist}_{\mathcal{X}}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) = 0.$$

Taken together, the four statements give the paper a single line of argument: a closed-loop PDE–ODE system, robust dissipativity, an attractor in the product space, and the cascade recovered as a singular limit. The first two are proved in the remainder of this section, the last two in Section 5.

It is convenient to recast (7) as a single abstract evolution equation on  $\mathcal{X}$ . Define

$$(15) \quad \mathcal{B}(u, y) = \lambda(y_1 u + y_2 N_1(u) - y_3 N_2(\nabla u) + y_4 \sin u), \quad G(u, y) = F(y) + \varepsilon e_4 \mathcal{P}(u).$$

With the operator  $A$  of Section 2, system (7) becomes the autonomous abstract problem

$$(16) \quad \dot{u} = Au + \mathcal{B}(u, y), \quad \dot{y} = G(u, y), \quad z(0) = z_0 \in \mathcal{X}.$$

**Lemma 4.5.** *For every  $\varrho > 0$  there is  $\ell(\varrho) > 0$  such that*

$$\|\mathcal{B}(u, y) - \mathcal{B}(w, \zeta)\| + |G(u, y) - G(w, \zeta)| \leq \ell(\varrho)(\|u - w\|_V + |y - \zeta|)$$

*whenever  $\|u\|_V, \|w\|_V \leq \varrho$  and  $|y|, |\zeta| \leq \varrho$ . One may take*

$$\ell(\varrho) = \lambda\varrho(2c_\Omega + L_*c_\Omega + L_*) + \lambda(c_\Omega\varrho + 3|\Omega|^{1/2}) + \text{Lip}_\varrho(F) + \varepsilon c_\Omega |\Omega|^{-1/2}.$$

*Proof.* Write

$$\begin{aligned} \mathcal{B}(u, y) - \mathcal{B}(w, \zeta) &= \lambda[y_1(u - w) + y_2(N_1(u) - N_1(w)) \\ &\quad - y_3(N_2(\nabla u) - N_2(\nabla w)) + y_4(\sin u - \sin w)] \\ &\quad + \lambda[(y_1 - \zeta_1)w + (y_2 - \zeta_2)N_1(w) \\ &\quad - (y_3 - \zeta_3)N_2(\nabla w) + (y_4 - \zeta_4)\sin w]. \end{aligned}$$

The first bracket is estimated by Lemma 3.2(b)–(d) together with (2), and the second by Lemma 3.2(a) and  $\|w\| \leq c_\Omega\varrho$ . For  $G$ , combine the local Lipschitz constant of  $F$  on  $\{|y| \leq \varrho\}$  with (10) and (2).  $\square$

Theorem 4.1 is now obtained by a fixed point argument.

*Proof of Theorem 4.1.* Fix  $z_0 \in \mathcal{X}$ , put  $\varrho = 2\|z_0\|_{\mathcal{X}} + 1$  and, for  $T > 0$ , set

$$\mathcal{Z}_T = \left\{ z = (u, y) \in C([0, T]; \mathcal{X}) : z(0) = z_0, \sup_{[0, T]} \|z(t)\|_{\mathcal{X}} \leq \varrho \right\},$$

which is a complete metric space for the sup-norm. Define  $\mathcal{T}z = (\tilde{u}, \tilde{y})$  by

$$\tilde{u}(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\mathcal{B}(u(s), y(s)) ds, \quad \tilde{y}(t) = y_0 + \int_0^t G(u(s), y(s)) ds.$$

By Lemma 3.2(a) we have  $\|\mathcal{B}(u, y)\| \leq \lambda\varrho(c_\Omega\varrho + 3|\Omega|^{1/2})$  on  $\mathcal{Z}_T$ , so the second estimate in (4) gives

$$\|\tilde{u}(t)\|_V \leq \|u_0\|_V + 2c_*T^{1/2}\lambda\varrho(c_\Omega\varrho + 3|\Omega|^{1/2}),$$

while  $|\tilde{y}(t) - y_0| \leq T \sup_{\mathcal{Z}_T} |G|$ . Hence  $\mathcal{T}(\mathcal{Z}_T) \subset \mathcal{Z}_T$  for  $T$  small. In the same way, Lemma 4.5 and (4) give

$$\|\mathcal{T}z - \mathcal{T}z'\|_{C([0, T]; \mathcal{X})} \leq (2c_*T^{1/2} + T)\ell(\varrho) \|z - z'\|_{C([0, T]; \mathcal{X})},$$

so that  $\mathcal{T}$  is a contraction for  $T$  small enough. The Banach fixed point theorem produces a unique mild solution on  $[0, T]$ , which extends to a maximal interval  $[0, T_{\max})$  by the usual continuation argument. The stated regularity follows from analytic smoothing [6, Chapter 3], and continuous dependence from Lemma 2.2 applied to the same estimates.  $\square$

We turn to the estimates that make the semiflow dissipative, starting from the energy identity satisfied by the parabolic component.

**Lemma 4.6** (Energy identity). *On  $(0, T_{\max})$ ,*

$$(17) \quad \frac{1}{2} \frac{d}{dt} \|u\|^2 = -D \|u\|_V^2 + \lambda y_1(t) \|u\|^2 + \lambda \int_{\Omega} (y_2 N_1(u) - y_3 N_2(\nabla u) + y_4 \sin u) u \, d\xi.$$

*Proof.* Multiply the first equation of (7) by  $u$ , integrate over  $\Omega$  and use  $u|_{\partial\Omega} = 0$ .  $\square$

**Lemma 4.7** (Absorbing set in  $H$ ). *Assume (H0)–(H1) and set*

$$\beta_0 = Dc_{\Omega}^{-2} + 2\lambda\mu, \quad k_1 = \frac{9\lambda^2 m_*^2 |\Omega| c_{\Omega}^2}{D}, \quad \varkappa = e^{2\lambda C_a}.$$

*Then, for  $t \geq s \geq t_*$ ,*

$$(18) \quad \|u(t)\|^2 \leq \varkappa \|u(s)\|^2 e^{-\beta_0(t-s)} + \frac{k_1 \varkappa}{\beta_0}.$$

*Proof.* By Lemma 3.2(a) and (14), the last term in (17) is bounded by  $3\lambda m_* \|u\|_{L^1} \leq 3\lambda m_* |\Omega|^{1/2} \|u\|$ . Using (2) in the form  $-2D \|u\|_V^2 \leq -2Dc_{\Omega}^{-2} \|u\|^2$  together with Young's inequality

$$6\lambda m_* |\Omega|^{1/2} \|u\| \leq Dc_{\Omega}^{-2} \|u\|^2 + \frac{9\lambda^2 m_*^2 |\Omega| c_{\Omega}^2}{D},$$

we obtain, with  $\psi = \|u\|^2$ ,

$$\psi'(t) \leq (-Dc_{\Omega}^{-2} + 2\lambda y_1(t)) \psi(t) + k_1.$$

Lemma 2.2 gives, for  $t \geq s \geq t_*$ ,

$$\psi(t) \leq \psi(s) \mathcal{E}(s, t) + k_1 \int_s^t \mathcal{E}(\tau, t) \, d\tau, \quad \mathcal{E}(\sigma, t) = \exp\left(-Dc_{\Omega}^{-2}(t - \sigma) + 2\lambda \int_{\sigma}^t y_1\right).$$

By (12) we have  $\mathcal{E}(\sigma, t) \leq \varkappa e^{-\beta_0(t-\sigma)}$  for  $t \geq \sigma \geq t_*$ , and (18) follows after integrating the exponential.  $\square$

**Corollary 4.8.** *Let  $\varrho_0^2 = 2k_1 \varkappa / \beta_0$ . Then the ball  $\{\|u\| \leq \varrho_0\}$  absorbs bounded subsets of  $H$  uniformly, in the sense that for every bounded  $B \subset \mathcal{X}$  there is  $t'_B \geq t_*$  such that  $\|u(t)\| \leq \varrho_0$  for all  $t \geq t'_B$  and all  $z_0 \in B$ .*

Combining Corollary 4.8 with (14) and Lemma 3.2(a) gives a bound that will be used repeatedly,

$$(19) \quad \|\mathcal{B}(u(t), y(t))\| \leq M_B := \lambda m_* (\varrho_0 + 3|\Omega|^{1/2}), \quad t \geq t'_B.$$

**Lemma 4.9** (Absorbing set in  $V$ ). *Assume (H0)–(H1). For  $t \geq t'_B$ ,*

$$(20) \quad \frac{d}{dt} \|u\|_V^2 \leq -Dc_{\Omega}^{-2} \|u\|_V^2 + a_2, \quad a_2 = \frac{M_B^2}{D}.$$

*Consequently  $\varrho_V^2 = 2a_2 c_{\Omega}^2 / D$  defines a bounded absorbing set  $\{\|u\|_V \leq \varrho_V\}$  in  $V$ .*

*Proof.* Multiply the parabolic equation by  $-\Delta u$  and integrate, which gives

$$\frac{1}{2} \frac{d}{dt} \|u\|_V^2 = -D \|\Delta u\|^2 - \int_{\Omega} \mathcal{B}(u, y) \Delta u \, d\xi.$$

By (19) and Young's inequality,

$$\frac{1}{2} \frac{d}{dt} \|u\|_V^2 \leq -\frac{D}{2} \|\Delta u\|^2 + \frac{M_B^2}{2D}.$$

Since  $\|\Delta u\|^2 \geq c_{\Omega}^{-2} \|u\|_V^2$  for  $u \in D(A)$ , estimate (20) follows, and Lemma 2.2 yields  $\limsup_{t \rightarrow \infty} \|u(t)\|_V^2 \leq a_2 c_{\Omega}^2 / D < \varrho_V^2$ .  $\square$

The proof of Theorem 4.2 now consists in assembling these estimates.

*Proof of Theorem 4.2.* Lemma 3.5 shows that  $y$  is global and eventually bounded by  $m_*$ , while Lemmas 4.7 and 4.9 show that  $\|u(t)\|_V$  stays bounded on bounded time intervals and is eventually bounded by  $\varrho_V$ . Hence  $T_{\max} = \infty$  and  $\Phi(t)z_0 = (u(t), y(t))$  is defined for all  $t \geq 0$ . The semigroup property follows from uniqueness in Theorem 4.1 and from the autonomy of (16), and joint continuity from continuous dependence. Finally

$$(21) \quad \mathcal{K} = \{(u, y) \in \mathcal{X} : \|u\|_V \leq \varrho_V, |y| \leq m_*\}$$

is a bounded absorbing set for  $\Phi$ . □

## 5. THE GLOBAL ATTRACTOR AND THE VANISHING-FEEDBACK LIMIT

Compactness comes from parabolic smoothing, measured in the fractional power spaces of Section 2 for the reason explained in Remark 2.1.

**Lemma 5.1** (Uniform bound in a fractional power space). *Assume (H0)–(H1) and fix  $\alpha \in (\frac{1}{2}, 1)$ . Let  $B \subset \mathcal{X}$  be bounded. Then there is  $C = C(\alpha, B) > 0$  such that*

$$(22) \quad \|u(t)\|_{X^\alpha} \leq C, \quad t \geq t'_B + 1, \quad z_0 \in B.$$

Consequently  $\Phi$  is asymptotically compact in  $\mathcal{X}$ , in the sense that for every bounded  $B \subset \mathcal{X}$ , every sequence  $t_n \rightarrow \infty$  and every  $z_n \in B$ , the sequence  $\Phi(t_n)z_n$  has a convergent subsequence in  $\mathcal{X}$ .

*Proof.* Let  $t \geq t'_B + 1$ . The variation-of-constants formula on  $[t - 1, t]$  reads

$$u(t) = e^A u(t - 1) + \int_{t-1}^t e^{(t-s)A} \mathcal{B}(u(s), y(s)) ds,$$

and on the absorbing set the forcing is uniformly bounded in  $H$  by (19). Applying  $(-A)^\alpha$ , using (5) and substituting  $r = t - s$ , we get

$$\|(-A)^\alpha u(t)\| \leq C_\alpha \|u(t - 1)\| + C_\alpha \int_0^1 r^{-\alpha} \|\mathcal{B}(u(t - r), y(t - r))\| dr \leq C_\alpha \left( \varrho_0 + \frac{M_{\mathcal{B}}}{1 - \alpha} \right),$$

the integral being convergent precisely because  $\alpha < 1$ . This is (22).

For the asymptotic compactness, write  $\Phi(t_n)z_n = (u^n, y^n)$ . The finite-dimensional components satisfy  $|y^n| \leq m_*$  for large  $n$  by (14), so that a subsequence converges in  $\mathbb{R}^4$  by Bolzano–Weierstrass. The parabolic components are bounded in  $X^\alpha$  by (22), and  $X^\alpha \Subset V$  by (6), so that a further subsequence of  $(u^n)$  converges in  $V$ . □

The existence of the attractor follows from the general theory.

*Proof of Theorem 4.3.* By Theorem 4.2,  $\Phi$  is a continuous semiflow on  $\mathcal{X}$  with the bounded absorbing set  $\mathcal{K}$  of (21), and by Lemma 5.1 it is asymptotically compact. The classical existence theorem for global attractors of dissipative semigroups ([16, Chapter I], [5, Chapter 3], [12, Chapter 10]) then yields a compact global attractor

$$\mathfrak{A} = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \Phi(t)\mathcal{K}} \subset V \times \mathbb{R}^4,$$

which is invariant and attracts bounded sets. Connectedness follows from the connectedness of  $\mathcal{K}$  and the continuity of  $\Phi$ . Finally, every  $z \in \mathfrak{A}$  can be written as  $z = \Phi(1)z'$  with  $z' \in \mathfrak{A} \subset \mathcal{K}$  by invariance, so that (22) applies and  $\mathfrak{A}$  is bounded in  $X^\alpha \times \mathbb{R}^4$ . □

**Remark 5.2** (Localisation). The proof gives the explicit localisation

$$\mathfrak{A} \subset \{(u, y) : \|u\|_V \leq \varrho_V, W(y) \leq c_0/c_1\},$$

with  $\varrho_V$  as in Lemma 4.9. Both radii are explicit functions of  $D, \lambda, \mu, C_a, |\Omega|, c_\Omega$  and the constants appearing in (H0).

We now let  $\varepsilon$  vary. From here on we assume, as in Theorem 4.4, that (H0)–(H1) hold uniformly for  $\varepsilon \in [0, \varepsilon_0]$  with  $\varepsilon$ -independent constants. All the quantities  $m_*, \varrho_0, M_B, \varrho_V$  and the absorbing set  $\mathcal{K}$  of (21) are then independent of  $\varepsilon \in [0, \varepsilon_0]$  as well.

**Lemma 5.3** (Uniform convergence on bounded time intervals). *For every  $T > 0$  there is  $C_T > 0$ , depending only on  $T$  and  $\mathcal{K}$ , such that*

$$(23) \quad \sup_{z_0 \in \mathcal{K}} \sup_{0 \leq t \leq T} \|\Phi_\varepsilon(t)z_0 - \Phi_0(t)z_0\|_{\mathcal{X}} \leq C_T \varepsilon, \quad 0 \leq \varepsilon \leq \varepsilon_0.$$

*Proof.* Fix  $z_0 \in \mathcal{K}$ , write  $(u_\varepsilon, y_\varepsilon) = \Phi_\varepsilon(t)z_0$  and  $(\bar{u}, \bar{y}) = \Phi_0(t)z_0$ , and set

$$E(t) = \|u_\varepsilon(t) - \bar{u}(t)\|_V + |y_\varepsilon(t) - \bar{y}(t)|.$$

Both trajectories stay in a fixed bounded set of  $\mathcal{X}$  on  $[0, T]$ , so that Lemma 4.5 applies with one constant  $\ell$ . Subtracting the two mild formulations and using the second estimate in (4),

$$\|u_\varepsilon(t) - \bar{u}(t)\|_V \leq c_* \ell \int_0^t (t-s)^{-1/2} E(s) ds.$$

For the finite-dimensional components, the difference of the two vector fields is  $F(y_\varepsilon) - F(\bar{y}) + \varepsilon e_4 \mathcal{P}(u_\varepsilon)$ , whose last term is bounded by  $\varepsilon$  thanks to (10), whence

$$|y_\varepsilon(t) - \bar{y}(t)| \leq \varepsilon T + \ell \int_0^t E(s) ds.$$

Since  $1 \leq T^{1/2}(t-s)^{-1/2}$  for  $0 \leq s \leq t \leq T$ , adding the two estimates gives

$$E(t) \leq C'_T \varepsilon + C'_T \int_0^t (t-s)^{-1/2} E(s) ds, \quad 0 \leq t \leq T,$$

and Lemma 2.3 with  $\gamma = \frac{1}{2}$  yields  $\sup_{[0, T]} E \leq C_T \varepsilon$ . The differential form Lemma 2.2 is not available here, because measuring the parabolic difference in  $V$  produces the weakly singular kernel  $(t-s)^{-1/2}$  rather than a bounded one.  $\square$

Theorem 4.4 follows in two steps.

*Proof of Theorem 4.4. Uniform precompactness.* Since all the constants of Lemmas 3.5–4.9 are independent of  $\varepsilon \in [0, \varepsilon_0]$ , so is the bound  $C$  of (22). By invariance, any  $(u, y) \in \mathfrak{A}_\varepsilon$  equals  $\Phi_\varepsilon(1)z'$  for some  $z' \in \mathfrak{A}_\varepsilon \subset \mathcal{K}$ , so that

$$\sup_{0 \leq \varepsilon \leq \varepsilon_0} \sup_{(u, y) \in \mathfrak{A}_\varepsilon} (\|u\|_{X^\alpha} + |y|) \leq C + m_* < \infty.$$

Because  $X^\alpha \Subset V$  and the  $y$ -component lives in a bounded subset of  $\mathbb{R}^4$ , the union  $\bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \mathfrak{A}_\varepsilon$  is relatively compact in  $\mathcal{X}$ . This is the point where boundedness alone would not suffice, since in an infinite-dimensional phase space a bounded family of attractors need not be precompact.

*Upper semicontinuity.* Let  $\eta > 0$ . Since  $\mathfrak{A}_0$  attracts bounded sets under  $\Phi_0$  and  $\mathcal{K}$  is bounded, there is  $T > 0$  with

$$\text{dist}_{\mathcal{X}}(\Phi_0(T)\mathcal{K}, \mathfrak{A}_0) \leq \eta/2.$$

For  $0 \leq \varepsilon \leq \varepsilon_0$  the invariance of  $\mathfrak{A}_\varepsilon$  gives  $\mathfrak{A}_\varepsilon = \Phi_\varepsilon(T)\mathfrak{A}_\varepsilon$  with  $\mathfrak{A}_\varepsilon \subset \mathcal{K}$ , hence

$$\text{dist}_{\mathcal{X}}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) \leq \sup_{z_0 \in \mathcal{K}} \|\Phi_\varepsilon(T)z_0 - \Phi_0(T)z_0\|_{\mathcal{X}} + \text{dist}_{\mathcal{X}}(\Phi_0(T)\mathcal{K}, \mathfrak{A}_0) \leq C_T\varepsilon + \frac{\eta}{2}$$

by Lemma 5.3. Choosing  $\varepsilon \leq \eta/(2C_T)$  makes the right-hand side at most  $\eta$ , which proves the claim.  $\square$

*Remark 5.4.* Theorem 4.4 identifies the one-way cascade as the singular limit  $\varepsilon = 0$  of the present interconnection. The convergence is upper semicontinuous only, and this is not an artefact of the proof. Lower semicontinuity generally requires additional structure, such as hyperbolicity of the equilibria of the limiting problem, and may genuinely fail [12, Chapter 10].

## 6. CONCLUSION

This paper has studied a parabolic equation and a four-dimensional oscillator that influence each other, the return channel consisting of a single bounded scalar observable of the parabolic state. For this closed loop we have proved local and global well-posedness, exhibited an explicit bounded absorbing set, constructed the compact global attractor in the product space  $V \times \mathbb{R}^4$ , and identified the classical one-way cascade as the singular limit  $\varepsilon \rightarrow 0$  of the interconnection.

Three features give these results their interest. Boundedness of the finite-dimensional component is here a theorem rather than a hypothesis, since it follows from a dissipativity estimate that is robust with respect to the feedback. The coupled dynamics is an autonomous semiflow on  $V \times \mathbb{R}^4$ , which makes the whole theory of global attractors available without any process or skew-product machinery. Finally, the attractor is bounded in a fractional power space uniformly in the feedback intensity, and this uniformity is precisely what yields the upper semicontinuous convergence to the attractor of the uncoupled problem.

The cost of closing the loop is, by contrast, very modest. The feedback is globally bounded and globally Lipschitz, so that the fixed point argument and the energy estimates go through with the same constants up to explicit corrections, all of which are computed along the way. The method is not tied to the particular observable, to the direction  $e_4$ , or to the specific nonlinearities of (7), and it applies verbatim to any interconnection sharing the two standing assumptions.

Several directions remain open. The finite-dimensionality of  $\mathfrak{A}$  and the existence of exponential attractors in the sense of [4] are natural next steps, as is the construction of explicit Lyapunov functions  $W$  satisfying (11) for concrete hyperchaotic fields, which, as Remark 3.9 shows, cannot be quadratic in general. A further question concerns sensitive dependence on initial conditions, which we state deliberately as an open problem. Two obstacles stand in the way. Fréchet differentiability of  $u \mapsto N_2(\nabla u)$  from  $V$  to  $H$  would require controlling the quadratic remainder  $\|\nabla h\|_{L^4}^2$ , which is not available for  $h \in V$ , so that a different functional setting or a smoother nonlinearity would be needed. Moreover, a positive upper Lyapunov exponent of the variational system does not by itself produce sensitive dependence for the nonlinear semiflow, since the admissible size of the initial perturbation may have to shrink faster than the linear growth accumulates. Establishing a bound on the nonlinear remainder that is uniform along the trajectory, and thereby quantifying the dependence of the exponents on  $\varepsilon$ , seems to us the most interesting continuation of this work.

**Competing interests.** The authors declare no competing interests.

## REFERENCES

- [1] A. Abou El Qassime, H. Nhaila, L. Bahatti, S. Zayrit, Advanced Speech Encryption Method Leveraging a Hidden Hyperchaotic Attractor and Its Synchronization with Robust Adaptive Sliding Mode Control, *Nonlinear Dyn.* 113 (2025), 15553–15578. <https://doi.org/10.1007/s11071-025-10948-0>.
- [2] T. Caraballo, X. Han, Random Dynamical Systems, in: *Applied Nonautonomous and Random Dynamical Systems*, SpringerBriefs in Mathematics, Springer, Cham, pp. 67–104, (2016). [https://doi.org/10.1007/978-3-319-49247-6\\_4](https://doi.org/10.1007/978-3-319-49247-6_4).

- [3] A.N. Carvalho, J.A. Langa, J.C. Robinson, *Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems*, Springer New York, 2013. <https://doi.org/10.1007/978-1-4614-4581-4>.
- [4] A. Eden, C. Foias, B. Nicolaenko, R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Masson, Paris; John Wiley & Sons, Chichester, 1994.
- [5] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, American Mathematical Society, 1988. <https://doi.org/10.1090/surv/025>.
- [6] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer Berlin Heidelberg, 1981. <https://doi.org/10.1007/bfb0089647>.
- [7] I. Karafyllis, M. Krstic, *Input-to-State Stability for PDEs*, Springer International Publishing, 2019. <https://doi.org/10.1007/978-3-319-91011-6>.
- [8] P.E. Kloeden, M. Rasmussen, *Nonautonomous Dynamical Systems*, American Mathematical Society, 2011. <https://doi.org/10.1090/surv/176>.
- [9] M. Krstic, A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*, Society for Industrial and Applied Mathematics, 2008. <https://doi.org/10.1137/1.9780898718607>.
- [10] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser Basel, 1995. <https://doi.org/10.1007/978-3-0348-9234-6>.
- [11] M. Marion, Finite-Dimensional Attractors Associated with Partly Dissipative Reaction-Diffusion Systems, *SIAM J. Math. Anal.* 20 (1989), 816–844. <https://doi.org/10.1137/0520057>.
- [12] J.C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge University Press, 2001.
- [13] O.E. Rössler, An Equation for Hyperchaos, *Phys. Lett. A* 71 (1979), 155–157. [https://doi.org/10.1016/0375-9601\(79\)90150-6](https://doi.org/10.1016/0375-9601(79)90150-6).
- [14] G.R. Sell, Y. You, *Dynamics of Evolutionary Equations*, Springer New York, 2002. <https://doi.org/10.1007/978-1-4757-5037-9>.
- [15] S. Sore, B.M. Ndiaye, Y. Simporé, Lyapunov Exponential Stability of the Shallow Water Equations in Trapezoidal Channel, *Pan-Am. J. Math.* 3 (2024), 9. <https://doi.org/10.28919/cpr-pajm/3-9>.
- [16] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer New York, 1997. <https://doi.org/10.1007/978-1-4612-0645-3>.