

A NEW APPROACH TO SOLVING MULTIOBJECTIVE SET-VALUED LINEAR FRACTIONAL OPTIMIZATION PROBLEMS BASED ON THE NULL SET CONCEPT

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ABSTRACT. We propose a hybrid solution approach for multiobjective set-valued linear fractional optimization problems, combining the null set concept with Dinkelbach's parametrization method. The approach proceeds in five computational stages. First, ideal values are computed for each fractional objective via scalar fractional programming. Second, the multiobjective set-valued fractional problem is converted into a multiobjective set-valued linear problem via Dinkelbach parametrization. Third, weighted-sum scalarization reduces it to a single-objective set-valued linear problem. Fourth, vectorization via the null set concept transforms this single-objective problem into a deterministic bi-objective linear problem solved by a single linear program. Fifth, an optimality test either certifies the current point or updates the parameter vector. A new biconditional theorem, in which the parameter is anchored at the point being tested, shows that Pareto optimality is preserved in both directions between the parametrized and the original problem, and yields a verifiable certificate: when the aggregate scalar objective vanishes at its minimizer, that minimizer is Pareto optimal. Under the additional assumption of a common vectorized denominator, the procedure is shown to terminate after finitely many iterations. We also introduce, for the first time in the set-valued fractional setting, notions of local and weak Pareto optimality and verify them on numerical examples. The effect of the weight vector on the recovered Pareto front is analyzed explicitly.

1. INTRODUCTION

Fractional programming—the optimization of ratios of functions—arises naturally whenever performance is measured relative to a resource or a risk. Prototypical examples include profit-to-investment and debt-to-equity ratios in corporate finance, output-to-workforce and revenue-to-cost ratios in production planning, nurses-to-patients and cost-per-patient ratios in healthcare management, and students-to-faculty and placement-to-admission ratios in academic resource allocation. In all these settings, the data are often *imprecise* or *range-valued*: budget figures are known only within intervals, demand forecasts come as ranges, and measurement errors introduce bounded uncertainty. Replacing scalar coefficients by compact sets leads naturally to *set-valued* fractional programs, where each objective value is itself a set rather than a number. When several such conflicting ratios must be optimized simultaneously, the problem becomes a *multiobjective set-valued linear fractional optimization problem*: no single feasible point minimizes all objectives at once, and one seeks instead the set of *Pareto optimal* solutions—those for which no objective can be improved without worsening another.

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The theory of linear fractional programs over the reals is well established. Dinkelbach [9] showed that a parametric linearization reduces the problem to a sequence of linear programs, providing the foundation for most subsequent algorithms. Nagih and Plateau [10] surveyed the breadth of applications and resolution methods; Kaur and Kohli [12] derived duality results for multiobjective fractional programs using convexifiers; and Wu [11] extended the theory to interval-valued multiobjective settings via convex cones.

A central obstacle specific to set-valued optimization is that the hyperspace $\Xi_{cc}(\mathcal{T})$ of all nonempty compact convex subsets of a normed space $\mathcal{T}$ is *not* a vector space: the algebraic difference $\mathcal{A} \ominus \mathcal{A}$ of a set with itself is not, in general, the singleton zero, preventing a direct transposition of classical optimality conditions. The mathematical foundations of set-valued optimization, including infimum and supremum in hyperspaces and ordering structures, are surveyed in the monograph [18] and the collected volume [19]. To circumvent the vector-space obstruction, Wu [6] introduced the notion of *null set* in the context of interval-valued problems; the concept was then formalized in full generality in [4], where two partial orders on $\Xi_{cc}(\mathcal{T})$ based on algebraic and Hukuhara differences were defined, and it was proved that every optimal solution of an associated scalarized problem is also optimal for the original set optimization problem. The same author further studied topological properties of the hyperspace in [8] and applied the null set idea to fuzzy optimization in [7]. The framework has since been extended in several directions: Moar, Kumar Sharma, and Lalitha [5] developed a nonlinear scalarization within this setting; Zhang [13] applied it to set-valued equilibrium problems; Sama and Some [14, 15] used it to solve fuzzy nonlinear optimization problems. More directly related to the present work, Lompo and Compaore [16] established that the null set concept yields an exact reformulation of single-objective set-valued linear fractional programs, and [17] extended this result to the quadratic set-valued case with full vectorization and optimality conditions.

Despite this body of work, the *multiobjective* set-valued fractional case has not been treated. The single-objective results of [16] do not extend directly: the Pareto structure requires combining Dinkelbach's parametrization with a scalarization step that preserves Pareto optimality, and the null-set vectorization must be shown to commute with the multiobjective ordering. This paper fills that gap.

The main contributions of this work can be summarized as follows. We introduce a complete mathematical framework for multiobjective set-valued linear fractional optimization, defining Pareto optimality and the ideal point directly in the set-valued setting, and for the first time in this context notions of local and weak Pareto optimality (Definitions 3.8 and 3.10). We establish that Dinkelbach's parametrization converts each fractional objective into an equivalent set-valued linear one without losing any Pareto optimal solution, provided the parameter is anchored at the point under test rather than at the ideal point. We then prove that the null-set vectorization map $\mathcal{L}$ reduces the resulting set-valued problem to a deterministic bi-objective linear program with *full bilateral preservation* of Pareto optimality (Theorem 3.22, biconditional), and derive the operational Corollary 3.23: a minimizer of the aggregate scalar φ_1 at which φ_1 vanishes, the parameter being anchored at that very point, is Pareto optimal for the original problem. These results are assembled into an explicit algorithm that practitioners can implement using any standard linear solver, each pass requiring a single linear program; we prove that it terminates after finitely many passes whenever the vectorized denominators coincide, and we exhibit the oscillation that may otherwise occur. Finally, we validate the approach on two instances with genuine interval-valued coefficients, analyze the effect of weight variation on the recovered Pareto front, and verify local and weak Pareto optimality of the computed solutions.

The remainder of the paper is organized as follows. Section 2 recalls the null set concept, the operations on $\Xi_{cc}(\mathcal{T})$, and the preference orders, and introduces the new notions of local and weak Pareto optimality for set-valued problems. Section 3 presents the main theoretical results (including the biconditional

Theorem 3.22 and the optimality certificate of Corollary 3.23) and the solution algorithm, together with its correctness, convergence and complexity analysis. Section 4 provides a numerical illustration, analyzes the effect of weight variation, and verifies local and weak Pareto optimality. Section 5 gives the conclusion and perspectives.

2. PRELIMINARIES

2.1. Set analysis. We recall the null set concept and some of its key properties. Throughout, Θ denotes a continuous linear functional on $\mathcal{T}$ with coefficient greater than or equal to 1, and we write $s(\mathcal{A}) := \sup_{\alpha \in \mathcal{A}} \Theta(\alpha)$ and $i(\mathcal{A}) := \inf_{\alpha \in \mathcal{A}} \Theta(\alpha)$ for brevity. We also use the scalar functional

$$(2.1) \quad S(\mathcal{A}) := s(\mathcal{A}) + i(\mathcal{A}), \quad \mathcal{A} \in \Xi_{cc}(\mathcal{T}),$$

which is the scalar attached to a compact convex set by the vectorization map of Proposition 3.19. Finally, for $\mathcal{A} \in \Xi_{cc}(\mathcal{T})$ we write $\mathcal{A} \approx \mathbf{0}$ whenever $\mathcal{A}$ belongs to the null set Ω defined in (2.2); equivalently, by Proposition 2.3(4), whenever $S(\mathcal{A}) = 0$.

Definition 2.1 ([4,6,8]). Let $(\mathcal{T}, \|\cdot\|)$ be a normed space and $\Xi_{cc}(\mathcal{T})$ the collection of all nonempty compact convex subsets of $\mathcal{T}$. For $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$ and $\lambda \in \mathbb{R}$, the *Minkowski sum* $\oplus$, *algebraic difference* $\ominus$, and *scalar multiplication* $\odot$ are defined by:

- $\mathcal{A} \oplus \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\};$
- $\lambda \odot \mathcal{A} = \{\lambda a \mid a \in \mathcal{A}\};$
- $\mathcal{A} \ominus \mathcal{B} = \mathcal{A} \oplus (-\mathcal{B}) = \{a - b \mid a \in \mathcal{A}, b \in \mathcal{B}\}.$

Remark 2.2. Throughout this paper we specialise to $\mathcal{T} = \mathbb{R}$ equipped with its natural total order. In this setting the continuous linear functional $\Theta : \mathbb{R} \rightarrow \mathbb{R}$ is the identity, so $s([a, b]) = b$ and $i([a, b]) = a$ for every compact interval $[a, b] \subseteq \mathbb{R}$. The largest and smallest elements of any $\mathcal{A} = [a, b] \in \Xi_{cc}(\mathbb{R})$ are $\alpha_{\max} = b$ and $\alpha_{\min} = a$, respectively, so that $\sup_{\alpha \in \mathcal{A}} \Theta(\alpha) = \Theta(\alpha_{\max}) = b$ and $\inf_{\alpha \in \mathcal{A}} \Theta(\alpha) = \Theta(\alpha_{\min}) = a$.

Proposition 2.3 ([4,6]). Let $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$ and $\lambda \in \mathbb{R}$. The following equalities hold:

- (1) $s(\mathcal{A} \oplus \mathcal{B}) = s(\mathcal{A}) + s(\mathcal{B})$ and $i(\mathcal{A} \oplus \mathcal{B}) = i(\mathcal{A}) + i(\mathcal{B})$.
- (2) $s(\lambda \odot \mathcal{A}) = \lambda s(\mathcal{A})$ if $\lambda \geq 0$, and $s(\lambda \odot \mathcal{A}) = \lambda i(\mathcal{A})$ if $\lambda < 0$.
- (3) $i(\lambda \odot \mathcal{A}) = \lambda i(\mathcal{A})$ if $\lambda \geq 0$, and $i(\lambda \odot \mathcal{A}) = \lambda s(\mathcal{A})$ if $\lambda < 0$.
- (4) $s(\mathcal{A} \ominus \mathcal{A}) = s(\mathcal{A}) - i(\mathcal{A})$ and $i(\mathcal{A} \ominus \mathcal{A}) = i(\mathcal{A}) - s(\mathcal{A})$.
- (5) $s(\lambda \odot \mathcal{A}) + i(\lambda \odot \mathcal{A}) = \lambda(s(\mathcal{A}) + i(\mathcal{A}))$.

Remark 2.4. $\Xi_{cc}(\mathcal{T})$ is not a vector space: the additive inverse of any $\mathcal{A}$ does not exist, since $\mathcal{A} \ominus \mathcal{A} \neq \{\theta_{\mathcal{T}}\}$. The singleton $\{\theta_{\mathcal{T}}\}$ acts as a neutral element for $\oplus$ (since $\mathcal{A} \oplus \{\theta_{\mathcal{T}}\} = \mathcal{A}$), but cannot serve as an additive inverse.

Definition 2.5 ([4,6]). The *null set* of $\Xi_{cc}(\mathcal{T})$ is

$$(2.2) \quad \Omega = \{\mathcal{A} \ominus \mathcal{A} \mid \mathcal{A} \in \Xi_{cc}(\mathcal{T})\}.$$

It plays the role of the zero element of $\Xi_{cc}(\mathcal{T})$.

To concretely illustrate the properties of the Minkowski difference and the notion of null set, we present a simple but revealing example.

Example 2.6. Let

$$\mathcal{A} = \text{conv}\{(1, 1), (3, 1), (2, 2)\},$$

as shown in Figure 1. We aim to compute $\mathcal{A} \ominus \mathcal{A}$, where $\ominus$ denotes the algebraic difference $\mathcal{A} \ominus \mathcal{B} = \{a - b : a \in \mathcal{A}, b \in \mathcal{B}\}$.

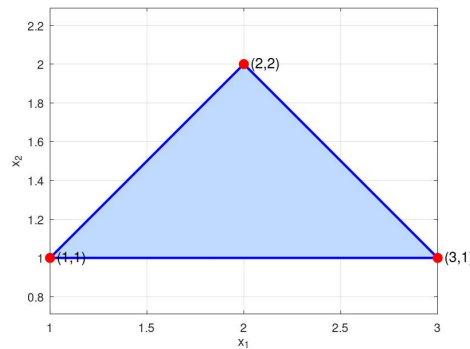


FIGURE 1. Convex hull $\mathcal{A}$.

Denoting by $v_1 = (1, 1)$, $v_2 = (3, 1)$ and $v_3 = (2, 2)$ the vertices of $\mathcal{A}$, and since $\mathcal{A}$ is a polytope, we have:

$$\mathcal{A} \ominus \mathcal{A} = \text{conv}\{v_i - v_j : i, j = 1, 2, 3\}.$$

The pairwise differences are summarized in the following table:

$v_i - v_j$	v_1	v_2	v_3
v_1	(0, 0)	(-2, 0)	(-1, -1)
v_2	(2, 0)	(0, 0)	(1, -1)
v_3	(1, 1)	(-1, 1)	(0, 0)

Taking the convex hull of the previous points, we obtain the hexagon:

$$\mathcal{A} \ominus \mathcal{A} = \text{conv}\{(-2, 0), (-1, -1), (1, -1), (2, 0), (1, 1), (-1, 1)\},$$

as shown in Figure 2.

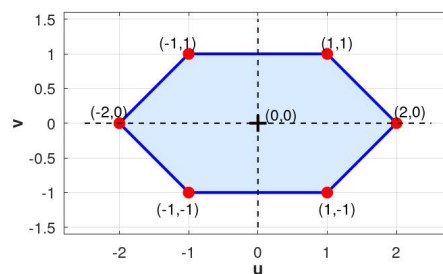


FIGURE 2. Convex hull $\mathcal{A} \ominus \mathcal{A}$.

We observe that $\mathcal{A} \ominus \mathcal{A}$ is a convex, compact set symmetric with respect to the origin, and that $\mathcal{A} \ominus \mathcal{A} \neq \{(0, 0)\}$. This set is therefore an element of the null set Ω , illustrating the fact that the difference of a set with itself is not necessarily equal to zero.

Definition 2.7 ([4,6,7]). Let V be an $\mathbb{R}$ -vector space and $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \rightarrow V$. We say that $\mathcal{L}$ is:

- *additive* if $\mathcal{L}(\mathcal{A} \oplus \mathcal{B}) = \mathcal{L}(\mathcal{A}) + \mathcal{L}(\mathcal{B})$ for all $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$;
- *homogeneous* if $\mathcal{L}(\lambda \odot \mathcal{A}) = \lambda \mathcal{L}(\mathcal{A})$ for all $\lambda \in \mathbb{R}$;
- *positively homogeneous* if the above holds for $\lambda \geq 0$;
- *linear* if additive and homogeneous;
- *homogeneous of degree k* if $\mathcal{L}(\lambda^k \odot \mathcal{A}) = \lambda^k \mathcal{L}(\mathcal{A})$ for all $\lambda \geq 0, k > 0$.

Proposition 2.8 ([4,6]). Let V be an $\mathbb{R}$ -vector space and $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \rightarrow V$ be additive and homogeneous.

- (1) If $-\mathcal{L}(\omega) = \mathcal{L}(-\omega)$ for all $\omega \in \Omega$, then $\mathcal{L}(\omega) = \theta_V$ for all $\omega \in \Omega$.
- (2) If $\mathcal{L}(\omega) = \theta_V$ for all $\omega \in \Omega$ and $\mathcal{L}$ is additive, then $\mathcal{L}(\mathcal{A} \ominus \mathcal{B}) = \mathcal{L}(\mathcal{A}) - \mathcal{L}(\mathcal{B})$.
- (3) If $\mathcal{L}$ is additive and the Hukuhara difference $\mathcal{A} \ominus_H \mathcal{B}$ exists, then $\mathcal{L}(\mathcal{A} \ominus_H \mathcal{B}) = \mathcal{L}(\mathcal{A}) - \mathcal{L}(\mathcal{B})$.

Definition 2.9 ([4,6]). A subset $\mathcal{C} \subseteq \Xi_{cc}(\mathcal{T})$ is said to be *convex* if $\lambda \odot \mathcal{A} \oplus (1 - \lambda) \odot \mathcal{B} \in \mathcal{C}$ for all $\mathcal{A}, \mathcal{B} \in \mathcal{C}$, $\lambda \in [0, 1]$; it is a *cone* if $\lambda \odot \mathcal{A} \in \mathcal{C}$ for all $\mathcal{A} \in \mathcal{C}$ and $\lambda > 0$. A *convex cone* is both a cone and convex. The canonical convex cone in $\Xi_{cc}(\mathcal{T})$ is

$$(2.3) \quad \mathcal{C} = \{C \in \Xi_{cc}(\mathcal{T}) : s(C) + i(C) \geq 0\}.$$

Proposition 2.10 ([4,6]). Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$ and $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \rightarrow V$ additive and positively homogeneous. Then $\mathcal{L}(\mathcal{C})$ is a convex cone in V .

Proof. For $\mathcal{A}, \mathcal{B} \in \mathcal{C}$ and $\lambda, \mu > 0$, we have $\lambda \odot \mathcal{A} \oplus \mu \odot \mathcal{B} \in \mathcal{C}$ (by closure of a convex cone under positive linear combinations). Since $\mathcal{L}$ is additive and positively homogeneous, $\mathcal{L}(\lambda \odot \mathcal{A} \oplus \mu \odot \mathcal{B}) = \lambda \mathcal{L}(\mathcal{A}) + \mu \mathcal{L}(\mathcal{B}) \in \mathcal{L}(\mathcal{C})$. Hence $\mathcal{L}(\mathcal{C})$ is a convex cone. $\square$

Proposition 2.11 ([4]). If $\mathcal{C}$ is a convex cone in $\Xi_{cc}(\mathcal{T})$, then $\mathcal{A} \oplus \mathcal{B} \in \mathcal{C}$ for all $\mathcal{A}, \mathcal{B} \in \mathcal{C}$.

Proof. By Definition 2.9, $s(\mathcal{A}) + i(\mathcal{A}) \geq 0$ and $s(\mathcal{B}) + i(\mathcal{B}) \geq 0$. Adding and applying Proposition 2.3(1): $0 \leq s(\mathcal{A} \oplus \mathcal{B}) + i(\mathcal{A} \oplus \mathcal{B})$, so $\mathcal{A} \oplus \mathcal{B} \in \mathcal{C}$. $\square$

2.2. Preference order.

Definition 2.12. Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$. Two binary relations on $\Xi_{cc}(\mathcal{T})$ are defined by:

- (1) $\mathcal{A} \preceq \mathcal{B}$ iff $\mathcal{B} \ominus \mathcal{A} \in \mathcal{C}$;
- (2) $\mathcal{A} \preceq_H \mathcal{B}$ iff $\mathcal{B} \ominus_H \mathcal{A}$ exists and $(\mathcal{B} \ominus_H \mathcal{A}) \oplus \omega \in \mathcal{C}$ for all $\omega \in \Omega$.

Proposition 2.13 ([4,6,14]). Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$. The relation $\preceq$ is reflexive (when $\Omega \subseteq \mathcal{C}$) and transitive, compatible with scalar multiplication and with set addition.

Proposition 2.14 ([4,6]). Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \rightarrow V$ be additive and positively homogeneous. The relation $\preceq_H$ is reflexive (when $\{\theta_{\mathcal{T}}\} \in \mathcal{C}$), transitive, and compatible with scalar multiplication and set addition.

Definition 2.15 ([4,6]). The kernel of $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \rightarrow V$ is $\ker \mathcal{L} = \{\mathcal{A} : \mathcal{L}(\mathcal{A}) = \theta_V\}$. When $\mathcal{C}$ is a convex cone and $\mathcal{L}$ is additive and positively homogeneous, two relations on $\mathcal{L}(\Xi_{cc}(\mathcal{T})) \subseteq V$ are defined: $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$ iff $\mathcal{L}(\mathcal{B}) - \mathcal{L}(\mathcal{A}) \in \mathcal{L}(\mathcal{C})$; and $\mathcal{L}(\mathcal{A}) \preceq_H \mathcal{L}(\mathcal{B})$ iff the same holds and $\mathcal{B} \ominus_H \mathcal{A}$ exists. Moreover, $\preceq_H$ implies $\preceq$.

Proposition 2.16 ([4,6]). Under the same hypotheses as Definition 2.15, the relations $\preceq$ and $\preceq_H$ on $\mathcal{L}(\Xi_{cc}(\mathcal{T}))$ are reflexive, transitive, and compatible with scalar multiplication and set addition.

Proposition 2.17 ([4,6,15]). Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \rightarrow V$ be additive and positively homogeneous, with $\Omega \subseteq \ker \mathcal{L}$. Then:

- (1) $\mathcal{A} \preceq \mathcal{B} \Rightarrow \mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$, and likewise for $\preceq_H$.
- (2) If additionally $\ker \mathcal{L} \subseteq \mathcal{C}$, then the implications reverse: $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B}) \Rightarrow \mathcal{A} \preceq \mathcal{B}$, and likewise for $\preceq_H$.

3. MAIN RESULTS

3.1. Problem formulation. The *multiobjective set-valued linear fractional optimization problem* is:

$$(3.1) \quad \begin{cases} \min f(x) = (f_1(x), \dots, f_n(x)), \\ g_k(x) \preceq 0, \quad k = 1, \dots, m, \\ x \in \mathbb{R}_+^p, \end{cases}$$

where

$$(3.2) \quad f_i(x) = \frac{\bigoplus_{j=1}^p A_{ij}x_j \oplus B_i}{\bigoplus_{j=1}^p C_{ij}x_j \oplus D_i} = \frac{P_i(x)}{Q_i(x)}, \quad i = \overline{1, n},$$

Here $A_{ij}, B_i, C_{ij}, D_i \in \Xi_{cc}(\mathcal{T})$, and P_i, Q_i are continuous set-valued functions with $Q_i(x) \succ 0$ for all i .

The *decision space* and *objective space* are:

$$(3.3) \quad G = \{x \in \mathbb{R}_+^p : g_k(x) \preceq 0, \quad k = 1, \dots, m\}, \quad \mathcal{F} = f(G).$$

The rules of set calculus, and division in particular, constitute the necessary foundation for the subsequent steps of our approach.

Definition 3.1. Let $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$ with $0 \notin \mathcal{B}$.

The inverse of the set $\mathcal{B}$ in the set sense is defined as follows:

$$(3.4) \quad \frac{1}{\mathcal{B}} = \left\{ \frac{1}{\beta} : \beta \in \mathcal{B} \right\} \in \Xi_{cc}(\mathcal{T}),$$

The set division is defined as follows:

$$(3.5) \quad \frac{\mathcal{A}}{\mathcal{B}} = \mathcal{A} \odot \frac{1}{\mathcal{B}} = \left\{ \frac{\alpha}{\beta} : \alpha \in \mathcal{A}, \beta \in \mathcal{B} \right\}.$$

This general definition applies to any compact convex set. We now specialize it to two particular geometric structures commonly used: boxes and polytopes.

Definition 3.2. Let $P = \prod_{i=1}^n [a_i, b_i]$ be a box in $\mathbb{R}^n$ such that $0 \notin [a_i, b_i]$ for all $i \in \{1, \dots, n\}$. We define the inverse of P by:

$$(3.6) \quad \frac{1}{P} = \prod_{i=1}^n \left[\frac{1}{b_i}, \frac{1}{a_i} \right],$$

and one verifies that $1/P \in \Xi_{cc}(\mathbb{R}^n)$.

To illustrate this definition, let us consider a concrete example of division between two boxes in $\mathbb{R}^3$.

Example 3.3. Let the boxes in $\mathbb{R}^3$ be defined by $A = [3, 4] \times [3, 4] \times [1, 2]$ and $B = [2, 3] \times [2, 3] \times [1, 2]$. Since $0 \notin B_i$ for all $i \in \{1, 2, 3\}$, the componentwise division is well-defined. By applying the interval division rule for positive intervals $\frac{[a, b]}{[c, d]} = \left[\frac{a}{d}, \frac{b}{c} \right]$ component by component, we obtain:

$$\begin{aligned} \frac{A_1}{B_1} &= \frac{[3, 4]}{[2, 3]} = \left[\frac{3}{3}, \frac{4}{2} \right] = [1, 2], \\ \frac{A_2}{B_2} &= \frac{[3, 4]}{[2, 3]} = \left[\frac{3}{3}, \frac{4}{2} \right] = [1, 2], \\ \frac{A_3}{B_3} &= \frac{[1, 2]}{[1, 2]} = \left[\frac{1}{2}, \frac{2}{1} \right] = \left[\frac{1}{2}, 2 \right]. \end{aligned}$$

Thus, the quotient box is written as:

$$(3.7) \quad \frac{A}{B} = [1, 2] \times [1, 2] \times \left[\frac{1}{2}, 2 \right].$$

Let us now turn to the case of polytopes, which constitute a more general class of convex sets, including in particular triangles and simplices.

Definition 3.4. Let $\mathcal{A} = \text{conv}\{v_1, \dots, v_k\} \subset \mathbb{R}^n$ be a polytope (in particular a triangle if $k = 3$, $n = 2$) such that, for every vertex $v_j = (v_{j,1}, \dots, v_{j,n})$ and every coordinate $i \in \{1, \dots, n\}$, we have $v_{j,i} \neq 0$. We define the (convexified) inverse of $\mathcal{A}$ by:

$$(3.8) \quad \frac{1}{\mathcal{A}} = \text{conv} \left\{ \left(\frac{1}{v_{j,1}}, \dots, \frac{1}{v_{j,n}} \right) : j = 1, \dots, k \right\},$$

that is, the convex hull of the coordinatewise inverses of each vertex of $\mathcal{A}$. By construction, $1/\mathcal{A} \in \Xi_{cc}(\mathbb{R}^n)$.

Let us apply this definition to the computation of the quotient of two triangles in the plane.

Example 3.5. Let $B = \text{conv}\{(4, 3), (7, 4), (3, 7)\}$ and $D = \text{conv}\{(2, 1), (3, 2), (1, 3)\}$. According to Definition 3.4, we obtain:

$$(3.9) \quad \frac{1}{D} = \text{conv} \left\{ \left(\frac{1}{2}, 1 \right), \left(\frac{1}{3}, \frac{1}{2} \right), \left(1, \frac{1}{3} \right) \right\}.$$

The division B/D is obtained by multiplying componentwise all vertices of B with those of $1/D$. The 9 possible products are:

Vertex of B	Vertex of $1/D$	Product
$(4, 3)$	$(1/2, 1)$	$(2, 3)$
$(4, 3)$	$(1/3, 1/2)$	$(4/3, 3/2)$
$(4, 3)$	$(1, 1/3)$	$(4, 1)$
$(7, 4)$	$(1/2, 1)$	$(7/2, 4)$
$(7, 4)$	$(1/3, 1/2)$	$(7/3, 2)$
$(7, 4)$	$(1, 1/3)$	$(7, 4/3)$
$(3, 7)$	$(1/2, 1)$	$(3/2, 7)$
$(3, 7)$	$(1/3, 1/2)$	$(1, 7/2)$
$(3, 7)$	$(1, 1/3)$	$(3, 7/3)$

Thus we obtain the following result:

$$(3.10) \quad \frac{B}{D} = \text{conv} \left\{ (2, 3), \left(\frac{4}{3}, \frac{3}{2} \right), (4, 1), \left(\frac{7}{2}, 4 \right), \left(\frac{7}{3}, 2 \right), \left(7, \frac{4}{3} \right), \left(\frac{3}{2}, 7 \right), \left(1, \frac{7}{2} \right), \left(3, \frac{7}{3} \right) \right\}.$$

Definition 3.6. x^* dominates x if $\frac{P_i(x^*)}{Q_i(x^*)} \preceq \frac{P_i(x)}{Q_i(x)}$ for all $i = 1, \dots, n$, with strict inequality for at least one index $i_0 \in \{1, \dots, n\}$.

Definition 3.7. $x^* \in G$ is Pareto optimal for (3.1) if no $x \in G$ dominates x^* .

Definition 3.8. $x^* \in G$ is locally Pareto optimal for (3.1) if there exists $r > 0$ such that x^* is Pareto optimal on $G \cap B(x^*, r)$.

Remark 3.9. To our knowledge, Definitions 3.8 and 3.10 are new in the set-valued fractional optimization setting: no analogue appears in the existing literature for problems with set-valued objectives. They generalise the classical notions of local and weak Pareto optimality to the ordering $\preceq$ induced by the cone $\mathcal{C}$ on $\Xi_{cc}(\mathcal{T})$.

Definition 3.10. $x \in G$ is *weakly Pareto optimal* for (3.1) if no $x' \in G$ satisfies $\frac{P_i(x')}{Q_i(x')} \prec \frac{P_i(x)}{Q_i(x)}$ for all $i = 1, \dots, n$.

Proposition 3.11. Every Pareto optimal solution of (3.1) is both locally Pareto optimal and weakly Pareto optimal.

Proof. Let $x^* \in G$ be Pareto optimal. For local Pareto optimality, take any $r > 0$: since no $x \in G$ dominates x^* , in particular no $x \in G \cap B(x^*, r) \subseteq G$ dominates x^* , so x^* is Pareto optimal on $G \cap B(x^*, r)$. For weak Pareto optimality, if some $x' \in G$ satisfied $P_i(x')/Q_i(x') \prec P_i(x^*)/Q_i(x^*)$ for all i , then x' would dominate x^* , contradicting Pareto optimality. $\square$

Definition 3.12. The element $Z^* = (Z_1^*, \dots, Z_n^*)$, obtained by minimizing each objective function f_i individually over the feasible set G with respect to $\preceq$, is called the *ideal set vector* (or set-valued ideal point) of the multiobjective optimization problem:

$$(3.11) \quad Z_i^* = \min_{x \in G} f_i(x), \quad i = 1, \dots, n.$$

The scalar parameter attached to the i -th objective is the optimal value of the *vectorized* ratio, namely

$$(3.12) \quad z_i^* = \min_{x \in G} \psi_i(x), \quad \psi_i(x) := \frac{S(P_i(x))}{S(Q_i(x))}, \quad i = 1, \dots, n,$$

which is well defined since $Q_i(x) \succ \mathbf{0}$ implies $S(Q_i(x)) > 0$. The vector $z^* = (z_1^*, \dots, z_n^*)$ is called the *ideal point* of problem (3.1).

Remark 3.13. The scalar z_i^* defined in (3.12) is exactly the optimal value returned by Step S1 of Algorithm 1, where the vectorized linear fractional program is solved by the Charnes–Cooper substitution. Since S is not multiplicative, one has in general $S(P_i(x)/Q_i(x)) \neq S(P_i(x))/S(Q_i(x))$; accordingly, z_i^* must not be identified with $S(Z_i^*)$.

3.2. Parametrization. The *parametrization* step (Dinkelbach [9]) converts each set-valued fractional objective into a set-valued linear one by replacing the ratio $P_i(x)/Q_i(x)$ by $P_i(x) \ominus z_i^* \odot Q_i(x)$:

$$(3.13) \quad \begin{cases} \min (P_i(x) \ominus z_i^* \odot Q_i(x)), & i = 1, \dots, n, \\ g_k(x) \preceq \mathbf{0}, & k = 1, \dots, m, \\ x \in \mathbb{R}_+^p. \end{cases}$$

Remark 3.14. In the classical Dinkelbach algorithm [9], the parameter is updated iteratively until convergence to the optimal value. Here we use z_i^* directly as the parameter value, which makes problem (3.13) an *exact reformulation* of the fractional problem (3.1) rather than a step of an iterative scheme. The ideal point z_i^* is computed once in Stage S1 (by solving a scalar linear-fractional program via the Charnes–Cooper substitution) and is then fixed throughout. Theorem 3.15 below rigorously justifies this reformulation.

Theorem 3.15 (Exactness of the parameter). Let $i \in \{1, \dots, n\}$, let z_i^* be given by (3.12) and set $R_i(x) = P_i(x) \ominus z_i^* \odot Q_i(x)$. Then

$$(3.14) \quad R_i(x) \succeq \mathbf{0} \quad \text{for all } x \in G, \quad \text{and} \quad \min_{x \in G} S(R_i(x)) = 0,$$

the minimum being attained exactly at the minimizers of ψ_i ; in particular $R_i(x) \approx \mathbf{0}$ whenever $\psi_i(x) = z_i^*$. Conversely, if $z \in \mathbb{R}$ satisfies $\min_{x \in G} S(P_i(x) \ominus z \odot Q_i(x)) = 0$, then $z = z_i^*$.

Proof. By Proposition 2.3, the functional S is additive, positively homogeneous and satisfies $S(\mathcal{A} \ominus \mathcal{B}) = S(\mathcal{A}) - S(\mathcal{B})$. Hence, for every $x \in G$,

$$(3.15) \quad S(R_i(x)) = S(P_i(x)) - z_i^* S(Q_i(x)) = S(Q_i(x))(\psi_i(x) - z_i^*).$$

Since $Q_i(x) \succ \mathbf{0}$ (assumption (H2)), we have $S(Q_i(x)) > 0$, and by (3.12) we have $\psi_i(x) \geq z_i^*$ for all $x \in G$. Therefore $S(R_i(x)) \geq 0$, that is $R_i(x) \succeq \mathbf{0}$, with equality if and only if $\psi_i(x) = z_i^*$. The minimum value 0 is thus attained exactly at the minimizers of ψ_i , and $S(R_i(x)) = 0$ means precisely $R_i(x) \in \Omega$, i.e. $R_i(x) \approx \mathbf{0}$. Conversely, replacing z_i^* by z in (3.15) yields $\min_{x \in G} S(Q_i(x))(\psi_i(x) - z) = 0$, which forces $\min_{x \in G} \psi_i(x) = z$, i.e. $z = z_i^*$. $\square$

Theorem 3.16. Let $x^* \in G$ and set $v_i^* = \psi_i(x^*)$ for $i = 1, \dots, n$, and $R_i^{v^*}(x) = P_i(x) \ominus v_i^* \odot Q_i(x)$. Then x^* is a Pareto optimal solution of the vectorized fractional problem

$$(3.16) \quad \min_{x \in G} (\psi_1(x), \dots, \psi_n(x))$$

if and only if x^* is a Pareto optimal solution of the parametrized problem

$$\min_{x \in G} (R_1^{v^*}(x), \dots, R_n^{v^*}(x)).$$

Moreover $S(R_i^{v^*}(x^*)) = 0$, i.e. $R_i^{v^*}(x^*) \approx \mathbf{0}$, for every i .

Proof. By Proposition 2.3, for every $x \in G$ and every i ,

$$(3.17) \quad S(R_i^{v^*}(x)) = S(P_i(x)) - v_i^* S(Q_i(x)) = S(Q_i(x))(\psi_i(x) - \psi_i(x^*)),$$

with $S(Q_i(x)) > 0$ by (H2). Taking $x = x^*$ gives $S(R_i^{v^*}(x^*)) = 0$, that is $R_i^{v^*}(x^*) \in \Omega$.

Since the order $\preceq$ compares two elements of $\Xi_{cc}(\mathcal{T})$ through the sign of the difference of their values under S , (3.17) shows that for every $x \in G$,

$$R_i^{v^*}(x) \preceq R_i^{v^*}(x^*) \iff S(R_i^{v^*}(x)) \leq 0 \iff \psi_i(x) \leq \psi_i(x^*),$$

the strict relations corresponding to one another in the same way because $S(Q_i(x))$ is strictly positive. Consequently, an $x \in G$ dominates x^* for (3.16) if and only if it dominates x^* for the parametrized problem, which is the announced equivalence. $\square$

Remark 3.17. The parameter has to be anchored at the point being tested, not at the ideal point: (3.17) is an equivalence only because the reference value $S(R_i^{v^*}(x^*))$ is zero. With the ideal value z_i^* in place of v_i^* one has

$$S(R_i^{z^*}(x^*)) = S(Q_i(x^*))(\psi_i(x^*) - z_i^*),$$

which vanishes only when x^* realizes the ideal value of the i -th ratio. This is what makes the parameter update of Step S5 in Algorithm 1 necessary.

3.3. Weighted-sum scalarization and vectorization. With the scalar parameter z_i^* of (3.12), expanding $R_i(x) = P_i(x) \ominus z_i^* \odot Q_i(x)$ by means of (3.2) turns each fractional objective into the set-valued linear function:

$$(3.18) \quad R_i(x) = \bigoplus_{j=1}^p (A_{ij} \ominus z_i^* \odot C_{ij}) x_j \oplus (B_i \ominus z_i^* \odot D_i), \quad i = \overline{1, n}.$$

The corresponding multiobjective set-valued linear problem is:

$$(3.19) \quad \begin{cases} \min \bigoplus_{j=1}^p (A_{ij} \ominus z_i^* \odot C_{ij}) x_j \oplus (B_i \ominus z_i^* \odot D_i), & i = \overline{1, n}; \\ g_k(x) \preceq \mathbf{0}, & k = \overline{1, m}; \\ x \in \mathbb{R}_+^p. \end{cases}$$

Applying weighted-sum scalarization with fixed weights $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$, yields the single-objective set-valued linear problem:

$$(3.20) \quad \begin{cases} \min \bigoplus_{i=1}^n \lambda_i \odot \left(\bigoplus_{j=1}^p (A_{ij} \ominus z_i^* \odot C_{ij}) x_j \oplus (B_i \ominus z_i^* \odot D_i) \right); \\ g_k(x) \preceq \mathbf{0}, \quad k = \overline{1, m}; \\ \lambda_i > 0, \quad \sum_{i=1}^n \lambda_i = 1; \\ x \in \mathbb{R}_+^p. \end{cases}$$

Remark 3.18. The weighted-sum scalarization in (3.20) is consistent with Pareto optimality provided the image set $\mathcal{F} = f(G) \subset \Xi_{cc}(\mathcal{T})^n$ is convex with respect to the ordering induced by $\mathcal{C}$. Under the linearity of f and the convexity of G , this convexity holds and every Pareto optimal solution of (3.19) can be recovered as a solution of (3.20) for an appropriate weight vector (λ_i) ; see, e.g., [3] for the classical scalar theory and [4] for the set-valued analogue in the single-objective case. In particular, if G is bounded and the image set is strictly convex with respect to $\preceq$, varying (λ_i) over the unit simplex traces the entire Pareto front.

To solve problem (3.20), we convert its set-valued objective function into two conflicting deterministic objective functions.

Proposition 3.19. *Let Θ be an increasing linear form on $(\mathcal{T}, \|\cdot\|)$. Define $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \rightarrow \mathbb{R}^2$ by:*

$$\mathcal{L}(A) = (s(A) + i(A), -s(A) - i(A)) = (S(A), -S(A)).$$

Let us set:

$$(3.21) \quad E(x) = \bigoplus_{i=1}^n \lambda_i \odot \left(\bigoplus_{j=1}^p (A_{ij} \ominus z_i^* \odot C_{ij}) x_j \oplus (B_i \ominus z_i^* \odot D_i) \right).$$

Then the composite function $\mathcal{L}(E(x))$ takes the form $(\varphi_1(x), \varphi_2(x))$ with $\varphi_1(x) = S(E(x))$ and $\varphi_2(x) = -\varphi_1(x)$, where:

$$(3.22) \quad \begin{aligned} \varphi_1(x) = \sum_{i=1}^n \lambda_i \left[\sum_{j=1}^p \left(s(A_{ij}) + i(A_{ij}) - z_i^* s(C_{ij}) - z_i^* i(C_{ij}) \right) x_j \right. \\ \left. + s(B_i) + i(B_i) - z_i^* i(D_i) - z_i^* s(D_i) \right], \end{aligned}$$

$$(3.23) \quad \begin{aligned} \varphi_2(x) = \sum_{i=1}^n \lambda_i \left[\sum_{j=1}^p \left(-s(A_{ij}) - i(A_{ij}) + z_i^* s(C_{ij}) + z_i^* i(C_{ij}) \right) x_j \right. \\ \left. - s(B_i) - i(B_i) + z_i^* i(D_i) + z_i^* s(D_i) \right]. \end{aligned}$$

Proof. By definition of $\mathcal{L}$, we directly have:

$$(3.24) \quad \mathcal{L}(A) = (s(A) + i(A), -s(A) - i(A)).$$

Using the linearity of $\mathcal{L}$ and the definition of $E(x)$, we expand the expression:

$$(3.25) \quad \mathcal{L}(E(x)) = \sum_{i=1}^n \lambda_i \mathcal{L} \left(\bigoplus_{j=1}^p (A_{ij} \ominus z_i^* \odot C_{ij}) x_j \oplus (B_i \ominus z_i^* \odot D_i) \right).$$

By applying Proposition 2.3 to each term of (3.25), we obtain the expressions for the bounds:

$$(3.26) \quad s(E(x)) = \sum_{i=1}^n \lambda_i \left[\sum_{j=1}^p (s(A_{ij}) - z_i^* i(C_{ij})) x_j + s(B_i) - z_i^* i(D_i) \right],$$

and

$$(3.27) \quad i(E(x)) = \sum_{i=1}^n \lambda_i \left[\sum_{j=1}^p (i(A_{ij}) - z_i^* s(C_{ij})) x_j + i(B_i) - z_i^* s(D_i) \right].$$

By summing (3.26) and (3.27), we obtain the first component $\varphi_1(x) = s(E(x)) + i(E(x))$:

$$\begin{aligned} \varphi_1(x) = \sum_{i=1}^n \lambda_i \left[\sum_{j=1}^p \left(s(A_{ij}) + i(A_{ij}) - z_i^* s(C_{ij}) - z_i^* i(C_{ij}) \right) x_j \right. \\ \left. + s(B_i) + i(B_i) - z_i^* i(D_i) - z_i^* s(D_i) \right]. \end{aligned}$$

Finally, since the second component is the opposite of the first, we have $\varphi_2(x) = -\varphi_1(x)$, which gives after distributing the minus sign:

$$\begin{aligned} \varphi_2(x) = \sum_{i=1}^n \lambda_i \left[\sum_{j=1}^p \left(-s(A_{ij}) - i(A_{ij}) + z_i^* s(C_{ij}) + z_i^* i(C_{ij}) \right) x_j \right. \\ \left. - s(B_i) - i(B_i) + z_i^* i(D_i) + z_i^* s(D_i) \right]. \end{aligned}$$

□

Remark 3.20. The identity $\varphi_2 = -\varphi_1$ is a structural consequence of the map $\mathcal{L}(A) = (S(A), -S(A))$: the second component always mirrors the first, so that the image of the feasible set under (φ_1, φ_2) is contained in the line $v = -u$ of $\mathbb{R}^2$. Consequently the componentwise order of $\mathbb{R}^2$ carries no information here—no feasible point dominates another for that order—and problem (3.35) has to be read with respect to the order induced by the image cone

$$(3.28) \quad \mathcal{K} := \mathcal{L}(\mathcal{C}) = \{(t, -t) : t \geq 0\},$$

which is precisely the order transported by $\mathcal{L}$ from $(\Xi_{cc}(\mathcal{T}), \preceq)$, as guaranteed by Proposition 2.10. With respect to $\mathcal{K}$ the solution set of (3.35) is exactly the set of minimizers of φ_1 over $\mathcal{G}$. The following theorem and corollary make this precise.

We now consider the constraints of problem (3.20), namely $g_k(x) \preceq \mathbf{0}$, $k = \overline{1, m}$. We have the following proposition.

Proposition 3.21. *Let $g_k : U \rightarrow \Xi_{cc}(\mathcal{T})$, for $k = \overline{1, m}$, be set-coefficient linear mappings. Applying the vectorization operator $\mathcal{L}$ to the set constraints $g_k(x) \preceq \mathbf{0}$ yields the vector constraints $\bar{g}_k(x) \leq 0$, for all $k = \overline{1, m}$.*

Proof. Since $\mathbf{0} \in \Omega$, we have:

$$(3.29) \quad s(\mathbf{0}) + i(\mathbf{0}) = 0.$$

The set constraints $g_k(x) \preceq \mathbf{0}$ with $k = \overline{1, m}$ mean, by definition of the partial order induced by the convex cone $\mathcal{C}$, that the set difference $\mathbf{0} \ominus g_k(x) \in \mathcal{C}$.

Now, according to Definition 2.9 of the convex cone $\mathcal{C}$, any set $\mathcal{D} \in \mathcal{C}$ satisfies the positivity property:

$$s(\mathcal{D}) + i(\mathcal{D}) \geq 0.$$

Applying this property to the set $\mathcal{D} = \mathbf{0} \ominus g_k(x) \in \mathcal{C}$, we obtain:

$$(3.30) \quad s(\mathbf{0} \ominus g_k(x)) + i(\mathbf{0} \ominus g_k(x)) \geq 0.$$

According to Proposition 2.3, we have:

$$(3.31) \quad s(\mathbf{0} \ominus g_k(x)) = s(\mathbf{0}) - i(g_k(x)) = -i(g_k(x)),$$

$$(3.32) \quad i(\mathbf{0} \ominus g_k(x)) = i(\mathbf{0}) - s(g_k(x)) = -s(g_k(x)).$$

Substituting (3.31) and (3.32) into inequality (3.30), we obtain:

$$(3.33) \quad -i(g_k(x)) - s(g_k(x)) \geq 0 \Rightarrow s(g_k(x)) + i(g_k(x)) \leq 0.$$

Thus we obtain:

$$(3.34) \quad \bar{g}_k(x) \leq 0.$$

□

The **deterministic bi-objective problem** is therefore:

$$(3.35) \quad \begin{cases} \min (\varphi_1(x), \varphi_2(x)); \\ \bar{g}_k(x) \leq 0, \quad k = 1, \dots, m; \\ x \in \mathbb{R}_+^p, \end{cases}$$

with feasible set $\mathcal{G} = \{x \in \mathbb{R}_+^p : \bar{g}_k(x) \leq 0, k = 1, \dots, m\}$.

Theorem 3.22. Let $\mathcal{K}$ be the cone defined in (3.28) and let $x^* \in \mathcal{G}$. The following statements are equivalent:

- (1) x^* is a $\mathcal{K}$ -minimal (Pareto optimal) solution of problem (3.35);
- (2) x^* minimizes φ_1 over $\mathcal{G}$;
- (3) x^* is an optimal solution of problem (3.20) for the given weight vector (λ_i) , that is $E(x^*) \preceq E(x)$ for all $x \in \mathcal{G}$.

Proof. (1) $\Leftrightarrow$ (2). Let $x, x^* \in \mathcal{G}$ and put $d = \varphi_1(x) - \varphi_1(x^*)$. Since $\varphi_2 = -\varphi_1$,

$$(\varphi_1(x), \varphi_2(x)) - (\varphi_1(x^*), \varphi_2(x^*)) = (d, -d),$$

and by (3.28) this vector belongs to $\mathcal{K}$ if and only if $d \geq 0$. Hence no $x \in \mathcal{G}$ satisfies $(\varphi_1(x), \varphi_2(x)) \prec_{\mathcal{K}} (\varphi_1(x^*), \varphi_2(x^*))$ if and only if $\varphi_1(x^*) \leq \varphi_1(x)$ for every $x \in \mathcal{G}$, that is, if and only if x^* minimizes φ_1 over $\mathcal{G}$.

(2) $\Leftrightarrow$ (3). By Proposition 3.19, $\varphi_1 = S \circ E$. For $\omega \in \Omega$ we have $S(\omega) = 0$ by Proposition 2.3(4), so $\Omega \subseteq \ker \mathcal{L}$; moreover $\ker \mathcal{L} = \{\mathcal{A} : S(\mathcal{A}) = 0\} \subseteq \mathcal{C}$ by (2.3). Proposition 2.17 therefore applies in both directions and gives, for all $x \in \mathcal{G}$,

$$\begin{aligned} E(x^*) \preceq E(x) &\iff \mathcal{L}(E(x^*)) \preceq \mathcal{L}(E(x)) \\ &\iff S(E(x^*)) \leq S(E(x)) \iff \varphi_1(x^*) \leq \varphi_1(x), \end{aligned}$$

the middle equivalence coming from

$$\mathcal{L}(E(x)) - \mathcal{L}(E(x^*)) = (d, -d) \in \mathcal{K} \iff d \geq 0.$$

Hence x^* solves (3.20) if and only if it minimizes φ_1 over $\mathcal{G}$. □

Corollary 3.23 (Optimality certificate). Let $\lambda_i > 0$ with $\sum_{i=1}^n \lambda_i = 1$, let $v \in \mathbb{R}^n$, and write $\varphi_1(\cdot; v) = \sum_{i=1}^n \lambda_i S(R_i^v(\cdot))$. Assume that x^* minimizes $\varphi_1(\cdot; v)$ over $\mathcal{G}$ and that $v_i = \psi_i(x^*)$ for every i . Then $\varphi_1(x^*; v) = 0$ and x^* is a Pareto optimal solution of (3.16).

Proof. By (3.17), $S(R_i^v(x^*)) = 0$ for every i , whence $\varphi_1(x^*; v) = 0$. Suppose some $x \in \mathcal{G}$ dominated x^* for (3.16), i.e. $\psi_i(x) \leq \psi_i(x^*) = v_i$ for all i , with a strict inequality for some i_0 . Then $S(R_{i_0}^v(x)) = S(Q_{i_0}(x))(\psi_{i_0}(x) - v_{i_0}) \leq 0$ for all i and $S(R_{i_0}^v(x)) < 0$, since $S(Q_{i_0}(x)) > 0$. As $\lambda_{i_0} > 0$, this gives $\varphi_1(x; v) < 0 = \varphi_1(x^*; v)$, contradicting the minimality of x^* . Hence no such x exists. $\square$

Remark 3.24. Corollary 3.23 is a *verifiable* certificate: once the linear program of Step S4 has been solved, one recomputes $v_i = \psi_i(x^*)$ and solves the same program again; an optimal value equal to zero proves Pareto optimality. Theorem 3.26 below shows that under the additional assumption (H7) this test is always passed after finitely many updates.

3.4. Solution algorithm. The following algorithm summarizes the resolution procedure. Stages S1 to S4 are performed once per value of the parameter vector v , and Stage S5 tests the optimality certificate of Corollary 3.23 and, if needed, updates v .

3.5. Convergence analysis of the algorithm.

3.5.1. Assumptions. Throughout this section, we assume that problem (3.1) satisfies the following assumptions.

(H1) Structure of the feasible set: The feasible set $G \subset \mathbb{R}^p$ is a non-empty compact convex polyhedron.

(H2) Strict positivity of denominators: For all $i \in \{1, \dots, n\}$ and all $x \in G$, the denominators are strictly positive in the sense of the order $\preceq$, that is, $Q_i(x) \succ 0$.

(H3) Properties of the evaluation functional: The linear form Θ used to define $s(\cdot)$ and $i(\cdot)$ satisfies: additivity, positive homogeneity, and monotonicity.

(H4) Convexity of the image set: The image set $\mathcal{F} = f(G)$ is convex with respect to the order induced by the cone $\mathcal{C}$. This condition is automatically satisfied when f is linear and G is convex.

(H5) Strictly positive weights: The weight vector satisfies $\lambda_i > 0$ for all i and $\sum_{i=1}^n \lambda_i = 1$.

(H6) Regularity of data: The coefficients A_{ij}, B_i, C_{ij}, D_i belong to $\Xi_{cc}(\mathcal{T})$, that is, they are compact convex sets in $\mathcal{T}$. The constraint functions g_k are linear.

(H7) Common denominator (used only for convergence): The vectorized denominators coincide, i.e. $S(Q_1(x)) = \dots = S(Q_n(x)) =: q(x)$ for all $x \in G$. This holds in particular when the objectives share the same denominator $Q_1 = \dots = Q_n$.

Assumptions (H1)–(H6) guarantee that the certificate of Corollary 3.23, when it is passed, is valid; assumption (H7) guarantees in addition that it is passed after finitely many iterations.

3.5.2. Correctness and convergence. The analysis splits into two statements: the certificate returned by Step S5 is always valid, and under (H7) it is always reached.

Theorem 3.25 (Correctness). *Under assumptions (H1)–(H6), if Algorithm 1 terminates, then the returned point x^* is a Pareto optimal solution of the vectorized fractional problem (3.16).*

Proof. Termination occurs at Step S5 with $\varphi_1(\hat{x}; v) = 0$, where $v_i = \psi_i(x^*)$ was computed from the previous iterate x^* and $\hat{x}$ minimizes $\varphi_1(\cdot; v)$ over $\mathcal{G}$. By Theorem 3.16 we have $\varphi_1(x^*; v) = 0$ as well, so x^* is itself a minimizer of $\varphi_1(\cdot; v)$ over $\mathcal{G}$. The pair (x^*, v) therefore satisfies the hypotheses of Corollary 3.23, which yields the Pareto optimality of x^* . Steps S1 and S4 are legitimate under (H2) and (H6): the Charnes–Cooper substitution [20] turns each vectorized fractional program into an equivalent linear program, and Proposition 3.21 shows that the set constraints vectorize into the linear constraints defining $\mathcal{G}$. $\square$

Algorithm 1 Solving the multiobjective linear fractional programming problem with set-valued coefficients

Require: Coefficient sets $A_{ij}, B_i, C_{ij}, D_i \in \Xi_{cc}(\mathcal{T})$; constraint functions g_k , for $k = 1, \dots, m$; weight vector $(\lambda_i)_{i=1}^n$ with $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$.

Ensure: On termination, a Pareto optimal solution x^* of the vectorized fractional problem (3.16).

```

1: Step S1: Computation of ideal points
2: for  $i = 1, \dots, n$  do
3:   Solve the vectorized linear fractional program (via the modified Charnes–Cooper transformation):
      $z_i^* \leftarrow \min_{x \in G} \psi_i(x) = \min_{x \in G} \frac{S(P_i(x))}{S(Q_i(x))}$ , as in (3.12).
4: end for
5:  $v \leftarrow z^*$ ; stop  $\leftarrow$  false
6: while stop = false do
7:   Step S2: Dinkelbach parametrization at the current parameter
8:   for  $i = 1, \dots, n$  do
9:     Define  $R_i^v(x) \leftarrow P_i(x) \ominus v_i \odot Q_i(x)$ .
10:  end for
11:  Formulate the parametrized multiobjective linear program with set-valued coefficients (3.19).
12:  Step S3: Weighted sum scalarization
13:  Compute the weighted aggregate:
      
$$E(x) \leftarrow \bigoplus_{i=1}^n \lambda_i \odot \left[ \bigoplus_{j=1}^p (A_{ij} \ominus v_i \odot C_{ij})x_j \oplus (B_i \ominus v_i \odot D_i) \right].$$

14:  This yields the mono-objective linear program with set-valued coefficients (3.20).
15:  Step S4: Vectorization via the null set and resolution
16:  Compute  $\varphi_1(x; v) \leftarrow s(E(x)) + i(E(x))$  and  $\varphi_2 \leftarrow -\varphi_1$ .
17:  Vectorize the constraints:  $\bar{g}_k(x) \leftarrow s(g_k(x)) + i(g_k(x))$ , so that  $g_k(x) \preceq \mathbf{0} \Leftrightarrow \bar{g}_k(x) \leq 0$ .
18:  Solve the linear program  $\text{LP}(p, m)$ :  $\hat{x} \leftarrow \arg \min_{x \in G} \varphi_1(x; v)$ .
19:  Step S5: Optimality test and parameter update
20:  if  $\varphi_1(\hat{x}; v) = 0$  then
21:    stop  $\leftarrow$  true;  $x^* \leftarrow$  the previous iterate (or  $\hat{x}$  at the first pass)
22:  else
23:     $v_i \leftarrow \psi_i(\hat{x}) = \frac{S(P_i(\hat{x}))}{S(Q_i(\hat{x}))}$  for  $i = 1, \dots, n$ 
24:  end if
25: end while
26: return  $x^*$ .
27: Validity: on exit  $v_i = \psi_i(x^*)$  and  $x^*$  minimizes  $\varphi_1(\cdot; v)$ , so Corollary 3.23 applies:  $x^*$  is Pareto optimal
    for (3.16).

```

Theorem 3.26 (Finite convergence under a common denominator). Assume (H1)–(H7) and write $q(x) = S(Q_i(x)) > 0$ for the common vectorized denominator. Then, for every weight vector (λ_i) with $\lambda_i > 0$,

$$(3.36) \quad \frac{\sum_{i=1}^n \lambda_i S(P_i(x))}{q(x)} = \sum_{i=1}^n \lambda_i \psi_i(x) =: \Theta_\lambda(x),$$

and Algorithm 1 coincides with the classical Dinkelbach scheme applied to the single ratio Θ_λ . It therefore terminates after finitely many iterations at a minimizer of Θ_λ over $\mathcal{G}$, which is a Pareto optimal solution of (3.16).

Proof. Under (H7) the aggregate objective of Step S3 reads

$$\varphi_1(x; v) = \sum_{i=1}^n \lambda_i (S(P_i(x)) - v_i q(x)) = \sum_{i=1}^n \lambda_i S(P_i(x)) - c q(x), \quad c = \sum_{i=1}^n \lambda_i v_i,$$

so it depends on v only through the scalar c ; and the update of Step S5 gives $c = \sum_i \lambda_i \psi_i(\hat{x}) = \Theta_\lambda(\hat{x})$. Algorithm 1 is thus exactly Dinkelbach's method for the linear fractional program $\min_{\mathcal{G}} \Theta_\lambda$, whose finite convergence over a polytope is classical [9]: the sequence (c^k) is strictly decreasing and takes its values among the finitely many vertices of $\mathcal{G}$. At termination x^* minimizes Θ_λ , which by (3.36) is the weighted sum of the ratios ψ_i with strictly positive weights; a minimizer of such a sum is Pareto optimal for (3.16). $\square$

Remark 3.27. Assumption (H7) cannot simply be dropped. When the denominators differ, $S(R_i^v(x)) = S(Q_i(x))(\psi_i(x) - v_i)$ carries a factor depending on both i and x , no scalar parameter is monotonically updated, and the iteration may oscillate between two vertices without ever reaching the certificate; this is illustrated in Example 4.1 for a narrow range of weights. A fixed point still exists—Theorem 3.16 is unconditional—but it need not be a vertex, so the linear program of Step S4 cannot return it. In practice one changes the weight vector (λ_i) , which is harmless since the weights are a free parameter of the method.

3.6. Complexity study of the algorithm. We analyze in this section the computational complexity of Algorithm 1. Let n denote the number of fractional objectives, p the number of decision variables, m the number of constraints, and $\mathcal{C}_{\text{LP}}(p, m)$ the complexity of solving a standard linear program with p variables and m constraints.

- Complexity of step S1

Under assumption (H6), step S1 of Algorithm 1, performed once, has a total complexity:

$$(3.37) \quad \mathcal{C}_{S1} = n \cdot \mathcal{C}_{\text{LP}}(p, m + 1) + \mathcal{O}(n).$$

- Complexity of steps S2 and S3

Under assumption (H6), the total complexity of steps S2 and S3 is:

$$(3.38) \quad \mathcal{C}_{S2} + \mathcal{C}_{S3} = \mathcal{O}(n \cdot p).$$

- Complexity of steps S4 and S5

Under assumption (H6), one pass through steps S4 and S5 costs:

$$(3.39) \quad \mathcal{C}_{S4} + \mathcal{C}_{S5} = \mathcal{O}(m) + \mathcal{C}_{\text{LP}}(p, m) + \mathcal{O}(n \cdot p),$$

the last term being the recomputation of $\psi_i(\hat{x})$, $i = 1, \dots, n$.

Theorem 3.28 (Global complexity of the algorithm). *Under assumption (H6), if Algorithm 1 performs K passes through the loop S2–S5, its total time complexity is:*

$$(3.40) \quad \mathcal{C}_{\text{total}} = (n + K) \cdot \mathcal{C}_{\text{LP}}(p, m) + \mathcal{O}(K n \cdot p + K m).$$

Under (H7), K is bounded by the number of vertices of $\mathcal{G}$, and the procedure remains polynomial for fixed n .

Proof. Step S1 is performed once and steps S2–S5 are performed K times, so

$$\begin{aligned} \mathcal{C}_{\text{total}} &= \mathcal{C}_{S1} + K[(\mathcal{C}_{S2} + \mathcal{C}_{S3}) + \mathcal{C}_{S4} + \mathcal{C}_{S5}] \\ &= [n \cdot \mathcal{C}_{\text{LP}}(p, m + 1) + \mathcal{O}(n)] + K[\mathcal{O}(n \cdot p) + \mathcal{O}(m) + \mathcal{C}_{\text{LP}}(p, m)]. \end{aligned}$$

Now, adding an additional constraint in a linear program modifies the complexity only by a term of order $\mathcal{O}(p)$, that is:

$$\mathcal{C}_{\text{LP}}(p, m + 1) = \mathcal{C}_{\text{LP}}(p, m) + \mathcal{O}(p).$$

Substituting and grouping terms:

$$\begin{aligned} \mathcal{C}_{\text{total}} &= n \cdot \mathcal{C}_{\text{LP}}(p, m) + n \cdot \mathcal{O}(p) + \mathcal{O}(n) + K \mathcal{C}_{\text{LP}}(p, m) + \mathcal{O}(K n \cdot p + K m) \\ &= (n + K) \cdot \mathcal{C}_{\text{LP}}(p, m) + \mathcal{O}(K n \cdot p + K m), \end{aligned}$$

where the terms $\mathcal{O}(n)$ and $n \cdot \mathcal{O}(p) = \mathcal{O}(n \cdot p)$ are absorbed into the remainder. $\square$

4. NUMERICAL ILLUSTRATION

Example 4.1 (Investment portfolio). A fund manager allocates capital between x_1 (corporate bonds, in millions of monetary units) and x_2 (technology stocks, in millions of monetary units). The decision criteria are *return-to-risk* ratios (akin to a generalized Sharpe ratio), leading to the fractional structure $P_i(x)/Q_i(x)$. Beyond the objectives, the manager is also subject to regulatory and risk management constraints whose coefficients are themselves uncertain: exposure to counterparty risk and available liquidity depend on the market regime (bull, stable, or bear), exactly like the returns and risks of the objectives.

The objectives f_i , $i = \overline{1, 2}$, and the constraints g_k , $k = \overline{1, 2}$, are set-coefficient mappings from $U \subseteq \mathbb{R}^p$ to $\Xi_{cc}(\mathcal{T})$, the denominators $\bigoplus_{j=1}^p C_{ij}x_j \oplus D_i$ being non-vanishing on U .

$$f_i(x) = \frac{\bigoplus_{j=1}^p A_{ij}x_j \oplus B_i}{\bigoplus_{j=1}^p C_{ij}x_j \oplus D_i}, \quad i = \overline{1, n},$$

Here A_{ij} , B_i , C_{ij} and D_i belong to $\Xi_{cc}(\mathcal{T})$, where $(\mathcal{T}, \|\cdot\|)$ is a normed space.

The multi-objective optimization problem is written as:

$$\begin{aligned} \min F(x) &= (f_1(x), f_2(x)) \\ \text{s.t.} \quad &\begin{cases} g_1(x) = x_1 \odot U_1 \oplus x_2 \odot U_2 \ominus W_0 \preceq \mathbf{0} \\ g_2(x) = F_0 \ominus (x_1 \odot V_1 \oplus x_2 \odot V_2) \preceq \mathbf{0} \\ x_1, x_2 \geq 0 \end{cases} \end{aligned} \quad (4.1)$$

The coefficients of the objective functions are:

$$\begin{aligned} A_{11} &= \text{conv}\{(4, 1), (3, 2), (2, 3)\}, & A_{12} &= \text{conv}\{(5, 1), (4, 2), (2, 4)\}, \\ B_1 &= \text{conv}\{(1, 1), (2, 1), (1, 2)\}, & C_{11} &= \text{conv}\{(1, 1), (2, 1), (1, 2)\}, \\ C_{12} &= \text{conv}\{(1, 2), (2, 2), (1, 3)\}, & D_1 &= \text{conv}\{(2, 2), (3, 2), (2, 3)\}, \\ A_{21} &= \text{conv}\{(1, 3), (2, 3), (1, 4)\}, & A_{22} &= \text{conv}\{(3, 1), (4, 1), (3, 2)\}, \\ B_2 &= \text{conv}\{(1, 1), (2, 1), (1, 2)\}, & C_{21} &= \text{conv}\{(1, 2), (2, 2), (1, 3)\}, \\ C_{22} &= \text{conv}\{(1, 1), (2, 1), (1, 2)\}, & D_2 &= \text{conv}\{(2, 2), (3, 2), (2, 3)\}. \end{aligned}$$

and the constraints are:

$$\begin{aligned} U_1 &= \text{conv}\{(1, 1), (2, 1), (1, 2)\}, & U_2 &= \text{conv}\{(2, 1), (3, 1), (2, 2)\}, \\ W_0 &= \text{conv}\{(6, 6), (8, 6), (6, 8)\}, & V_1 &= \text{conv}\{(2, 1), (3, 1), (2, 2)\}, \\ V_2 &= \text{conv}\{(1, 2), (2, 2), (1, 3)\}, & F_0 &= \text{conv}\{(3, 3), (4, 3), (3, 4)\}. \end{aligned}$$

Vectorization. Recall that for any compact convex set $\mathcal{A} \in \Xi_{cc}(\mathcal{T})$, the operator $\mathcal{L}$ assigns to each convex set the following result:

$$\mathcal{L}(\mathcal{A}) = (s(\mathcal{A}) + i(\mathcal{A}), -s(\mathcal{A}) - i(\mathcal{A})).$$

With the scalar functional S of (2.1), this reads $\mathcal{L}(\mathcal{A}) = (S(\mathcal{A}), -S(\mathcal{A}))$.

By evaluating S on the coefficients of the table, we obtain the scalar coefficients:

TABLE 1. Computation of $S(\mathcal{A}) = s(\mathcal{A}) + i(\mathcal{A})$

Coefficient	Vertices	$s(\mathcal{A})$	$i(\mathcal{A})$	$S(\mathcal{A})$
A_{11}	$\text{conv}\{(4, 1), (3, 2), (2, 3)\}$	5	5	10
A_{12}	$\text{conv}\{(5, 1), (4, 2), (2, 4)\}$	6	6	12
B_1	$\text{conv}\{(1, 1), (2, 1), (1, 2)\}$	3	2	5
C_{11}	$\text{conv}\{(1, 1), (2, 1), (1, 2)\}$	3	2	5
C_{12}	$\text{conv}\{(1, 2), (2, 2), (1, 3)\}$	4	3	7
D_1	$\text{conv}\{(2, 2), (3, 2), (2, 3)\}$	5	4	9
A_{21}	$\text{conv}\{(1, 3), (2, 3), (1, 4)\}$	5	4	9
A_{22}	$\text{conv}\{(3, 1), (4, 1), (3, 2)\}$	5	4	9
B_2	$\text{conv}\{(1, 1), (2, 1), (1, 2)\}$	3	2	5
C_{21}	$\text{conv}\{(1, 2), (2, 2), (1, 3)\}$	4	3	7
C_{22}	$\text{conv}\{(1, 1), (2, 1), (1, 2)\}$	3	2	5
D_2	$\text{conv}\{(2, 2), (3, 2), (2, 3)\}$	5	4	9

And for the constraint coefficients we have:

TABLE 2. Computation of $S(\mathcal{A}) = s(\mathcal{A}) + i(\mathcal{A})$ for constraint sets

Coefficient	Vertices	$s(\mathcal{A})$	$i(\mathcal{A})$	$S(\mathcal{A})$
U_1	$\text{conv}\{(1, 1), (2, 1), (1, 2)\}$	3	2	5
U_2	$\text{conv}\{(2, 1), (3, 1), (2, 2)\}$	4	3	7
W_0	$\text{conv}\{(6, 6), (8, 6), (6, 8)\}$	14	12	26
V_1	$\text{conv}\{(2, 1), (3, 1), (2, 2)\}$	4	3	7
V_2	$\text{conv}\{(1, 2), (2, 2), (1, 3)\}$	4	3	7
F_0	$\text{conv}\{(3, 3), (4, 3), (3, 4)\}$	7	6	13

Algorithm steps.

Step S1: Ideal points.

$$(4.2) \quad \begin{aligned} & \min f_1(x), \\ & \text{s.t.} \quad \begin{cases} g_1(x) = x_1 \odot U_1 \oplus x_2 \odot U_2 \ominus W_0 \preceq \mathbf{0}, \\ g_2(x) = F_0 \ominus (x_1 \odot V_1 \oplus x_2 \odot V_2) \preceq \mathbf{0}, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Using the modified Charnes-Cooper transformation we obtain:

$$(4.3) \quad \begin{aligned} \min \tilde{f}_1(y, t) &= A_{11} \odot y_1 \oplus A_{12} \odot y_2 \oplus B_1 \odot t, \\ \text{s.t.} \quad &\begin{cases} C_{11} \odot y_1 \oplus C_{12} \odot y_2 \oplus D_1 \odot t \succeq 1, \\ \tilde{g}_1(y, t) = y_1 \odot U_1 \oplus y_2 \odot U_2 \ominus W_0 \odot t \preceq \mathbf{0}, \\ \tilde{g}_2(y, t) = F_0 \odot t \ominus (y_1 \odot V_1 \oplus y_2 \odot V_2) \preceq \mathbf{0}, \\ y_1, y_2 \geq 0, t > 0. \end{cases} \end{aligned}$$

Using vectorization and scalarization we obtain:

$$(4.4) \quad \begin{aligned} \min \tilde{f}_1(y, t) &= 10y_1 + 12y_2 + 5t, \\ \text{s.t.} \quad &\begin{cases} 5y_1 + 7y_2 + 9t \geq 1 \\ 5y_1 + 7y_2 - 26t \leq 0 \\ 13t - (7y_1 + 7y_2) \leq 0 \\ y_1, y_2 \geq 0, t > 0. \end{cases} \end{aligned}$$

Using Dantzig's simplex algorithm the following solution is obtained:

$$(y_1^*, y_2^*, t^*) = \left(0, \frac{13}{154}, \frac{1}{22}\right)$$

Thus $x^{(1)} = (y_1^*/t^*, y_2^*/t^*) = (0, 13/7)$ and the ideal value of the first objective is $z_1^* = \frac{191}{154}$.
And for the ideal value of the second objective function:

$$(4.5) \quad \begin{aligned} \min f_2(x), \\ \text{s.t.} \quad &\begin{cases} g_1(x) = x_1 \odot U_1 \oplus x_2 \odot U_2 \ominus W_0 \preceq \mathbf{0}, \\ g_2(x) = F_0 \ominus (x_1 \odot V_1 \oplus x_2 \odot V_2) \preceq \mathbf{0}, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Using the same reasoning we obtain:

$$(4.6) \quad \begin{aligned} \min \tilde{f}_2(y, t) &= 9y_1 + 9y_2 + 5t, \\ \text{s.t.} \quad &\begin{cases} 7y_1 + 5y_2 + 9t \geq 1, \\ 5y_1 + 7y_2 - 26t \leq 0, \\ 13t - (7y_1 + 7y_2) \leq 0, \\ y_1, y_2 \geq 0, t > 0. \end{cases} \\ (y_1^*, y_2^*, t^*) &= \left(\frac{13}{154}, 0, \frac{1}{22}\right). \end{aligned}$$

Thus $x^{(2)} = (13/7, 0)$ and the ideal value of the second objective is $z_2^* = \frac{76}{77}$.

Step S2: Dinkelbach parametrization and vectorization.

$$(4.7) \quad R_1(x) = (A_{11} \odot x_1 \oplus A_{12} \odot x_2 \oplus B_1) \ominus z_1^* \odot (C_{11} \odot x_1 \oplus C_{12} \odot x_2 \oplus D_1),$$

$$(4.8) \quad R_2(x) = (A_{21} \odot x_1 \oplus A_{22} \odot x_2 \oplus B_2) \ominus z_2^* \odot (C_{21} \odot x_1 \oplus C_{22} \odot x_2 \oplus D_2).$$

Using vectorization we have:

Computation of $S(R_1(x))$ and $S(R_2(x))$.

$$\begin{aligned} S(R_1(x)) &= S(A_{11})x_1 + S(A_{12})x_2 + S(B_1) - z_1^*[S(C_{11})x_1 + S(C_{12})x_2 + S(D_1)] \\ &= 10x_1 + 12x_2 + 5 - \frac{191}{154}(5x_1 + 7x_2 + 9) \\ &= \frac{585}{154}x_1 + \frac{511}{154}x_2 - \frac{949}{154}, \end{aligned}$$

and

$$\begin{aligned} S(R_2(x)) &= S(A_{21})x_1 + S(A_{22})x_2 + S(B_2) - z_2^*[S(C_{21})x_1 + S(C_{22})x_2 + S(D_2)] \\ &= 9x_1 + 9x_2 + 5 - \frac{76}{77}(7x_1 + 5x_2 + 9) \\ &= \frac{161}{77}x_1 + \frac{313}{77}x_2 - \frac{299}{77}. \end{aligned}$$

Step S3: Scalarization. With scalarization parameters $\lambda_1 = \lambda_2 = 1/2$, we apply the linear combination to the two vectorized residuals:

$$\begin{aligned} \varphi_1(x) &= \frac{1}{2}S(R_1(x)) + \frac{1}{2}S(R_2(x)) \\ &= \frac{1}{2} \left(\frac{585}{154}x_1 + \frac{511}{154}x_2 - \frac{949}{154} \right) + \frac{1}{2} \left(\frac{322}{154}x_1 + \frac{626}{154}x_2 - \frac{598}{154} \right) \\ &= \frac{1}{308} [907x_1 + 1137x_2 - 1547] \\ &= \frac{907}{308}x_1 + \frac{1137}{308}x_2 - \frac{1547}{308}. \end{aligned}$$

Thus we obtain the following bi-objective problem:

$$(4.9) \quad \begin{aligned} &\min (\varphi_1(x), \varphi_2(x)) \\ \text{s.t.} \quad &\begin{cases} 5x_1 + 7x_2 - 26 \leq 0, \\ 13 - (7x_1 + 7x_2) \leq 0, \\ x_1, x_2 \geq 0, \end{cases} \end{aligned}$$

with $\varphi_2(x) = -\varphi_1(x)$

The obtained deterministic bi-objective problem. With $\varphi_1(x) = \frac{907}{308}x_1 + \frac{1137}{308}x_2 - \frac{1547}{308}$ obtained at Step S3 and $\varphi_2(x) = -\varphi_1(x)$, the deterministic bi-objective problem reads:

$$(4.10) \quad \begin{cases} \min \left(\frac{907}{308}x_1 + \frac{1137}{308}x_2 - \frac{1547}{308}, -\frac{907}{308}x_1 - \frac{1137}{308}x_2 + \frac{1547}{308} \right), \\ 5x_1 + 7x_2 \leq 26, \quad 7x_1 + 7x_2 \geq 13, \quad x_1 \geq 0, \quad x_2 \geq 0. \end{cases}$$

Resolution and Pareto analysis. The feasible set $\mathcal{G} = \{(x_1, x_2) : 5x_1 + 7x_2 \leq 26, 7x_1 + 7x_2 \geq 13, x_1 \geq 0, x_2 \geq 0\}$ is a trapezoid with vertices $V_1(13/7, 0)$, $V_2(26/5, 0)$, $V_3(0, 26/7)$ and $V_4(0, 13/7)$. Since φ_1 is linear, its extrema over $\mathcal{G}$ are attained at vertices; the corresponding values are collected in Table 3.

TABLE 3. Values of (φ_1, φ_2) at the vertices of $\mathcal{G}$ for $\lambda_1 = \lambda_2 = 1/2$.

Vertex	x_1	x_2	$\varphi_1(x)$	$\varphi_2(x)$
V_1	13/7	0	$481/1078 \approx 0.4462$	$-481/1078$
V_4	0	13/7	$988/539 \approx 1.8330$	$-988/539$
V_3	0	26/7	$18733/2156 \approx 8.6888$	$-18733/2156$
V_2	26/5	0	$15847/1540 \approx 10.2903$	$-15847/1540$

Since $\varphi_2 = -\varphi_1$, the image of $\mathcal{G}$ under (φ_1, φ_2) is the segment

$$\{(t, -t) : t = \varphi_1(x), x \in \mathcal{G}\} = \left\{ (t, -t) : t \in \left[\frac{481}{1078}, \frac{15847}{1540} \right] \right\},$$

so that, as observed in Remark 3.20, no feasible point dominates another for the componentwise order of $\mathbb{R}^2$. With respect to the order induced by the cone $\mathcal{K}$ of (3.28), Theorem 3.22 shows that the solution set of (4.10) is $\arg \min_{x \in \mathcal{G}} \varphi_1(x; v^0) = \{V_1\}$, i.e. $\hat{x} = (13/7, 0)$ with $\varphi_1(\hat{x}; v^0) = 481/1078 \approx 0.4462$.

Step S5: optimality test and parameter update. Since $\varphi_1(\hat{x}; v^0) \neq 0$, the certificate of Corollary 3.23 is not yet available and the parameter is updated:

$$v^1 = (\psi_1(\hat{x}), \psi_2(\hat{x})) = \left(\frac{165}{128}, \frac{76}{77} \right).$$

The second component is unchanged, since $\hat{x}$ already realizes the ideal value z_2^* . Repeating Steps S2–S4 with v^1 gives

$$\varphi_1(x; v^1) = \frac{1}{2} \left[\frac{455}{128} x_1 + \frac{381}{128} x_2 - \frac{845}{128} \right] + \frac{1}{2} \left[\frac{161}{77} x_1 + \frac{313}{77} x_2 - \frac{299}{77} \right],$$

whose two coefficients are again strictly positive, so that the minimum is attained on the edge $7x_1 + 7x_2 = 13$ at V_1 , with

$$\min_{x \in \mathcal{G}} \varphi_1(x; v^1) = \varphi_1(V_1; v^1) = 0.$$

The test of Step S5 is passed at the second pass, and the algorithm returns $x^* = V_1 = (13/7, 0)$. By Corollary 3.23, x^* is a Pareto optimal solution of (3.16). Consistently with Theorem 3.15, x^* realizes the ideal value of the second objective, since $S(R_2^{z^*}(x^*)) = 0$, whereas $S(R_1^{z^*}(x^*)) = 481/539$.

Verification of non-domination. Let $x_A = (13/7, 0)$ (returned for $\lambda_1 = 1/2$) and $x_B = (0, 13/7)$ (returned for $\lambda_1 = 7/8$, see below). The vectorized ratios are

$$\begin{aligned} \psi_1(x_A) &= \frac{165}{128} \approx 1.2891, & \psi_2(x_A) &= \frac{76}{77} \approx 0.9870, \\ \psi_1(x_B) &= \frac{191}{154} \approx 1.2403, & \psi_2(x_B) &= \frac{19}{16} = 1.1875. \end{aligned}$$

Thus x_B is strictly better than x_A for the first objective and strictly worse for the second: neither point dominates the other, which confirms that both are Pareto optimal for (3.16). A direct computation shows that the whole edge $[x_A, x_B]$ of $\mathcal{G}$ is non-dominated.

Effect of weight variation. The aggregate objective depends on the weight vector (λ_1, λ_2) with $\lambda_2 = 1 - \lambda_1$. From $\varphi_1(\cdot; \lambda_1) = \lambda_1 S(R_1) + \lambda_2 S(R_2)$ we obtain

$$(4.11) \quad \varphi_1(x; \lambda_1) = \frac{(263\lambda_1 + 322)x_1 + (626 - 115\lambda_1)x_2 - (351\lambda_1 + 598)}{154}.$$

Both coefficients are strictly positive for every $\lambda_1 \in [0, 1]$, so the first iterate lies on the edge $7x_1 + 7x_2 = 13$, at V_1 or at V_4 according to whether $263\lambda_1 + 322 < 626 - 115\lambda_1$, i.e. $\lambda_1 < 152/189$. Running the loop S2–S5 to termination yields three regimes:

- $\lambda_1 \leq 9728/12577 \approx 0.7735$: the test of Step S5 is passed at the second pass and the algorithm returns $x^* = V_1 = (13/7, 0)$;
- $\lambda_1 \geq 1463/1759 \approx 0.8317$: it is passed at the second pass and the algorithm returns $x^* = V_4 = (0, 13/7)$;
- $9728/12577 < \lambda_1 < 1463/1759$: the iterates alternate between V_1 and V_4 without reaching the certificate.

The last case is precisely the situation described in Remark 3.27: the present example does not satisfy (H7), since $S(Q_1(x)) = 5x_1 + 7x_2 + 9$ and $S(Q_2(x)) = 7x_1 + 5x_2 + 9$ differ. A fixed point of the parameter map does exist in that range—it lies in the relative interior of the edge $[V_1, V_4]$, where the two coefficients of $\varphi_1(\cdot; v)$ become equal—but the linear program of Step S4 always returns a vertex and never reaches it. Since the weights are a free parameter, it suffices to move λ_1 out of this narrow band.

Table 4 presents three representative weight vectors, the point returned by the algorithm and the certified value of φ_1 . According to Corollary 3.23, each entry is Pareto optimal for (3.16).

TABLE 4. Effect of weight variation: point returned by Algorithm 1, number of passes, and certified value.

λ_1	λ_2	Returned x^*	Passes	$\varphi_1(x^*; v)$
1/4	3/4	(13/7, 0)	2	0
1/2	1/2	(13/7, 0)	2	0
7/8	1/8	(0, 13/7)	2	0

The two points obtained are the vertices V_1 and V_4 of the Pareto front of (3.16), whose whole edge $[V_1, V_4]$ is non-dominated: varying (λ_i) therefore explores the extreme points of this front.

Local and weak Pareto optimality. We illustrate the notions of local and weak optimality on the solution $x^* = (0, 13/7)$ returned with $\lambda_1 = 7/8$. At termination the certified parameter is $v = (\psi_1(x^*), \psi_2(x^*)) = (\frac{191}{154}, \frac{19}{16})$ and

$$\varphi_1(x; v) = \frac{33607x_1 + 32389x_2 - 60151}{9856}, \quad \varphi_1(x^*; v) = 0.$$

Local Pareto optimality. A direction $d = (d_1, d_2)$ is feasible at x^* if and only if $d_1 \geq 0$ and $d_1 + d_2 \geq 0$ (the constraints $x_1 \geq 0$ and $7x_1 + 7x_2 \geq 13$ being active at x^*). For $\tau > 0$ small enough,

$$\begin{aligned} \varphi_1(x^* + \tau d; v) - \varphi_1(x^*; v) &= \tau \frac{33607d_1 + 32389d_2}{9856} \\ &\geq \tau \frac{(33607 - 32389)d_1}{9856} = \tau \frac{87d_1}{704} \geq 0. \end{aligned}$$

No feasible point in a neighborhood of x^* therefore improves φ_1 , so x^* cannot be dominated in any neighborhood. According to Definition 3.8, $x^* = (0, 13/7)$ is locally Pareto optimal.

Weak Pareto optimality. Since $x^* = (0, 13/7)$ attains the minimal value 0 of $\varphi_1(\cdot; v)$ over $\mathcal{G}$, no $x' \in \mathcal{G}$ satisfies $\varphi_1(x'; v) < 0$; by (3.17) this rules out the existence of an x' with $\psi_i(x') < \psi_i(x^*)$ for every i . Hence $x^* = (0, 13/7)$ is weakly Pareto optimal according to Definition 3.10.

Example 4.2 (Interval-valued coefficients). The objective functions have interval-valued coefficients:

$$f_1(x) = \frac{[1, 2] x_1 \oplus [0, 1] x_2 \oplus [2, 4]}{[1, 1] x_1 \oplus [1, 1] x_2 \oplus [1, 3]}, \quad f_2(x) = \frac{[0, 1] x_1 \oplus [1, 2] x_2 \oplus [1, 3]}{[1, 1] x_1 \oplus [1, 1] x_2 \oplus [1, 3]}.$$

The coefficients of the decision variables in the denominators are degenerate intervals, $C_{ij} = [1, 1]$, but the constant term $D_1 = D_2 = [1, 3]$ is a proper interval, so that $Q_1(x) = Q_2(x) = (x_1 + x_2) \odot [1, 1] \oplus [1, 3]$. One checks that $Q_i(x) \succ \mathbf{0}$ for all $x \in G$.

Feasible set:

$$G = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 4, x_1 \geq 0, x_2 \geq 0\}.$$

Step S1: Modified Charnes–Cooper transformation and ideal values.

- Ideal value of the first objective function

$$(4.12) \quad \begin{cases} Z^* = \min f_1(x) = \frac{[1, 2] x_1 \oplus [0, 1] x_2 \oplus [2, 4]}{[1, 1] x_1 \oplus [1, 1] x_2 \oplus [1, 3]}, \\ x_1 + x_2 \leq 4, \\ x_1, x_2 \geq 0; \end{cases}$$

Using the modified Charnes–Cooper transformation, we obtain:

$$(4.13) \quad \begin{cases} Z^* = \min f_1(x) = [1, 2] y_1 \oplus [0, 1] y_2 \oplus [2, 4] t \\ y_1 + y_2 \leq 4t, \\ [1, 1] y_1 \oplus [1, 1] y_2 \oplus [1, 3] t \geq 1, \\ y_1, y_2 \geq 0; t > 0; \end{cases}$$

and vectorization and scalarization yield:

$$(4.14) \quad \begin{cases} z^* = \min f_1(x) = 3y_1 + y_2 + 6t \\ y_1 + y_2 - 4t \leq 0, \\ 2y_1 + 2y_2 + 4t \geq 1, \\ y_1, y_2 \geq 0; t > 0; \end{cases}$$

Using Dantzig's simplex algorithm, we obtain the following solution:

$$(4.15) \quad (y_1^*, y_2^*, t^*) = \left(0, \frac{1}{3}, \frac{1}{12}\right)$$

Thus the ideal value of the first objective is $z_1^* = \frac{5}{6}$, attained at $x^{(1)} = (0, 4)$.

- Ideal value of the second objective function

$$(4.16) \quad \begin{cases} Z^* = f_2(x) = \frac{[0, 1] x_1 \oplus [1, 2] x_2 \oplus [1, 3]}{[1, 1] x_1 \oplus [1, 1] x_2 \oplus [1, 3]}, \\ x_1 + x_2 \leq 4, \\ x_1, x_2 \geq 0; \end{cases}$$

Using the modified Charnes–Cooper transformation, we obtain:

$$(4.17) \quad \begin{cases} Z^* = \min f_2(x) = [0, 1] y_1 \oplus [1, 2] y_2 \oplus [1, 3] t \\ y_1 + y_2 \leq 4t, \\ [1, 1] y_1 \oplus [1, 1] y_2 \oplus [1, 3] t \geq 1, \\ y_1, y_2 \geq 0; t > 0; \end{cases}$$

and vectorization and scalarization yield:

$$(4.18) \quad \begin{cases} z^* = \min f_2(x) = y_1 + 3y_2 + 4t \\ y_1 + y_2 - 4t \leq 0, \\ 2y_1 + 2y_2 + 4t \geq 1, \\ y_1, y_2 \geq 0; t > 0; \end{cases}$$

Using Dantzig's simplex algorithm, we obtain the following solution:

$$(4.19) \quad (y_1^*, y_2^*, t^*) = \left(\frac{1}{3}, 0, \frac{1}{12}\right)$$

Thus the ideal value of the second objective is $z_2^* = \frac{2}{3}$, attained at $x^{(2)} = (4, 0)$.

Step S2: Parametrization. For $i = 1$ ($z_1^* = 5/6$, $C_{1j} = [1, 1]$, $D_1 = [1, 3]$):

$$\begin{aligned} A_{11} \ominus z_1^* \odot C_{11} &= [1, 2] \ominus \frac{5}{6}[1, 1] = \left[\frac{1}{6}, \frac{7}{6}\right], \\ A_{12} \ominus z_1^* \odot C_{12} &= [0, 1] \ominus \frac{5}{6}[1, 1] = \left[-\frac{5}{6}, \frac{1}{6}\right], \\ B_1 \ominus z_1^* \odot D_1 &= [2, 4] \ominus \frac{5}{6}[1, 3] = [2, 4] \ominus \left[\frac{5}{6}, \frac{5}{2}\right] = \left[-\frac{1}{2}, \frac{19}{6}\right]. \end{aligned}$$

For $i = 2$ ($z_2^* = 2/3$, $C_{2j} = [1, 1]$, $D_2 = [1, 3]$):

$$\begin{aligned} A_{21} \ominus z_2^* \odot C_{21} &= [0, 1] \ominus \frac{2}{3}[1, 1] = \left[-\frac{2}{3}, \frac{1}{3}\right], \\ A_{22} \ominus z_2^* \odot C_{22} &= [1, 2] \ominus \frac{2}{3}[1, 1] = \left[\frac{1}{3}, \frac{4}{3}\right], \\ B_2 \ominus z_2^* \odot D_2 &= [1, 3] \ominus \frac{2}{3}[1, 3] = [1, 3] \ominus \left[\frac{2}{3}, 2\right] = \left[-1, \frac{7}{3}\right]. \end{aligned}$$

Steps S3–S4: Scalarization and vectorization. We use equal weights $\lambda_1 = \lambda_2 = \frac{1}{2}$.

$$\begin{aligned} s(E(x)) &= \frac{1}{2} \left[\frac{7}{6}x_1 + \frac{1}{6}x_2 + \frac{19}{6} \right] + \frac{1}{2} \left[\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3} \right] \\ &= \frac{1}{2} \left[\frac{9}{6}x_1 + \frac{9}{6}x_2 + \frac{33}{6} \right] = \frac{3}{4}x_1 + \frac{3}{4}x_2 + \frac{11}{4}; \end{aligned}$$

and

$$\begin{aligned} i(E(x)) &= \frac{1}{2} \left[\frac{1}{6}x_1 - \frac{5}{6}x_2 - \frac{1}{2} \right] + \frac{1}{2} \left[-\frac{2}{3}x_1 + \frac{1}{3}x_2 - 1 \right] \\ &= \frac{1}{2} \left[-\frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{3}{2} \right] = -\frac{1}{4}x_1 - \frac{1}{4}x_2 - \frac{3}{4}. \end{aligned}$$

The two deterministic scalar objectives are therefore given by:

$$\varphi_1(x) = s(E) + i(E) = \frac{1}{2}(x_1 + x_2) + 2, \quad \varphi_2(x) = -\varphi_1(x) = -\frac{1}{2}(x_1 + x_2) - 2.$$

Resulting deterministic bi-objective problem. The constraints vectorize trivially:

$$(4.20) \quad \begin{cases} \min \left(\frac{1}{2}(x_1 + x_2) + 2, -\frac{1}{2}(x_1 + x_2) - 2 \right), \\ x_1 + x_2 \leq 4, \quad x_1 \geq 0, \quad x_2 \geq 0. \end{cases}$$

Step S5: optimality test and parameter updates. This example satisfies assumption (H7): the two denominators coincide, $S(Q_1(x)) = S(Q_2(x)) = q(x) = 2x_1 + 2x_2 + 4$. Theorem 3.26 therefore guarantees that the loop terminates after finitely many passes, and that the returned point minimizes $\Theta_\lambda = \lambda_1\psi_1 + \lambda_2\psi_2$ over G . We follow the iterations for $\lambda_1 = \lambda_2 = 1/2$.

First pass ($v^0 = z^* = (5/6, 2/3)$). The objective of (4.20) is $\varphi_1(x; v^0) = \frac{1}{2}(x_1 + x_2) + 2$, minimized at $\hat{x} = (0, 0)$ with value $2 \neq 0$; the parameter is updated to

$$v^1 = (\psi_1(0, 0), \psi_2(0, 0)) = \left(\frac{6}{4}, \frac{4}{4}\right) = \left(\frac{3}{2}, 1\right).$$

Second pass ($v^1 = (3/2, 1)$). Here $S(R_1^{v^1}(x)) = -2x_2$ and $S(R_2^{v^1}(x)) = -x_1 + x_2$, so

$$\varphi_1(x; v^1) = -\frac{1}{2}x_1 - \frac{1}{2}x_2,$$

whose minimum over G is -2 , attained on the whole edge $x_1 + x_2 = 4$; the simplex method returns $\hat{x} = (4, 0)$. Since $-2 \neq 0$, the parameter is updated to

$$v^2 = (\psi_1(4, 0), \psi_2(4, 0)) = \left(\frac{18}{12}, \frac{8}{12}\right) = \left(\frac{3}{2}, \frac{2}{3}\right).$$

Third pass ($v^2 = (3/2, 2/3)$). Now $S(R_1^{v^2}(x)) = -2x_2$ and $S(R_2^{v^2}(x)) = -\frac{1}{3}x_1 + \frac{5}{3}x_2 + \frac{4}{3}$, whence

$$\varphi_1(x; v^2) = -\frac{1}{6}x_1 - \frac{1}{6}x_2 + \frac{2}{3},$$

whose minimum over G is 0 , attained on the edge $x_1 + x_2 = 4$. The test of Step S5 is passed and the algorithm returns $x^* = (4, 0)$, with $\Theta_{1/2}(x^*) = \frac{13}{12}$.

Resolution and Pareto analysis. By Corollary 3.23, $x^* = (4, 0)$ is a Pareto optimal solution of (3.16). This can be checked directly: $\psi_2(x) = 2/3$ forces $x_1 = 5x_2 + 4$, which together with $x_1 + x_2 \leq 4$ gives $x = (4, 0)$, so x^* is the unique point of G realizing the ideal value z_2^* and cannot be dominated. More generally, on the edge $x_1 + x_2 = 4$ one has $q \equiv 12$, $\psi_1 = (2x_1 + 10)/12$ increasing in x_1 and $\psi_2 = (16 - 2x_1)/12$ decreasing, while $\psi_1(x) + \psi_2(x) = (4s + 10)/(2s + 4)$, $s = x_1 + x_2$, is strictly decreasing in s ; hence the whole edge $x_1 + x_2 = 4$ is the Pareto front of (3.16) and no interior point of G belongs to it.

Since $\varphi_2 = -\varphi_1$, the image of G under $(\varphi_1(\cdot; v^0), \varphi_2(\cdot; v^0))$ is the segment

$$\{(t, -t) : t = \frac{1}{2}(x_1 + x_2) + 2, (x_1, x_2) \in G\} = \{(t, -t) : t \in [2, 4]\},$$

traversed as $x_1 + x_2$ varies from 0 to 4 : as observed in Remark 3.20, no feasible point dominates another for the componentwise order of $\mathbb{R}^2$, which is why the optimality test of Step S5, and not the bi-objective structure alone, selects the solution.

Table 5 reports the values of (φ_1, φ_2) at representative feasible points.

TABLE 5. Values of (φ_1, φ_2) at the initial parameter $v^0 = z^*$, at representative feasible points of (4.20).

x_1	x_2	$x_1 + x_2$	$\varphi_1(x)$	$\varphi_2(x)$
0	0	0	2	-2
1	0	1	5/2	-5/2
1	1	2	3	-3
2	1	3	7/2	-7/2
2	2	4	4	-4

Both $(2, 2)$ and $(0, 4)$ lie on the edge $x_1 + x_2 = 4$ and are therefore Pareto optimal by the discussion above. We check this directly on the set-valued objectives themselves.

Initial objectives at $(2, 2)$:

$$f_1(2, 2) = \frac{[1, 2] \cdot 2 \oplus [0, 1] \cdot 2 \oplus [2, 4]}{2 \cdot [1, 1] \oplus 2 \cdot [1, 1] \oplus [1, 3]} = \frac{[4, 10]}{[5, 7]} = \left[\frac{4}{7}, 2\right],$$

$$f_2(2, 2) = \frac{[0, 1] \cdot 2 \oplus [1, 2] \cdot 2 \oplus [1, 3]}{[5, 7]} = \frac{[3, 9]}{[5, 7]} = \left[\frac{3}{7}, \frac{9}{5}\right].$$

Initial objectives at $(0, 4)$:

$$f_1(0, 4) = \frac{[0, 1] \cdot 4 \oplus [2, 4]}{[5, 7]} = \frac{[2, 8]}{[5, 7]} = \left[\frac{2}{7}, \frac{8}{5}\right],$$

$$f_2(0, 4) = \frac{[1, 2] \cdot 4 \oplus [1, 3]}{[5, 7]} = \frac{[5, 11]}{[5, 7]} = \left[\frac{5}{7}, \frac{11}{5}\right].$$

Verification of non-domination. Using the scalar functional $S(A) = s(A) + i(A)$:

$$S(f_1(0, 4)) = \frac{8}{5} + \frac{2}{7} = \frac{66}{35}, \quad S(f_1(2, 2)) = 2 + \frac{4}{7} = \frac{18}{7} = \frac{90}{35}.$$

Since $\frac{66}{35} < \frac{90}{35}$, the point $(0, 4)$ is strictly better than $(2, 2)$ for f_1 .

$$S(f_2(0, 4)) = \frac{11}{5} + \frac{5}{7} = \frac{16}{5} = \frac{102}{35}, \quad S(f_2(2, 2)) = \frac{9}{5} + \frac{3}{7} = \frac{78}{35}.$$

Since $\frac{78}{35} < \frac{102}{35}$, the point $(2, 2)$ is strictly better than $(0, 4)$ for f_2 . Thus neither $(0, 4)$ nor $(2, 2)$ dominates the other, in accordance with the fact that both belong to the Pareto front.

Effect of weight variation. Since assumption (H7) holds, Theorem 3.26 identifies the point returned by the algorithm as a minimizer over G of

$$(4.21) \quad \Theta_\lambda(x) = \lambda_1 \psi_1(x) + \lambda_2 \psi_2(x) = \frac{(2\lambda_1 + 1)x_1 + (3 - 2\lambda_1)x_2 + (2\lambda_1 + 4)}{2x_1 + 2x_2 + 4},$$

with $\lambda_2 = 1 - \lambda_1$. Evaluating (4.21) at the three vertices of G gives $\Theta_\lambda(0, 0) = (6\lambda_1 + 12)/12$, $\Theta_\lambda(4, 0) = (10\lambda_1 + 8)/12$ and $\Theta_\lambda(0, 4) = (16 - 6\lambda_1)/12$. The vertex $(0, 0)$ is never optimal for $\lambda_1 < 1$, and the comparison of the two others yields three regimes:

- $\lambda_1 < 1/2$: the algorithm returns $x^* = (4, 0)$ (a heavy weight on ψ_2 pushes x_1 to its upper bound);
- $\lambda_1 = 1/2$: $\Theta_{1/2}$ is constant on the edge $x_1 + x_2 = 4$, every point of which is optimal;
- $\lambda_1 > 1/2$: the algorithm returns $x^* = (0, 4)$ (a heavy weight on ψ_1 pushes x_2 to its upper bound).

Table 6 presents three representative weight vectors, the returned solution and the corresponding value of Θ_λ . According to Corollary 3.23, each entry is Pareto optimal for (3.16).

TABLE 6. Effect of weight variation on the solution returned for (4.20).

λ_1	λ_2	Returned x^*	$\Theta_\lambda(x^*)$
1/8	7/8	(4, 0)	37/48 $\approx$ 0.7708
1/2	1/2	edge $x_1 + x_2 = 4$	13/12 $\approx$ 1.0833
7/8	1/8	(0, 4)	43/48 $\approx$ 0.8958

The solutions obtained are distinct points of the Pareto front of (3.16)—which is exactly the edge $x_1 + x_2 = 4$ —showing that the variation of (λ_i) allows for an efficient exploration of this front.

Local and weak Pareto optimality. We illustrate the new set-valued notions introduced in Definitions 3.8 and 3.10 using the solution $x^* = (0, 4)$ returned with $\lambda_1 = 7/8$.

Local Pareto optimality. For any $r > 0$, a point $(\varepsilon, 4 - \varepsilon) \in G \cap B(x^*, r)$ with $0 < \varepsilon < r$ satisfies

$$\begin{aligned} P_1(\varepsilon, 4 - \varepsilon) &= [\varepsilon + 2, 2\varepsilon + (4 - \varepsilon) + 4] = [\varepsilon + 2, \varepsilon + 8], \\ Q_1(\varepsilon, 4 - \varepsilon) &= [5, 7]. \end{aligned}$$

Hence $S(f_1(\varepsilon, 4 - \varepsilon)) = (\varepsilon + 8)/5 + (\varepsilon + 2)/7 = (12\varepsilon + 66)/35 > S(f_1(0, 4)) = 66/35$, so $f_1(\varepsilon, 4 - \varepsilon) \succ f_1(0, 4)$. No neighboring feasible point strictly improves f_1 , so $x^* = (0, 4)$ cannot be dominated in any neighborhood. According to Definition 3.8, $x^* = (0, 4)$ is locally Pareto-optimal for (3.1).

Weak Pareto optimality. Since $x^* = (0, 4)$ attains the ideal value $z_1^* = 5/6$ of the vectorized ratio ψ_1 (see (3.12)), Theorem 3.15 gives $S(R_1^{z_1^*}(0, 4)) = 0 = \min_{x \in G} S(R_1^{z_1^*}(x))$, so no $x' \in G$ satisfies $\psi_1(x') < \psi_1(0, 4)$. Thus, no $x' \in G$ simultaneously dominates all objectives, and $x^* = (0, 4)$ is weakly Pareto-optimal according to Definition 3.10. According to Proposition 3.11, these two properties also follow from global Pareto optimality.

5. CONCLUSION AND PERSPECTIVES

We have proposed a systematic approach—ideal-value computation, Dinkelbach parametrization, weighted-sum scalarization, null-set vectorization, and an optimality test on the parameter—for multiobjective set-valued linear fractional optimization. Pareto optimality is preserved in the biconditional sense (Theorems 3.16 and 3.22) once the parameter is anchored at the point under test, and Corollary 3.23 turns this into a certificate that can be checked by solving one additional linear program. Theorem 3.26 shows that the certificate is reached after finitely many passes whenever the vectorized denominators coincide; Remark 3.27 and Example 4.1 show that this assumption cannot simply be dropped. The deterministic linear program (3.35) solved at each pass can be handled by any off-the-shelf LP solver. We also introduced, for the first time in the set-valued fractional optimization setting, notions of local and weak Pareto optimality and verified them on the numerical examples.

From a computational complexity standpoint, Stage S1 requires solving n scalar linear-fractional programs, each reducible to a linear program via the Charnes–Cooper substitution, hence solvable in polynomial time. Each pass through Stages S2–S5 costs one deterministic linear program $\text{LP}(p, m)$ plus $\mathcal{O}(np)$ arithmetic operations for the parameter update. Under (H7) the number of passes is bounded by the number of vertices of $\mathcal{G}$, so the overall procedure runs in polynomial time for fixed n .

As established in Remark 3.20 and Corollary 3.23, the vectorization map $\mathcal{L}$ induces $\varphi_2 = -\varphi_1$, so practical selection reduces to minimizing φ_1 over $\mathcal{G}$ —a single LP per pass—and the optimality test decides when the current point is Pareto optimal for the original problem. Different weight vectors (λ_i) yield different Pareto optimal solutions, effectively tracing the Pareto front (Example 4.2). These findings motivate the following research directions.

Four natural extensions are worth pursuing. First, designing a parameter update that converges without assumption (H7): the fixed point whose existence Theorem 3.16 guarantees may lie in the relative interior of a face, where a vertex-returning linear solver cannot reach it, and a bundle or bisection strategy on the weight vector would be a natural remedy. Second, investigating *alternative vectorization maps* that produce a richer Pareto structure in the deterministic problem while still preserving the Pareto-lifting property established in Theorem 3.22. Third, extending the approach to the *nonlinear* set-valued case via the nonlinear scalarization of [5] and of [19]. Fourth, validating the framework on further *real-world instances*: Example 4.1

demonstrates feasibility on a stylized portfolio model; extensions to multi-asset, multi-period, or robust settings, as well as the study of locally Pareto optimal solutions (Definition 3.8) for non-convex feasible sets, remain open.

Competing interests. The authors declare no competing interests.

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