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ON GOULD-HOPPER-BASED TYPE 2 DEGENERATE POLY-FROBENIUS-EULER POLYNOMIALS

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ABSTRACT. In this paper, we introduce the Gould-Hopper-based type 2 degenerate poly-Frobenius-Euler polynomials and investigate some of its fundamental properties such as addition formula, explicit formula, implicit summation formulae, operational identities and some symmetric identities. We then establish connections of these polynomials with Stirling numbers of the first kind.

1. INTRODUCTION

The Frobenius-Euler polynomials of order α are defined through the following generating function (see [1,15]):

$$\left(\frac{1-u}{e^t-u}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x;u)\frac{t^n}{n!}, \qquad u \in \mathbb{C},\ \ u \neq 1. \tag{1.1}$$

When $x = 0$ in (1.1), $H_n^{(\alpha)}(0;u) = H_n^{(\alpha)}(u)$ gives the Frobenius-Euler numbers of order α. If, in addition, $\alpha = 1$, then $H_n^{(1)}(u) = H_n(u)$ gives the Frobenius-Euler numbers. On the other hand, setting $\alpha = 1$ in (1.1), the Frobenius-Euler polynomials, $H_n^{(1)}(x;u) = H_n(x;u)$ are obtained. Moreover, setting $u = -1$ yields the classical Euler polynomials, $H_n^{(1)}(x;-1) = E_n(x)$ (see [16,17]).

In 2024, a Type 2 poly-Frobenius-Euler polynomials were introduced by Corcino *et al.* in [7] via the following generating function:

$$\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e^t-u)} e^{xt} = \sum_{n=0}^{\infty} H_n^{(k)}(x;u)\frac{t^n}{n!},$$

where $\mathrm{Ei}_k(x)$ denotes the polyexponential function.

Polyexponential function, the inverse of the polylogarithm function, was introduced and defined in [14] through the following series:

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \qquad k \in \mathbb{Z}.$$

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For $k=1$,

$$\mathrm{Ei}_1(x)=\sum_{n=1}^{\infty}\frac{x^n}{n!}=e^x-1.$$

In 2015, Kim *et al.* [13] introduced the degenerate forms of Frobenius-Euler polynomials called degenerate Frobenius-Euler polynomials through the following generating function:

$$\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t)=\sum_{n=0}^{\infty}H_{n,\lambda}(x;u)\frac{t^n}{n!},\qquad \lambda\in\mathbb{R}\text{ (or }\mathbb{C}), \tag{1.2}$$

where $e_\lambda^x(t)$ is the degenerate exponential function [2] given by

$$e_\lambda^x(t)=(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \tag{1.3}$$

and $(x)_{n,\lambda}$ is the degenerate $\lambda-$falling factorial defined by (see [9,12])

$$(x)_{n,\lambda}=\begin{cases}x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), & n=1,2,\dots\\ 1, & n=0\end{cases},\qquad \lambda\in\mathbb{C}.$$

Setting $\lambda=1$, the λ-falling factorial reverts to the classical falling factorial $(x)_n$. Setting $x=0$ in (1.2), the degenerate Frobenius-Euler numbers $H_{n,\lambda}(u)$ were obtained.

In 2025, Corcino *et al.* [6] introduced the type 2 degenerate poly-Frobenius-Euler polynomials, denoted by $H_{n,\lambda}^{(k)}(x;u)$, using polyexponential function. For $\lambda,u\in\mathbb{C}$ with $u\neq 1$, the polynomials $H_{n,\lambda}^{(k)}(x;u)$ have the following generating function:

$$\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}e_\lambda^x(t)=\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}. \tag{1.4}$$

Setting $x=0$ in (1.4), the type 2 degenerate poly-Frobenius-Euler numbers, $H_{n,\lambda}^{(k)}(u)$,

$$\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}=\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(u)\frac{t^n}{n!}, \tag{1.5}$$

were obtained. In addition, setting $k=1$ yields the degenerate Frobenius-Euler polynomials $H_{n,\lambda}(x;u)$ [13]. As $\lambda\to 0$, (1.4) reduces to the type 2 poly-Frobenius-Euler polynomials $H_n^{(k)}(x;u)$ [7].

Another well-known family of numbers, the Stirling numbers of the first kind, $S_1(n,k)$, is given by (see [4–6])

$$(x)_n=\sum_{k=0}^{n}S_1(n,k)x^k,$$

and defined by the following exponential generating function:

$$\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}=\frac{[\log(1+t)]^k}{k},\qquad \text{(see [4–6,12])}.$$

Another type of polynomial that is widely investigated by many researchers is the one introduced by Gould and Hopper in 1962 which was defined through the following generating function [10]:

$$e^{xt+yt^j}=\sum_{n=0}^{\infty}\mathcal{G}_n^{(j)}(x,y)\frac{t^n}{n!},\qquad j\in\mathbb{Z}^+.$$

The degenerate version of the Gould-Hopper polynomials were then defined through the following generating function [8]:

$$e_\lambda^x(t)e_\lambda^y(t^j)=\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(j)}(x,y)\frac{t^n}{n!}\qquad j\in\mathbb{Z}^+. \tag{1.6}$$

As $\lambda\to 0$, (1.6) reduces to Gould-Hopper polynomials $\mathcal{G}_n^{(j)}(x,y)$ and setting $j=2$ gives the degenerate 2-variable Hermite Kampé de Fériet polynomials $\mathcal{G}_{n,\lambda}^{(2)}(x,y)$ (see [11]).

2. Gould-Hopper-Based Type 2 Degenerate poly-Frobenius-Euler Polynomials

Definition 2.1. For $k\in\mathbb{Z}$, $j\in\mathbb{Z}^+$, $n\geq 0$ and $u\neq 1$, the *Gould-Hopper-based type 2 degenerate poly-Frobenius-Euler polynomials* , denoted by ${}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x,y;u)$, are defined through the following generating function:

$$\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x,y;u)\frac{t^n}{n!}=\frac{\mathrm{Ei}_k\left(\log(1+(1-u)t)\right)}{t(e_\lambda(t)-u)}e_\lambda^x(t)e_\lambda^y(t^j), \tag{2.1}$$

where $\mathrm{Ei}_k(z)$ denotes the polyexponential function.

The Gould-Hopper-based degenerate Frobenius-Euler polynomials, which are obtained from (2.1) by setting $k=1$, are then defined by

$$\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(j)}(x,y;u)\frac{t^n}{n!}=\frac{(1-u)}{e_\lambda(t)-u}e_\lambda^x(t)e_\lambda^y(t^j). \tag{2.2}$$

A special case of (2.1) is obtained by setting $j=1$, namely,

$$\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,1)}(x,y;u)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{\mathrm{E\,i}_k\left(\log(1+(1-u)t)\right)}{t(e_\lambda(t)-u)}(x+y)_{n,\lambda}\frac{t^n}{n!}. \tag{2.3}$$

In the case of $j=2$, (2.1) reduces to the Hermite-based type 2 degenerate poly-Frobenius-Euler polynomials which are defined through the following generating function:

$$\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,2)}(x,y;u)\frac{t^n}{n!}=\frac{\mathrm{Ei}_k\left(\log(1+(1-u)t)\right)}{t(e_\lambda(t)-u)}e_\lambda^x(t)e_\lambda^y(t^2). \tag{2.4}$$

The difference operator $\triangle_{\lambda,x}$ with respect to x is defined as follows [3] :

$$\triangle_{\lambda,x}f(x)=\frac{1}{\lambda}\left[f(x+\lambda)-f(x)\right],\qquad \lambda\neq 0. \tag{2.5}$$

It is easy to see that $\triangle_{\lambda,x}e_\lambda^x(t)=te_\lambda^x(t)$ and $\triangle_{\lambda,y}e_\lambda^y(t^j)=t^je_\lambda^y(t^j)$. With this, we obtain the following difference rules.

Theorem 2.2. (Difference Rules). *For $k\in\mathbb{Z}$, $j\in\mathbb{Z}^+$ and $n\geq 0$,*

$$\triangle_{\lambda,x}\,{}_{\mathcal{G}}H_{n+1,\lambda}^{(k,j)}(x,y;u)=(n+1)\,{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x,y;u), \tag{2.6}$$

$$\triangle_{\lambda,y}\,{}_{\mathcal{G}}H_{n+j,\lambda}^{(k,j)}(x,y;u)=(n+j)_j\,{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x,y;u). \tag{2.7}$$

The following Lemma will be useful in the succeeding properties.

Lemma 2.3. [17] *For any positive integer j, the following elementary series manipulation holds:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/j]}A(k,n-jk), \tag{2.8}$$

where $[\cdot]$ denotes the largest integer which does not exceed the number in the square brackets.

Theorem 2.4. (Addition Formulas). For $n \geq 0$, we have

$$(2.9) \qquad (i) \quad {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x+v,y+w;u) = \sum_{m=0}^{n} \binom{n}{m} {}_{\mathcal{G}}H^{(k,j)}_{n-m,\lambda}(x,y;u)\mathcal{G}^{(j)}_{m,\lambda}(v,w),$$

$$(2.10) \qquad (ii) \quad {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x+v,y;u) = \sum_{m=0}^{n} \binom{n}{m} {}_{\mathcal{G}}H^{(k,j)}_{m,\lambda}(v,y;u)\,(x)_{n-m,\lambda}\,,$$

$$(2.11) \qquad (iii) \quad {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y+w;u) = \sum_{m=0}^{[n/j]} {}_{\mathcal{G}}H^{(k,j)}_{n-jm,\lambda}(x,w;u)\,(y)_{m,\lambda}\,\frac{n!}{m!(n-jm)!}.$$

Proof.

$$\begin{aligned}(i) \quad \sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x+v,y+w;u)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_k\left(\log(1+(1-u)t)\right)}{t(e_\lambda(t)-u)} e^x_\lambda(t)e^y_\lambda(t^j) \times e^v_\lambda(t)e^w_\lambda(t^j)\\
&= \sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}^{(j)}_{n,\lambda}(v,w)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} {}_{\mathcal{G}}H^{(k,j)}_{n-m,\lambda}(x,y;u)\mathcal{G}^{(j)}_{m,\lambda}(v,w)\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficients of $\frac{t^n}{n!}$ give the desired result.

In proving (ii) and (iii), we make use of identity (1.3).

$$\begin{aligned}(ii) \quad \sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x+v,y;u)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(v,y;u)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x)_{n,\lambda}\,\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} {}_{\mathcal{G}}H^{(k,j)}_{m,\lambda}(v,y;u)\,(x)_{n-m,\lambda}\,\frac{t^n}{n!}.\\
(iii) \quad \sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y+w;u)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,w;u)\frac{t^n}{n!}\sum_{m=0}^{\infty}(y)_{m,\lambda}\,\frac{(t^j)^m}{m!}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{[n/j]} {}_{\mathcal{G}}H^{(k,j)}_{n-jm,\lambda}(x,w;u)\,(y)_{m,\lambda}\,\frac{n!}{m!(n-jm)!}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\dfrac{t^n}{n!}$ completes the proof. $\square$

Corollary 2.5. *Setting $y=0$ in* (2.9), *we have*

$$(2.12) \qquad {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x+v,w;u) = \sum_{m=0}^{n}\binom{n}{m} H^{(k)}_{n-m,\lambda}(x;u)\mathcal{G}^{(j)}_{m,\lambda}(v,w).$$

Corollary 2.6. *Setting $j=2$ and $y=0$ in* (2.9), *the following holds:*

$$(2.13) \qquad {}_{\mathcal{G}}H^{(k,2)}_{n,\lambda}(x+v,w;u) = \sum_{m=0}^{n}\binom{n}{m} H^{(k)}_{n-m,\lambda}(x;u)\mathcal{H}_{m,\lambda}(v,w),$$

where $\mathcal{H}_{m,\lambda}(v,w)$ is the degenerate 2-variable Hermite Kampé de Fériet polynomials.

Corollary 2.7. *Setting $w=0$ in* (2.11), *we have*

$$(2.14) \qquad {}_{\mathcal{G}}H^{(k)}_{n,\lambda}(x,y;u) = \sum_{m=0}^{[n/j]} H^{(k)}_{n-jm,\lambda}(x;u)\,(y)_{m,\lambda}\,\frac{n!}{m!(n-jm)!}.$$

It can be seen from (1.5) and (1.6) that

$$\frac{\mathrm{Ei}_k\left(\log(1+(1-u)t)\right)}{t(e_\lambda(t)-u)}e^x_\lambda(t)e^y_\lambda(t^j)=\sum_{n=0}^{\infty}H^{(k)}_{n,\lambda}(u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}^{(j)}_{n,\lambda}(x,y)\frac{t^n}{n!}. \tag{2.15}$$

Application of the Cauchy product of power series in the RHS of (2.15) yields

$$\sum_{n=0}^{\infty}H^{(k)}_{n,\lambda}(u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}^{(j)}_{n,\lambda}(x,y)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}H^{(k)}_{n-m,\lambda}(u)\mathcal{G}^{(j)}_{m,\lambda}(x,y)\frac{t^n}{n!}.$$

Then comparing the coefficients of both sides completes the proof of the next theorem.

Theorem 2.8. (Explicit Summation Formula). *For* $n\geq 0$*,*

$$_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u)=\sum_{m=0}^{n}\binom{n}{m}H^{(k)}_{n-m,\lambda}(u)\mathcal{G}^{(j)}_{m,\lambda}(x,y), \tag{2.16}$$

Other explicit formulas were also obtained and given in the next theorem.

Theorem 2.9. *For* $n\geq 0$*,*

$$(i)\qquad {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u)=\sum_{m=0}^{[n/j]}H^{(k)}_{n-jm,\lambda}(x;u)\frac{\lambda^m n!}{m!(n-jm)!}\left(\frac{y}{\lambda}\right)_m,$$

$$\begin{aligned}(ii)\qquad {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u)&=\sum_{m=0}^{[n/j]}\sum_{p=0}^{n-jm}\binom{n-jm}{p}\left(\frac{x}{\lambda}\right)_{n-p-jm}H^{(k)}_{p,\lambda}(u)\\&\quad\times(\lambda)^{n-p-jm}\,\frac{(\lambda)^m\,n!}{m!(n-jm)!}\left(\frac{y}{\lambda}\right)_m.\end{aligned}$$

Proof. Using (1.5) and (1.3), we obtain

$$\begin{aligned}(i)\qquad \sum_{n=0}^{\infty}{}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}H^{(k)}_{n,\lambda}(x;u)\frac{t^n}{n!}\times(1+\lambda t^j)^{y/\lambda}\\&=\sum_{n=0}^{\infty}H^{(k)}_{n,\lambda}(x;u)\frac{t^n}{n!}\sum_{m=0}^{\infty}\left(\frac{y}{\lambda}\right)_m\lambda^m\frac{t^{jm}}{m!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{[n/j]}H^{(k)}_{n-jm,\lambda}(x;u)\frac{\lambda^m n!}{m!(n-jm)!}\left(\frac{y}{\lambda}\right)_m\frac{t^n}{n!}.\end{aligned} \tag{2.17}$$

Similarly, for (ii) we have

$$(ii)\ \sum_{n=0}^{\infty}{}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u)\frac{t^n}{n!}=\sum_{n=0}^{\infty}H^{(k)}_{n,\lambda}(u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\left(\frac{x}{\lambda}\right)_n\frac{(\lambda t)^n}{n!}\sum_{m=0}^{\infty}\left(\frac{y}{\lambda}\right)_m\frac{\left(\lambda t^j\right)^m}{m!}.$$

Application of Cauchy product on the summations gives

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\sum_{m=0}^{[n/j]}\sum_{p=0}^{n-jm}\binom{n-jm}{p}H^{(k)}_{p,\lambda}(u)\left(\frac{x}{\lambda}\right)_{n-p-jm}(\lambda)^{n-p-jm}\\&\quad\times\frac{(\lambda)^m\,n!}{m!(n-jm)!}\left(\frac{y}{\lambda}\right)_m\frac{t^n}{n!}.\end{aligned} \tag{2.18}$$

Comparing coefficients on both sides of (2.17) and (2.18) completes the proof. □

Corollary 2.10. *For* $j=2$ *in* (2.16)*, we have*

$$_{\mathcal{G}}H^{(k,2)}_{n,\lambda}(x,y;u)=\sum_{m=0}^{n}\binom{n}{m}H^{(k)}_{n-m,\lambda}(u)\mathcal{H}_{m,\lambda}(x,y). \tag{2.19}$$

Theorem 2.11. *For all integers $a>0, b>0$, $a\neq b$ and $n\geq 0$, the following symmetry identity holds:*

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}b^m a^{n-m}\,{}_{\mathcal{G}}H_{n-m,\lambda}^{(k,j)}(bx,b^jy;u)\,{}_{\mathcal{G}}H_{m,\lambda}^{(k,j)}(ax,a^jy;u)\\&\quad=\sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m}\,{}_{\mathcal{G}}H_{n-m,\lambda}^{(k,j)}(ax,a^jy;u)\,{}_{\mathcal{G}}H_{m,\lambda}^{(k,j)}(bx,b^jy;u).\end{aligned}\tag{2.20}$$

Proof. Let

$$\begin{aligned}\mathcal{A}(t)=&\frac{\mathrm{Ei}_k\left(\log(1+(1-u)at)\right)}{at(e_\lambda(at)-u)}\frac{\mathrm{Ei}_k\left(\log(1+(1-u)bt)\right)}{bt(e_\lambda(bt)-u)}e_\lambda^{ax}(bt)e_\lambda^{bx}(at)\\&\times e_\lambda^{a^jy}((bt)^j)e_\lambda^{b^jy}((at)^j).\end{aligned}$$

By rearranging the factors in $\mathcal{A}(t)$, we obtain

$$\begin{aligned}\mathcal{A}(t)&=\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(bx,b^jy;u)\frac{a^nt^n}{n!}\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(ax,a^jy;u)\frac{b^nt^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}b^ma^{n-m}\,{}_{\mathcal{G}}H_{n-m,\lambda}^{(k,j)}(bx,b^jy;u)\,{}_{\mathcal{G}}H_{m,\lambda}^{(k,j)}(ax,a^jy;u)\frac{t^n}{n!}.\end{aligned}\tag{2.21}$$

Also $A(t)$ can be expressed as

$$\begin{aligned}\mathcal{A}(t)&=\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(ax,a^jy;u)\frac{b^nt^n}{n!}\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(bx,b^jy;u)\frac{a^nt^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^mb^{n-m}\,{}_{\mathcal{G}}H_{n-m,\lambda}^{(k,j)}(ax,a^jy;u)\,{}_{\mathcal{G}}H_{m,\lambda}^{(k,j)}(bx,b^jy;u)\frac{t^n}{n!}.\end{aligned}\tag{2.22}$$

By equating (2.21) and (2.22) and comparing coefficients on both sides, we obtain the desired result. □

Corollary 2.12. *For $b=1$ in* (2.20), *we have*

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}a^{n-m}\,{}_{\mathcal{G}}H_{n-m,\lambda}^{(k,j)}(x,y;u)\,{}_{\mathcal{G}}H_{m,\lambda}^{(k,j)}(ax,a^jy;u)\\&\quad=\sum_{m=0}^{n}\binom{n}{m}a^m\,{}_{\mathcal{G}}H_{n-m,\lambda}^{(k,j)}(ax,a^jy;u)\,{}_{\mathcal{G}}H_{m,\lambda}^{(k,j)}(x,y;u).\end{aligned}\tag{2.23}$$

Theorem 2.13. *For all integers $a>0, b>0$, $a\neq b$ and $n\geq 0$, the following symmetry identity holds:*

$$\begin{aligned}&\sum_{p=0}^{n}\binom{n}{p}b^pa^{n-p}\,{}_{\mathcal{G}}H_{n-p,\lambda}^{(k,j)}(bx,b^jy;u)\sum_{m=0}^{[p/j]}H_{p-jm,\lambda}^{(k)}(ax;u)\left(a^jy\right)_{m,\lambda}\frac{p!}{(p-jm)!m!}\\&\quad=\sum_{p=0}^{n}\binom{n}{p}a^pb^{n-p}\,{}_{\mathcal{G}}H_{n-p,\lambda}^{(k,j)}(ax,a^jy;u)\sum_{m=0}^{[p/j]}H_{p-jm,\lambda}^{(k)}(bx;u)\left(b^jy\right)_{m,\lambda}\frac{p!}{(p-jm)!m!}.\end{aligned}\tag{2.24}$$

Proof. Let

$$\begin{aligned}\mathcal{D}(t)=&\frac{\mathrm{Ei}_k\left(\log(1+(1-u)at)\right)}{at(e_\lambda(at)-u)}e_\lambda^{bx}(at)e_\lambda^{b^jy}(a^jt^j)\frac{\mathrm{Ei}_k\left(\log(1+(1-u)bt)\right)}{bt(e_\lambda(bt)-u)}e_\lambda^{ax}(bt)\\&\times e_\lambda^{a^jy}(b^jt^j).\end{aligned}$$

By using (1.3), $\mathcal{D}(t)$ can be expressed as

$$\mathcal{D}(t)=\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(bx,b^jy;u)\frac{a^nt^n}{n!}\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(ax;u)\frac{b^nt^n}{n!}\sum_{m=0}^{\infty}\left(a^jy\right)_{m,\lambda}\frac{b^{jm}t^{jm}}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}b^p a^{n-p}\,{}_{\mathcal{G}}H_{n-p,\lambda}^{(k,j)}(bx,b^jy;u)\sum_{m=0}^{[p/j]}H_{p-jm,\lambda}^{(k)}(ax;u)$$

$$\times\left(a^jy\right)_{m,\lambda}\frac{p!}{(p-jm)!m!}\frac{t^n}{n!}. \tag{2.25}$$

In a similar manner, it can be shown that

$$\mathcal{D}(t) = \sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(ax,a^jy;u)\frac{b^nt^n}{n!}\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(bx;u)\frac{a^nt^n}{n!}\sum_{m=0}^{\infty}\left(b^jy\right)_{m,\lambda}\frac{a^{jm}t^{jm}}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}a^p b^{n-p}\,{}_{\mathcal{G}}H_{n-p,\lambda}^{(k,j)}(ax,a^jy;u)\sum_{m=0}^{[p/j]}H_{p-jm,\lambda}^{(k)}(bx;u)$$

$$\times\left(b^jy\right)_{m,\lambda}\frac{p!}{(p-jm)!m!}\frac{t^n}{n!}. \tag{2.26}$$

Therefore, by equating (2.25) and (2.26), we obtain the desired result. □

Theorem 2.14. *For $n \geq 0$, the following implicit summation formula holds:*

$${}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x,y;u) = \sum_{m=0}^{n}\binom{n}{m}\left(\frac{v}{\lambda}\right)_{n-m}{}_{\mathcal{G}}H_{m,\lambda}^{(k,j)}(x-v,y;u)\,(\lambda)^{n-m}. \tag{2.27}$$

Proof. Using (2.1), we have

$$\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x,y;u)\frac{t^n}{n!} = \frac{\mathrm{Ei}_k\left(\log(1+(1-u)t)\right)}{t(e_\lambda(t)-u)}e_\lambda^{x-v}(t)e_\lambda^{y}(t^j)e_\lambda^{v}(t)$$

$$= \sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x-v,y;u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\left(\frac{v}{\lambda}\right)_n(\lambda)^n\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}{}_{\mathcal{G}}H_{m,\lambda}^{(k,j)}(x-v,y;u)\left(\frac{v}{\lambda}\right)_{n-m}(\lambda)^{n-m}\frac{t^n}{n!}.$$

Comparing coefficients gives the desired result. □

The following identity [16] will be useful in the proof of the next theorem:

$$\sum_{m,n=0}^{\infty}f(m+n)\frac{x^m}{m!}\frac{y^n}{n!} = \sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}. \tag{2.28}$$

Theorem 2.15. *For $n \geq 0$, the following implicit summation formula holds:*

$${}_{\mathcal{G}}H_{m+n,\lambda}^{(k,j)}(w,y;u) = \sum_{l,p=0}^{m,n}\binom{m}{l}\binom{n}{p}(w-x)_{l+p,\lambda}\,{}_{\mathcal{G}}H_{m+n-l-p,\lambda}^{(k,j)}(x,y;u). \tag{2.29}$$

Proof. Applying (2.28), we have

$$\sum_{m,n=0}^{\infty}{}_{\mathcal{G}}H_{m+n,\lambda}^{(k,j)}(w,y;u)\frac{v^m}{m!}\frac{t^n}{n!} = \sum_{N=0}^{\infty}{}_{\mathcal{G}}H_{N,\lambda}^{(k,j)}(w,y;u)\frac{(v+t)^N}{N!}$$

$$= e_\lambda^{(w-x)}(v+t)\sum_{m,n=0}^{\infty}{}_{\mathcal{G}}H_{m+n,\lambda}^{(k,j)}(x,y;u)\frac{v^m}{m!}\frac{t^n}{n!}$$

$$= \sum_{l,p=0}^{\infty}(w-x)_{l+p,\lambda}\frac{v^l}{l!}\frac{t^p}{p!}\sum_{m,n=0}^{\infty}{}_{\mathcal{G}}H_{m+n,\lambda}^{(k,j)}(x,y;u)\frac{v^m}{m!}\frac{t^n}{n!}$$

$$= \sum_{m,n=0}^{\infty} \sum_{l,p=0}^{m,n} \binom{m}{l} \binom{n}{p} (w-x)_{l+p,\lambda} \, {}_{\mathcal{G}}H^{(k,j)}_{m+n-l-p,\lambda}(x,y;u) \frac{v^m}{m!} \frac{t^n}{n!}.$$

□

3. Connection with Stirling Numbers of the First Kind

In this section, we present some connections of Gould-Hopper-based type 2 degenerate poly-Frobenius-Euler polynomials with the Stirling numbers of the first kind.

Theorem 3.1. *For $n > 0$,*

$$ {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u) = \frac{1}{n+1} \sum_{r=0}^{n+1} \binom{n+1}{r} \sum_{m=1}^{r} S_1(r,m) \frac{(1-u)^{r-1}}{m^{k-1}} \, {}_{\mathcal{G}}H^{(j)}_{n+1-r,\lambda}(x,y;u). \tag{3.1}$$

Proof. It can be seen in [6] that

$$\begin{aligned} \mathrm{Ei}_k(\log(1+(1-u)t)) &= \sum_{m=1}^{\infty} \frac{(\log(1+(1-u)t))^m}{(m-1)!m^k} \\ &= \sum_{m=1}^{\infty} \left[\sum_{n=m}^{\infty} S_1(n,m)(1-u)^n \frac{t^n}{n!} \right] \frac{1}{m^{k-1}}. \end{aligned} \tag{3.2}$$

Thus,

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u) \frac{t^n}{n!} \\ &= \frac{1}{t} \sum_{m=1}^{\infty} \left[\sum_{n=m}^{\infty} S_1(n,m)(1-u)^n \frac{t^n}{n!} \right] \frac{1}{m^{k-1}} \cdot \frac{1}{1-u} \frac{1-u}{(e_\lambda(t)-u)} e^x_\lambda(t) e^y_\lambda(t^j) \\ &= \frac{1}{t} \sum_{n=0}^{\infty} \sum_{m=1}^{n} S_1(n,m) \frac{(1-u)^{n-1}}{m^{k-1}} \frac{t^n}{n!} \sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(j)}_{n,\lambda}(x,y;u) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \sum_{m=1}^{r} S_1(r,m) \frac{(1-u)^{r-1}}{m^{k-1}} \, {}_{\mathcal{G}}H^{(j)}_{n-r,\lambda}(x,y;u) \frac{t^{n-1}}{n(n-1)!}. \end{aligned} \tag{3.3}$$

Reindexing the RHS of (3.2) gives

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{n+1} \binom{n+1}{r} \sum_{m=1}^{r} \frac{1}{n+1} S_1(r,m) \frac{(1-u)^{r-1}}{m^{k-1}} \, {}_{\mathcal{G}}H^{(j)}_{n+1-r,\lambda}(x,y;u) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of both sides gives the desired result. □

Theorem 3.2. *For $n \geq 0$, the following relations holds:*

$$\begin{aligned} &\sum_{l=0}^{n} \binom{n}{l} (1)_{l,\lambda} \, {}_{\mathcal{G}}H^{(k,j)}_{n-l,\lambda}(x,y;u) - u \sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u) \\ &= \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} S_1(l+1,m+1) \frac{(1-u)^{l+1}}{(l+1)} \mathcal{G}^{(j)}_{n-l,\lambda}(x,y). \end{aligned} \tag{3.4}$$

Proof. Note that (2.1) can be written as

$$ t(e_\lambda(t)-u) \sum_{n=0}^{\infty} {}_{\mathcal{G}}H^{(k,j)}_{n,\lambda}(x,y;u) \frac{t^n}{n!} = \mathrm{Ei}_k\left(\log(1+(1-u)t)\right) e^x_\lambda(t) e^y_\lambda(t^j). \tag{3.5}$$

The LHS of (3.5) can be expressed as

$$LHS = t\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\binom{n}{l}(1)_{l,\lambda}\,{}_{\mathcal{G}}H_{n-l,\lambda}^{(k,j)}(x,y;u) - u\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x,y;u)\right]\frac{t^n}{n!}. \tag{3.6}$$

while the RHS as

$$\begin{aligned}
RHS &= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}S_1(n,m)\frac{(1-u)^n t^n}{n!}\frac{1}{m^{k-1}}\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(j)}(x,y)\frac{t^n}{n!}\\
&= \sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}\frac{1}{(m+1)^{k-1}}S_1(n,m+1)(1-u)^n\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(j)}(x,y)\frac{t^n}{n!}\\
&= t\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{(m+1)^{k-1}}S_1(n+1,m+1)\frac{(1-u)^{n+1}}{(n+1)}\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(j)}(x,y)\frac{t^n}{n!}\\
&= t\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{1}{(m+1)^{k-1}}S_1(l+1,m+1)\frac{(1-u)^{l+1}}{(l+1)}\mathcal{G}_{n-l,\lambda}^{(j)}(x,y)\frac{t^n}{n!}. \qquad (3.7)
\end{aligned}$$

The result follows after equating (3.6) and (3.7). □

Theorem 3.3. *For $n \geq 0$,*

$$\begin{aligned}
{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x,y;u) &= \sum_{r=0}^{\infty}\left(\frac{1}{-u}\right)(u)^{-r}\sum_{p=0}^{n}\binom{n}{p}\sum_{l=0}^{p}\binom{p}{l}\sum_{m=0}^{l}\frac{1}{(m+1)^{k-1}}S_1(l+1,m+1)\\
&\quad\times\frac{(1-u)^{l+1}}{(l+1)}(r)_{p-l,\lambda}\,\mathcal{G}_{n-p,\lambda}^{(j)}(x,y) \qquad (3.8)
\end{aligned}$$

Proof. By (2.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}{}_{\mathcal{G}}H_{n,\lambda}^{(k,j)}(x,y;u)\frac{t^n}{n!} &= \frac{1}{t}\mathrm{Ei}_k\left(\log(1+(1-u)t)\right)(e_\lambda(t)-u)^{-1}e_\lambda^x(t)e_\lambda^y(t^j)\\
&= \frac{1}{t}\sum_{n=0}^{\infty}\sum_{m=0}^{n}S_1(n+1,m+1)(1-u)^{n+1}\frac{t^{n+1}}{(n+1)!}\frac{1}{(m+1)^{k-1}}\\
&\quad\times\sum_{r=0}^{\infty}\left(\frac{1}{-u}\right)(u)^{-r}e_\lambda^r(t)\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(j)}(x,y)\frac{t^n}{n!}\\
&= \sum_{r=0}^{\infty}\left(\frac{1}{-u}\right)(u)^{-r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}S_1(n+1,m+1)\frac{(1-u)^{n+1}}{(n+1)}\frac{t^n}{n!}\frac{1}{(m+1)^{k-1}}\\
&\quad\times\sum_{n=0}^{\infty}(r)_{n,\lambda}\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(j)}(x,y)\frac{t^n}{n!}\\
&= \sum_{r=0}^{\infty}\left(\frac{1}{-u}\right)(u)^{-r}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}S_1(l+1,m+1)\frac{(1-u)^{l+1}}{(l+1)}\frac{1}{(m+1)^{k-1}}\\
&\quad\times(r)_{n-l,\lambda}\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(j)}(x,y)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\left(\frac{1}{-u}\right)(u)^{-r}\sum_{p=0}^{n}\binom{n}{p}\sum_{l=0}^{p}\binom{p}{l}\sum_{m=0}^{l}\frac{1}{(m+1)^{k-1}}S_1(l+1,m+1)\\
&\quad\times\frac{(1-u)^{l+1}}{(l+1)}(r)_{p-l,\lambda}\,\mathcal{G}_{n-p,\lambda}^{(j)}(x,y)\frac{t^n}{n!}.
\end{aligned}$$

The proof is complete. □

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