ISSN: 2832-4293

Pan-American Journal of Mathematics 5 (2026), 10
https://doi.org/10.28919/cpr-pajm/5-10
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A ROBUST APPROACH FOR FUZZY MULTIOBJECTIVE LINEAR OPTIMIZATION VIA BANACH SPACE EMBEDDING AND OPTIMAL TRANSFORMATION TECHNIQUE

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ABSTRACT. This paper proposes an innovative approach to solving fuzzy multiobjective linear optimization problems by combining the embedding theorem and an optimal transformation technique. The objective functions and fuzzy constraints are first embedded in a Banach space, where they are represented as parametric biobjective vectors. The application of a Riemann integral operator then allows them to be transformed into a deterministic and non-parametric form. Appropriate weights are introduced in order to limit compensatory effects and to ensure a non-empty admissible domain, leading to a weighted deterministic multiobjective problem of the same dimension. This problem is then aggregated into a single-objective nonlinear model, solvable by classical methods. The proposed transformations ensure the equivalence of the models and the preservation of Pareto optimality. The numerical results show an improvement in robustness and performance, with balanced and realistic solutions.

1. Introduction

Decision-making in a fuzzy environment represents a major challenge in multiobjective optimization, mainly due to the uncertainty and imprecision inherent in complex systems. Since the seminal work of Bellman and Zadeh [3] on fuzzy decision making and the theoretical developments of Zadeh [6–8] on linguistic variables, fuzzy optimization has become an essential tool for modeling conflicting criteria under ambiguous conditions (Klir and Yuan [5], de Barros et al. [4]). To solve such problems, the literature has explored various strategies for transformation into deterministic models. Classical approaches include the use of deviation degree measures (Cheng et al. [10]), weighted max-min methods (Dong et al. [11, 28]), or possibilistic programming (Iskander [17]). More recently, researchers have proposed methods based on interval approximation (Sharma and Aggarwal [33]) or statistical approaches for objective aggregation (Nahar et al. [16]).

A major theoretical advance lies in the use of the embedding theorem in Banach spaces. Originally introduced by Puri and Ralescu [1] and re-examined by Wu [19, 27], Ma [2] and Solatikia [18], this concept allows the fuzzy number space to be projected into a deterministic functional framework, thus facilitating the use of classical real analysis. This framework made it possible to define rigorous Karush-Kuhn-Tucker

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Submitted on Apr. 8, 2026.

2020 *Mathematics Subject Classification.* Primary 90C29, 90C70; Secondary 46E15, 03E72.

Key words and phrases. fuzzy optimization; multiobjective optimization; embedding theorem; optimal transformation; Pareto optimality.

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(KKT) optimality conditions for fuzzy-valued functions (Wu [15]) and advanced scalarization concepts (Wu [14]). However, despite these advances, several limitations remain in the existing literature. On the one hand, statistical aggregation methods (Nahar et al. [16]) or harmonic mean methods (Fathya and Hassanien [31]) can dilute the influence of certain critical objectives through excessive simplification of the data. On the other hand, the transition to an exploitable Pareto-optimal solution often remains computationally heavy, particularly for large-scale problems (Lodwick [21], Nayak and Ojha [23]). Ultimately, few methods allow the decision-maker to finely adjust the robustness of the system with respect to uncertainties in the lower and upper bounds simultaneously.

In this context, this paper proposes an innovative hybrid approach called RFMOOA (Robust Fuzzy Multi-Objective Optimization Algorithm). Unlike existing works (Wu [19], Hernandez-Jimenez et al. [20]), our contribution is distinguished by the systematic use of the Riemann integral operator for precise defuzzification, coupled with an optimal transformation technique O_{TT} based on quadratic aggregation that ensures the generation of dense and well-distributed Pareto fronts. This model introduces a tuning lever between nominal performance and structural safety through the assignment of specific weights to the model's bounds.

The remainder of this paper is organized as follows: Section 2 presents the preliminaries on fuzzy numbers and the embedding theorem. Section 3 details the formulation of our approach, including the theoretical proofs of equivalence and optimality, as well as the numerical results. Finally, Section 4 concludes this work.

2. Preliminaries

2.1. **Fuzzy numbers.** This part presents the notion of fuzzy numbers and the embedding theorem.

Definition 2.1. [3, 4, 6–8, 11, 20]

Let $\mathcal{X}$ be a set. A fuzzy subset $\tilde{a}$ of $\mathcal{X}$ is characterized by a membership function $\mu_{\tilde{a}} : \mathcal{X} \to [0, 1]$ and represented by a set of ordered pairs defined as follows :

$$\tilde{a} = \{(x, \mu_{\tilde{a}}(x))/x \in \mathcal{X}\}. \tag{2.1}$$

The value $\mu_{\tilde{a}}(x) \in [0, 1]$ represents the degree of membership of x to the fuzzy set and is interpreted as the extent to which x belongs to $\tilde{a}$.

Definition 2.2. [3, 4, 20] Let $\tilde{a}$ be a fuzzy set on $\mathcal{X}$ and $\alpha \in [0, 1]$. The $\alpha - level$ set of $\tilde{a}$ is the classical set noted $\tilde{a}_\alpha$ and is defined by:

$$\tilde{a}_\alpha = \{x \in \mathcal{X}, \mu_{\tilde{a}}(x) \geq \alpha\}. \tag{2.2}$$

We will identify $\mathcal{X}$ to $\mathbb{R}$.

Definition 2.3. [4, 5, 13, 25] Let $\tilde{a}$ a fuzzy subset of $\mathbb{R}$. Then $\tilde{a}$ is called a fuzzy number if the following conditions are satisfied:

(1) $\tilde{a}$ is normal, i.e, $\mu_{\tilde{a}}(x) = 1$, for some $x \in \mathbb{R}$;
(2) $\tilde{a}$ is convex, i.e., the membership function $\mu_{\tilde{a}}(x)$ is quasi-concave;
(3) $\mu_{\tilde{a}}(x)$ is upper semicontinuous, i.e., $\tilde{a}_\alpha$ is a closed subset of $\mathbb{R}$ for some $\alpha \in \mathbb{R}$;
(4) the 0-level set $\tilde{a}_0$, is a compact subset of $\mathbb{R}$.

We note by $\mathcal{F}(\mathbb{R})$, the set of all fuzzy numbers. Indeed, if $\tilde{a} \in \mathcal{F}(\mathbb{R})$, the α-level set of $\tilde{a}$ is a compact and convex subset of $\mathbb{R}$, then $\tilde{a}_\alpha$ is a closed interval, denoted $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ for $\alpha \in [0, 1]$. Indeed, we write $\tilde{a} \preceq \tilde{b}$ if and only if $\tilde{a}^L_\alpha \leq \tilde{b}^L_\alpha$ and $\tilde{a}^U_\alpha \leq \tilde{b}^U_\alpha$ for all $\alpha \in [0, 1]$. Then $\preceq$ is a partial ordering on $\mathcal{F}(\mathbb{R})$. On the other hand, we say that $\tilde{a} \prec \tilde{b}$ if and only if for all α,

$$\begin{cases} \tilde{a}^L_\alpha < \tilde{b}^L_\alpha \\ \tilde{a}^U_\alpha \leq \tilde{b}^U_\alpha \end{cases} \quad \text{or} \quad \begin{cases} \tilde{a}^L_\alpha \leq \tilde{b}^L_\alpha \\ \tilde{a}^U_\alpha < \tilde{b}^U_\alpha \end{cases} \quad \text{or} \quad \begin{cases} \tilde{a}^L_\alpha < \tilde{b}^L_\alpha \\ \tilde{a}^U_\alpha < \tilde{b}^U_\alpha \end{cases}$$

Remark 2.4. If $\tilde{a} \in \mathcal{F}(\mathbb{R})$, then $\tilde{a}^L_\alpha$ is increasing with respect to α and $\tilde{a}^U_\alpha$ is decreasing with respect to α.

Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$ and $\odot$ be any binary operation $\oplus$, $\ominus$, or $\otimes$ between $\tilde{a}$ and $\tilde{b}$. The membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\xi_{\tilde{a}\odot\tilde{b}}(z) = \sup_{xy=z} \min\left\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\right\}$$

where $\odot = \oplus$, $\ominus$, or $\otimes$ and $\cdot = +$, $-$, or $\times$ according to the extension principle proposed by Zadeh [?, 3]. Then, we have the following well-known result.

Definition 2.5. [4, 5, 12, 20] Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$ of α-levels respectively $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ and $\tilde{b}_\alpha = [\tilde{b}^L_\alpha, \tilde{b}^U_\alpha]$ for $\alpha \in$ [0,1]. Then, we have:

(1) $\tilde{a} \oplus \tilde{b} \in \mathcal{F}(\mathbb{R})$ and

$$\begin{aligned}(\tilde{a} \oplus \tilde{b})_\alpha &= \tilde{a}_\alpha \oplus \tilde{b}_\alpha \\ &= [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{b}^U_\alpha + \tilde{b}^U_\alpha],\end{aligned}$$

(2) $\tilde{a} \ominus \tilde{b} \in \mathcal{F}(\mathbb{R})$ and

$$\begin{aligned}(\tilde{a} \ominus \tilde{b})_\alpha &= (\tilde{a} \oplus (-\tilde{b}))_\alpha \\ &= \tilde{a}_\alpha \oplus (-\tilde{b})_\alpha \\ &= [\tilde{a}^L_\alpha - \tilde{b}^U_\alpha, \tilde{a}^U_\alpha - \tilde{b}^L_\alpha]\end{aligned}$$

(3) $\lambda\tilde{a} \in \mathcal{F}(\mathbb{R})$ and

$$(\lambda\tilde{a})_\alpha = \begin{cases} [\lambda\tilde{a}^L_\alpha, \lambda\tilde{a}^U_\alpha] \ si \ \lambda \geq 0, \\ [\lambda\tilde{a}^U_\alpha, \lambda\tilde{a}^L_\alpha] \ si \ \lambda < 0. \end{cases} .$$

We say that $\tilde{a}$ is a crisp number with value m if its membership function is given by

$$\mu_{\tilde{a}}(r) = \begin{cases} 1 \ \text{si} \ \ r = m \\ 0 \ \text{otherwise.} \end{cases} \tag{2.3}$$

We also use the notation $\tilde{1}_m$ to represent the crisp number with value m. So, we see that $\left(\tilde{1}_m\right)^L(\alpha) = \left(\tilde{1}_m\right)^U(\alpha) = m$ pour tout $\alpha \in [0, 1]$. Let us remark that a real number m can be regarded as a crisp number $\tilde{1}_m$.

Now we are going to embed $\mathcal{F}(\mathbb{R})$ into a Banach space isometrically and isomorphically. Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, The Hausdorff metric is defined by

$$d_H(A, B) = \max\left\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \ \sup_{b\in B} \inf_{a\in A} \|a - b\|\right\} \tag{2.4}$$

According to Puri and Ralescu [1], we define the metric $d_{\mathcal{F}}$ in $\mathcal{F}(\mathbb{R})$ as

$$d_{\mathcal{F}}(\tilde{a}, \tilde{b}) = \sup_{0\leq\alpha\leq1} d_H(\tilde{a}_\alpha, \tilde{b}_\alpha)$$

where

$$d_H(\tilde{a}_\alpha, \tilde{b}_\alpha) = \max\left\{\left|\tilde{a}^L_\alpha - \tilde{b}^L_\alpha\right|, \left|\tilde{a}^U_\alpha - \tilde{b}^U_\alpha\right|\right\}.$$

The space $\overline{C}[0,1]$ is the set of all real-valued bounded functions f on $[0,1]$ such that f is left-continuous for any $t \in [0,1]$ and right-continuous at 0, and f has a right limit for any $t \in [0,1)$. Then, $\left(\overline{C}[0,1], ||\cdot||\right)$ is a Banach space with the norm defined by

$$||f|| = \sup_{t\in[0,1]} |f(t)|\,.$$

Furthermore, $\left(\overline{C}[0,1] \times \overline{C}[0,1], ||\cdot||\right)$ is also a Banach space with the norm defined by

$$||(f,g)|| = \max\{||f||\,, ||g||\}\,,$$

where $(f,g) \in \overline{C}[0,1] \times \overline{C}[0,1]$ (see Wu [19] and Ma [2]). Let's put $\tilde{a}^L_\alpha = a^L(\alpha)$ and $\tilde{a}^U_\alpha = a^U(\alpha)$ functions of $\alpha \in [0,1]$. Then we can define the embedding function $\eta : \mathcal{F}(\mathbb{R}) \to \overline{C}[0,1] \times \overline{C}[0,1]$ by $\eta\left(a^L(\alpha), a^U(\alpha)\right)$. Then the embedding theorem is presented below:

Theorem 2.6. *(Embedding theorem [2, 24]) The function $\eta : \mathcal{F}(\mathbb{R}) \to \overline{C}[0,1] \times \overline{C}[0,1]$ is defined by $\eta(\tilde{a}) = \left(a^L(\alpha), a^U(\alpha)\right)$. Then, the following properties hold true*

(1) *η is injective*
(2) $\eta\left((\tilde{1}_{\{s\})} \otimes \tilde{a}) \oplus (\tilde{1}_{\{t\}} \otimes \tilde{b})\right) = s\eta(\tilde{a}) + t\eta(\tilde{b})$ *for all* $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$, $s \geq 0$ *and* $t \geq 0$
(3) $d_{\mathcal{F}}(\tilde{a}, \tilde{b}) = ||\ \eta(\tilde{a}) - \eta(\tilde{b})\ ||$

That is to say, $\tilde{a} \in \mathcal{F}(\mathbb{R})$ can be embedded into $\overline{C}[0,1] \times \overline{C}[0,1]$ isometrically and isomorphically.

Proof. see [2,18,24] □

The above theorem says that each element in $\mathcal{F}(\mathbb{R})$ can be regarded as an element in $\overline{C}[0,1]\times\overline{C}[0,1]$,more precisely $\tilde{a}$ in $\mathcal{F}(\mathbb{R})$ can be identified with an element $\left(a^L(\alpha), a^U(\alpha)\right)$ in $\overline{C}[0,1] \times \overline{C}[0,1]$, where $a^L(\alpha) = \tilde{a}^L_\alpha$ and $a^U(\alpha) = \tilde{a}^U_\alpha$ and this identication is isometric and isomorphic. The function η is called the defuzzification function.

Proposition 2.7. *Let η be the function defined in Theorem 2.6 and let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. Then*

$$\tilde{a} \preceq \tilde{b} \iff \eta(\tilde{a}) \leq \eta(\tilde{b}), \quad \textit{and} \quad \tilde{a} \prec \tilde{b} \iff \eta(\tilde{a}) < \eta(\tilde{b}).$$

Proof. From the definition of $\preceq$, we have

$$\tilde{a} \preceq \tilde{b} \iff a^L(\alpha) \leq b^L(\alpha) \text{ and } a^U(\alpha) \leq b^U(\alpha),\ \forall\alpha \in [0,1].$$

This is equivalent to $\eta(\tilde{a}) \leq \eta(\tilde{b})$.

The strict case follows analogously. □

2.2. **Formulation of problem.** A fuzzy multi objective linear optimization problem is defined as follows:

$$\begin{aligned} &\max\ \tilde{f}(x) = \left(\tilde{f}_1(x), \tilde{f}_2(x), \ldots, \tilde{f}_p(x)\right) \\ &\quad \text{s.t.} \quad x \in \tilde{\Omega} = \left\{x \in \mathbb{R}^n_+ : \tilde{g}_j(x) \preceq \tilde{0},\ j = 1, \ldots, m\right\}. \end{aligned} \tag{2.5}$$

where $\tilde{\Omega}$ is a nonempty compact feasible set, and $\tilde{f}_i$ $(i = 1, \ldots, p)$ and $\tilde{g}_j$ $(j = 1, \ldots, m)$ are linear functions with fuzzy coefficients.

Definition 2.8. [15] A point $\overline{x} \in \tilde{\Omega}$ is Pareto optimal for (2.5) if there is no $x \in \tilde{\Omega}$ such that

$$\tilde{f}_i(\overline{x}) \preceq \tilde{f}_i(x)\ \forall i, \quad \text{and} \quad \tilde{f}_j(\overline{x}) \prec \tilde{f}_j(x) \text{ for some } j.$$

3. Main results

This section is devoted to the presentation of our new approach and didactic example.

3.1. **New approach.** Our approach is based on the use of embedding theorem and optimal transformation technique that can be summarized in five main step as follows :

Step 1 :	`Original problem`. Consider a fuzzy linear multiobjective optimization problem with given fuzzy data.
Step 2 :	`Defuzzification`. Convert the original fuzzy multiobjective optimization problem into an equivalent deterministic multiobjective optimization problem using the embedding theorem.
Step 3:	`Optimal transformation`. Apply an optimal transformation technique based on normalization by individual optimal values and a non-linear transformation to convert the multi-objective problem into a single-objective problem
Step 4 :	`Resolution`. Solve the resulting non-linear single-objective problem using an appropriate optimization method, such as quadratic programming or numerical techniques.
Step 5 :	`Validation`. The final result is a Pareto-optimal solution (or best compromise solution) for the original fuzzy problem..

Original problem.: Consider problem (2.5) with fuzzy coefficients.

Defuzzification.: Let η be the defuzzification function defined in Theorem 2.6. Then, we consider the following optimization problem by applying the embedding function η to problem (2.5):

$$\begin{aligned} &\max \eta(\tilde{f}(x)) = \left(\eta(\tilde{f}_1(x)), \eta(\tilde{f}_2(x)), ..., \eta(\tilde{f}_p(x))\right), \\ &st \\ &\eta(\tilde{g}_j(x)) \preceq \eta(\tilde{0}), \quad j = 1, ..., m, \\ &x \in \mathbb{R}^n_+. \end{aligned} \tag{3.1}$$

According to problems (3.1), we consider the following parametric multi-objective optimization problem:

$$\begin{aligned} &\max \left(f_1^L(x,\alpha), f_1^U(x,\alpha)\right), \\ &\max \left(f_2^L(x,\alpha), f_2^U(x,\alpha)\right), \\ &\vdots \\ &\max \left(f_p^L(x,\alpha), f_p^U(x,\alpha)\right), \\ &st \\ &\left(g_j^L(x,\alpha), g_j^U(x,\alpha)\right) \leq \left(0, 0\right) \quad j = 1, ..., m, \\ &0 \leq \alpha \leq 1 \\ &x \in \mathbb{R}^n_+. \end{aligned} \tag{3.2}$$

Let $S^\alpha = \{x \in \mathbb{R}^n_+ : \left(g_j^L(x,\alpha), g_j^U(x,\alpha)\right) \leq \left(0, 0\right), \alpha \in (0,1), j = 1, ..., m\}$

Proposition 3.1. *A feasible solution $\overline{x}$ is called an $\overline{\alpha}$-Pareto optimal solution of problem (3.2) if there exists no $x \in S^\alpha$, $x \neq \overline{x}$, such that, for some $\overline{\alpha} \in [0,1]$:*

$$\begin{aligned} &f_i^L(\overline{x}, \overline{\alpha}) \leq f_i^L(x, \overline{\alpha}), \\ &f_i^U(\overline{x}, \overline{\alpha}) \leq f_i^U(x, \overline{\alpha}), \quad \forall i = 1, \ldots, p, \end{aligned} \tag{3.3}$$

and

$$f_j^L(\overline{x},\overline{\alpha}) < f_j^L(x,\overline{\alpha}),$$
$$f_j^U(\overline{x},\overline{\alpha}) < f_j^U(x,\overline{\alpha}), \quad \textit{for at least one } j \in \{1,\ldots,p\}. \tag{3.4}$$

Proof. Let $\overline{x}$ be a Pareto optimal solution of problem (2.5), i.e., there does not exist another feasible solution $x \in \tilde{\Omega}$ such that

$$\tilde{f}_i(\overline{x}) \preceq \tilde{f}_i(x), \quad \forall i, \quad \text{and} \quad \tilde{f}_i(\overline{x}) \prec \tilde{f}_i(x) \quad \text{for at least one } j.$$

Applying the function η (see Proposition 2.7), this is equivalent to

$$\eta(\tilde{f}_i(\overline{x})) \leq \eta(\tilde{f}_i(x)), \quad \forall i, \quad \text{and} \quad \eta(\tilde{f}_i(\overline{x})) < \eta(\tilde{f}_i(x)) \quad \text{for at least one } j. \tag{3.5}$$

By definition of η, Equation (3.5) can be rewritten as

$$\big(f_i^L(\overline{x},\alpha), f_i^U(\overline{x},\alpha)\big) \leq \big(f_i^L(x,\alpha), f_i^U(x,\alpha)\big), \quad \forall i,$$
$$\big(f_j^L(\overline{x},\alpha), f_j^U(\overline{x},\alpha)\big) < \big(f_j^L(x,\alpha), f_j^U(x,\alpha)\big), \quad \text{for at least one } j,$$

for all $\alpha \in [0,1]$.

Fixing $\alpha = \overline{\alpha} \in [0,1]$, we obtain

$$\begin{cases} f_i^L(\overline{x},\overline{\alpha}) \leq f_i^L(x,\overline{\alpha}), \\ f_i^U(\overline{x},\overline{\alpha}) \leq f_i^U(x,\overline{\alpha}), \end{cases} \quad \forall i, \quad \text{and} \quad \begin{cases} f_j^L(\overline{x},\overline{\alpha}) < f_j^L(x,\overline{\alpha}), \\ f_j^U(\overline{x},\overline{\alpha}) < f_j^U(x,\overline{\alpha}), \end{cases} \quad \text{for at least one } j. \tag{3.6}$$

Hence, for any Pareto optimal solution $\overline{x}$ of problem (3.2) and fixed $\overline{\alpha} \in [0,1]$, Equation (3.6) holds.

Therefore, we conclude that $\overline{x}$ is also an $\overline{\alpha}$-Pareto optimal solution. □

Theorem 3.2. *A feasible solution $\overline{x} \in S^\alpha$ is an $\overline{\alpha}$-Pareto optimal solution of problem (3.2) for $\overline{\alpha} \in [0,1]$ if and only if $\overline{x}$ is a Pareto optimal solution of problem (2.5).*

Proof. **(If part)** Let $\overline{x} \in S^\alpha$ be an $\overline{\alpha}$-Pareto optimal solution of problem (3.2), i.e., there exists no $x \in S^\alpha$ such that

$$f_i^L(\overline{x},\overline{\alpha}) \leq f_i^L(x,\overline{\alpha}), \quad f_i^U(\overline{x},\overline{\alpha}) \leq f_i^U(x,\overline{\alpha}), \quad \forall i = 1,\ldots,p,$$

with strict inequality for at least one index j.

Suppose, for contradiction, that $\overline{x}$ is not a Pareto optimal solution of problem (2.5). Then there exists $x \in \tilde{\Omega}$ such that

$$\tilde{f}_i(\overline{x}) \preceq \tilde{f}_i(x), \quad \forall i, \quad \text{and} \quad \tilde{f}_j(\overline{x}) \prec \tilde{f}_j(x) \quad \text{for at least one } j.$$

Applying the function η (Proposition 2.7), we obtain

$$\eta(\tilde{f}_i(\overline{x})) \leq \eta(\tilde{f}_i(x)), \quad \forall i, \quad \eta(\tilde{f}_j(\overline{x})) < \eta(\tilde{f}_j(x)) \quad \text{for at least one } j.$$

By the definition of η, this implies

$$f_i^L(\overline{x},\overline{\alpha}) \leq f_i^L(x,\overline{\alpha}), \quad f_i^U(\overline{x},\overline{\alpha}) \leq f_i^U(x,\overline{\alpha}), \quad \forall i,$$

and

$$f_j^L(\overline{x},\overline{\alpha}) < f_j^L(x,\overline{\alpha}), \quad f_j^U(\overline{x},\overline{\alpha}) < f_j^U(x,\overline{\alpha}), \quad \text{for at least one } j,$$

which contradicts the assumption that $\overline{x}$ is an $\overline{\alpha}$-Pareto optimal solution. Therefore, $\overline{x}$ must be a Pareto optimal solution of problem (2.5).

(Only if part) Conversely, let $\overline{x} \in \tilde{\Omega}$ be a Pareto optimal solution of problem (2.5). Suppose, for contradiction, that $\overline{x}$ is not an $\overline{\alpha}$-Pareto optimal solution of problem (3.2). Then there exists $x \in S^{\alpha}$ such that

$$f_i^L(\overline{x}, \overline{\alpha}) \leq f_i^L(x, \overline{\alpha}), \quad f_i^U(\overline{x}, \overline{\alpha}) \leq f_i^U(x, \overline{\alpha}), \quad \forall i,$$

with strict inequality for at least one index j.

By the definition of η, this implies

$$\eta(\tilde{f}_i(\overline{x})) \leq \eta(\tilde{f}_i(x)), \quad \forall i, \quad \eta(\tilde{f}_j(\overline{x})) < \eta(\tilde{f}_j(x)) \quad \text{for at least one } j.$$

According to Proposition 2.7, this means

$$\tilde{f}_i(\overline{x}) \preceq \tilde{f}_i(x), \quad \forall i, \quad \tilde{f}_j(\overline{x}) \prec \tilde{f}_j(x) \quad \text{for at least one } j,$$

which contradicts the assumption that $\overline{x}$ is Pareto optimal for problem (2.5).

Hence, $\overline{x}$ is an $\overline{\alpha}$-Pareto optimal solution of problem (3.2) for all $\overline{\alpha} \in [0, 1]$. □

In order to proceed, we first recall the following classical result from real analysis.

Proposition 3.3. *[9] Every function that is monotonic on a closed interval is Riemann integrable on that interval.*

Indeed, the functions f_i^L and f_i^U, for $i = 1, \ldots, p$, are respectively increasing and decreasing on $[0, 1]$, i.e., they are monotonic on $[0, 1]$. Therefore, by Proposition 3.3, they are Riemann integrable over $[0, 1]$, and we can define

$$F_i^L(x) = \int_0^1 f_i^L(x, \alpha)\, d\alpha, \quad i = 1, \ldots, p, \tag{3.7}$$

$$F_i^U(x) = \int_0^1 f_i^U(x, \alpha)\, d\alpha, \quad i = 1, \ldots, p. \tag{3.8}$$

Remark 3.4. According to Proposition 2.7, each fuzzy constraint $\tilde{g}_j(x) \preceq \tilde{0}$, for $j = 1, \ldots, m$, implies that

$$\eta(\tilde{g}_j(x)) \leq \eta(\tilde{0}), \quad j = 1, \ldots, m.$$

Equivalently, for all $\alpha \in [0, 1]$,

$$g_j^L(x, \alpha) \leq 0 \quad \text{and} \quad g_j^U(x, \alpha) \leq 0.$$

Since $g_j^L(x, \alpha) \leq g_j^U(x, \alpha)$, considering only the upper bound may lead to an empty feasible set when α approaches 1, even if the original fuzzy feasible set is non-empty.

To avoid this issue, we consider the average over all levels of α and introduce a weighting of the bounds.

Moreover, g_j^L is increasing and g_j^U is decreasing over $[0, 1]$, which ensures that both functions are Riemann integrable. The corresponding integrals are defined as

$$G_j^L(x) = \int_0^1 g_j^L(x, \alpha)\, d\alpha, \quad G_j^U(x) = \int_0^1 g_j^U(x, \alpha)\, d\alpha, \quad j = 1, \ldots, m. \tag{3.9}$$

To guarantee that the defuzzified feasible set remains non-empty whenever the original fuzzy feasible set is non-empty, we define the scalar weighted constraints by

$$G_j^{\omega}(x) = \omega_1 G_j^U(x) + \omega_2 G_j^L(x) \leq 0, \quad j = 1, \ldots, m, \tag{3.10}$$

with $\omega_1, \omega_2 \in [0, 1]$ and $\omega_1 + \omega_2 = 1$.

Thus, the defuzzified feasible set can be written as

$$S^{\omega} = \left\{ x \in \mathbb{R}_+^n : G_j^{\omega}(x) \leq 0,\ j = 1, \ldots, m \right\}.$$

Finally, the fuzzy objective function can also be aggregated using the same weights to obtain a scalar weighted objective:

$$F_i^{\omega}(x) = \omega_1 F_i^U(x) + \omega_2 F_i^L(x), \quad i = 1, \ldots, p.$$

Using Remark 3.4, problem (3.2) can thus be reformulated as a single-objective optimization problem.

$$\begin{aligned} &\max_{x \in \mathbb{R}_+^n} F_i^{\omega}(x) = \Big(\omega_1 F_i^L(x) + \omega_2 F_i^U(x)\Big) \quad i = 1, \ldots, p, \\ &\text{s.t. } G_j^{\omega}(x) \leq 0, \quad j = 1, \ldots, m, \end{aligned} \tag{3.11}$$

where $\omega = (\omega_1, \omega_2) \in [0,1]^2$ are positive weights for the bi-objective functions such that $\omega_1 + \omega_2 = 1$.

Remark 3.5. Problems (3.2) and (3.11) have the same feasible set for all $\alpha \in [0,1]$.

Proof. Let us consider the parametric multi-objective optimization problem (3.2), defined for each $\alpha \in [0,1]$, subject to the following constraints:

$$g_j^L(x,\alpha) \leq 0 \quad \text{and} \quad g_j^U(x,\alpha) \leq 0, \quad j = 1, \ldots, m.$$

We denote by the feasible set of this parametric problem:

$$S^{\alpha} = \{x \in \mathbb{R}_+^n : \big(g_j^L(x,\alpha), g_j^U(x,\alpha)\big) \leq \big(0,0\big) \ \ j = 1, \ldots, m\}.$$

To convert this problem into a scalar-constrained optimization problem, we consider the weighted integrals of the parametric functions:

$$G_j^{\omega}(x) = \omega_1 \int_0^1 g_j^U(x,\alpha)\, d\alpha + \omega_2 \int_0^1 g_j^L(x,\alpha)\, d\alpha, \quad j = 1, \ldots, m,$$

avec $\omega_1, \omega_2 \in [0,1]$ et $\omega_1 + \omega_2 = 1$. The associated feasible set is given by:

$$S^{\omega} = \{x \in \mathbb{R}_+^n : G_j^{\omega}(x) \leq 0,\ j = 1, \ldots, m\}.$$

Step 1: $S^{\alpha} \subseteq S^{\omega}$.

If $x \in S^{\alpha}$, $\forall\, \alpha \in [0,1]$, then $g_j^L(x,\alpha) \leq 0$ and $g_j^U(x,\alpha) \leq 0$, $\forall\, \alpha$. By integrating over $[0,1]$ and applying a weighted combination, we obtain:

$$G_j^{\omega}(x) = \omega_1 \int_0^1 g_j^U(x,\alpha)\, d\alpha + \omega_2 \int_0^1 g_j^L(x,\alpha)\, d\alpha \leq 0,$$

hence $x \in S^{\omega}$.

Step 2: $S^{\omega} \subseteq S^{\alpha}$.

Conversely, let us assume that $x \in S^{\omega}$, i.e. $G_j^{\omega}(x) \leq 0$. Due to the monotonicity of the parametric functions, where g_j^U is decreasing and g_j^L is increasing on $[0,1]$, the integral values correspond to the extremal values of the constraints:

- For g_j^U, being decreasing, the integral value is greater than or equal to all its pointwise values $g_j^U(x,\alpha)$, therefore
$$g_j^U(x,\alpha) \le G_j^\omega(x) \le 0$$
.
- For g_j^L, being increasing, the integral value is less than or equal to all its pointwise values $g_j^L(x,\alpha)$, therefore
$$g_j^L(x,\alpha) \le G_j^\omega(x) \le 0$$
.

Consequently, the original constraints are satisfied for every α, and therefore $x \in S^\alpha$. In conclusion, the two inclusions imply that
$$S^\alpha = S^\omega.$$
Hence, the weighted integral transformation exactly preserves the feasible set of the parametric problem for every $\alpha \in [0,1]$.

□

Theorem 3.6. *A solution $\overline{x}^\omega$, with $\omega \in [0,1]$, is a Pareto optimal solution of the weighted problem (3.11) if and only if it is a $\overline{\alpha}$-Pareto optimal solution of problem (3.2).*

Proof. Suppose that $\overline{x}^\omega$ is a $\overline{\alpha}$-Pareto optimal solution of problem (3.2), i.e., there does not exist any $x \neq \overline{x}^\omega$ such that, for $0 \le \overline{\alpha} \le 1$,
$$\begin{cases} f_i^L(\overline{x}^\omega,\overline{\alpha}) \le f_i^L(x,\overline{\alpha}), \\ f_i^U(\overline{x}^\omega,\overline{\alpha}) \le f_i^U(x,\overline{\alpha}) \end{cases}, \ \forall i, \quad \begin{cases} f_j^L(\overline{x}^\omega,\overline{\alpha}) < f_j^L(x,\overline{\alpha}), \\ f_j^U(\overline{x}^\omega,\overline{\alpha}) < f_j^U(x,\overline{\alpha}) \end{cases} \text{for at least one } j.$$

Since the functions f_i^L and f_i^U, $i = 1,\dots,p$, are monotonic on $[0,1]$, by Proposition 3.3 they are Riemann integrable. Therefore, integrating over $[0,1]$ gives
$$\begin{cases} F_i^L(\overline{x}^\omega) = \int_0^1 f_i^L(\overline{x}^\omega,\alpha)d\alpha \le \int_0^1 f_i^L(x,\alpha)d\alpha = F_i^L(x), \\ F_i^U(\overline{x}^\omega) = \int_0^1 f_i^U(\overline{x}^\omega,\alpha)d\alpha \le \int_0^1 f_i^U(x,\alpha)d\alpha = F_i^U(x) \end{cases}, \ \forall i,$$
and
$$\begin{cases} F_j^L(\overline{x}^\omega) < F_j^L(x), \\ F_j^U(\overline{x}^\omega) < F_j^U(x) \end{cases} \text{for at least one } j.$$

Multiplying by the positive weights $(\omega_1,\omega_2) \in [0,1]^2$ such that $\omega_1 + \omega_2 = 1$, we obtain
$$\omega_1 F_i^L(\overline{x}^\omega) + \omega_2 F_i^U(\overline{x}^\omega) \le \omega_1 F_i^L(x) + \omega_2 F_i^U(x), \ \forall i,$$
and
$$\omega_1 F_j^L(\overline{x}^\omega) + \omega_2 F_j^U(\overline{x}^\omega) < \omega_1 F_j^L(x) + \omega_2 F_j^U(x) \text{ for at least one } j.$$

By definition, these inequalities satisfy the Pareto optimality conditions of the deterministic weighted problem (3.11). Moreover, by Remark 3.5, the feasible sets of problems (3.2) and (3.11) are identical.

Hence, we conclude that $\overline{x}^\omega$ is a Pareto optimal solution of the weighted problem (3.11).

Conversely, suppose that $\overline{x}^\omega$ is a Pareto optimal solution of the weighted problem (3.11), i.e., there does not exist any $x \neq \overline{x}^\omega$ such that
$$F_i^\omega(\overline{x}^\omega) = \omega_1 F_i^L(\overline{x}^\omega) + \omega_2 F_i^U(\overline{x}^\omega) \le \omega_1 F_i^L(x) + \omega_2 F_i^U(x) = F_i^\omega(x), \quad \forall i,$$
with strict inequality for at least one index j.

By the linearity of the integral and the positivity of the weights, this implies

$$F_i^L(\overline{x}^\omega) \le F_i^L(x), \quad F_i^U(\overline{x}^\omega) \le F_i^U(x), \quad \forall i,$$

with strict inequalities for at least one j.

Since the functions f_i^L and f_i^U are monotonic on $[0,1]$, it follows from Proposition 3.3 that, for all $\alpha \in [0,1]$,

$$f_i^L(\overline{x}^\omega, \alpha) \le f_i^L(x, \alpha), \quad f_i^U(\overline{x}^\omega, \alpha) \le f_i^U(x, \alpha), \quad \forall i,$$

with strict inequalities for at least one j.

Hence, by definition, $\overline{x}^\omega$ is a $\overline{\alpha}$-Pareto optimal solution of the problem (3.2). □

In the sequel, we consider the weighted deterministic multiobjective problem

$$\max_{x \in S^\omega} \left(F_1^\omega(x), F_2^\omega(x), \ldots, F_p^\omega(x) \right), \tag{3.12}$$

where the feasible set associated with the weighted problem is defined by

$$S^\omega = \left\{ x \in \mathbb{R}_+^n : G_j^\omega(x) \le 0, \ j = 1, \ldots, m \right\}.$$

Optimal transformation. Let us consider the individual maximization of each objective function:

$$\begin{aligned} \max_{x \in S^\omega} F_1^\omega &= \phi_1, \\ \max_{x \in S^\omega} F_2^\omega &= \phi_2, \\ &\vdots \\ \max_{x \in S^\omega} F_p^\omega &= \phi_p, \end{aligned}$$

where ϕ_i, $i = 1, \ldots, p$, represents the optimal value of the i-th objective function. We define:

$$O_{TT} = \min\{|\phi_1|, |\phi_2|, \ldots, |\phi_p|\}. \tag{3.13}$$

Each weighted objective function, defined by the following expression

$$F_i^\omega(x) = \omega_1 F_i^L(x) + \omega_2 F_i^U(x), \quad i = 1, \ldots, p$$

is subjected to a normalization procedure based on its individual optimal value ϕ_i, thereby ensuring dimensionless comparability among objectives with heterogeneous scales. To reduce the dimensionality of the determistic multiobjective problem, we employ O_{TT}. This formalism leads to the aggregation of normalized functions into a single nonlinear scalar model, thereby ensuring a rigorous transition to a single-objective problem.

The optimal transformation of problem (3.12) is

$$\max_{x \in S^\omega} \ F^\omega(x) = \frac{1}{O_{TT}} \sum_{i=1}^{p} \left(\frac{F_i^\omega(x)}{\phi_i} \right)^2, \tag{3.14}$$

avec

$$S^\omega = \left\{ x \in \mathbb{R}_+^n \ \middle| \ G_j^\omega(x) \le 0, \ j = 1, \ldots, m \right\}.$$

By converting the linear multiobjective problem into a nonlinear single-objective program that is strictly convex and differentiable with respect to the normalized components, the normalized quadratic aggregation ensures that each optimal solution accurately reflects the individual performances of the objectives. This structure amplifies the differences between solutions while limiting compensatory effects, thereby enhancing the discriminative power of the model and promoting the attainment of well-balanced solutions..

Theorem 3.7. *Let $\omega \in [0,1]$ be fixed. If $\overline{x}^\omega \in S^\omega$ is an optimal solution of the nonlinear single-objective problem (3.14) obtained via the normalized quadratic transformation , then $\overline{x}^\omega$ is a Pareto optimal solution of the associated weighted multiobjective problem (3.12).*

Proof. Let $\omega \in [0,1]$ be fixed. Suppose that $\overline{x}^\omega \in S^\omega$ is an optimal solution of the nonlinear single-objective problem (3.14) obtained via the normalized quadratic transformation.

By definition, the normalized quadratic aggregation is

$$F^\omega(x) = \frac{1}{O_{TT}} \sum_{i=1}^{p} \left(\frac{F_i^\omega(x)}{\phi_i} \right)^2.$$

Assume, for contradiction, that $\overline{x}^\omega$ is not Pareto optimal for the associated weighted multi-objective problem (3.12). Then there exists some $x \in S^\omega$ such that

$$F_i^\omega(x) \geq F_i^\omega(\overline{x}^\omega), \quad \forall i = 1, \ldots, p,$$

with strict inequality for at least one index j.

Dividing by the positive ϕ_i and squaring, we obtain

$$\left(\frac{F_i^\omega(x)}{\phi_i} \right)^2 \geq \left(\frac{F_i^\omega(\overline{x}^\omega)}{\phi_i} \right)^2, \quad \forall i,$$

with strict inequality for at least one i, which implies

$$F^\omega(x) = \frac{1}{O_{TT}} \sum_{i=1}^{p} \left(\frac{F_i^\omega(x)}{\phi_i} \right)^2 > \frac{1}{O_{TT}} \sum_{i=1}^{p} \left(\frac{F_i^\omega(\overline{x}^\omega)}{\phi_i} \right)^2 = F^\omega(\overline{x}^\omega).$$

This contradicts the optimality of $\overline{x}^\omega$ for the nonlinear single-objective problem (3.14).

Hence, $\overline{x}^\omega$ is a Pareto optimal solution of the weighted multi-objective problem (3.12). □

Resolution.: Problem (3.14) is formulated as a single-objective nonlinear optimization problem. The transformed problem is solved by any single-objective nonlinear programming method, whether classical or contemporary.

Validation.: We complete the resolution of the fuzzy optimization problem by establishing the following theorem.

Theorem 3.8. *Let $\Omega \subset [0,1]$ be a set of fixed weights. For every $\omega \in \Omega$, if $\overline{x}^\omega \in S^\omega$ is an optimal solution of the nonlinear single-objective problem* (3.14) *obtained via the normalized quadratic transformation, then $\overline{x}^\omega$ is a Pareto optimal solution of the original fuzzy multi-objective problem* (2.5).

Proof. Based on the proofs of Theorems 3.6, 3.7, and 3.2, the demonstration is complete. □

Remark 3.9. By varying the weight vector ω, we can generate the entire Pareto optimal front of problem (2.5).

3.2. **Proposed Algorithm.** The algorithm of our method, called RFMOOA, is proposed as follows:

Algorithm 1 Robust Fuzzy Multi-Objective Optimization Algorithm (RFMOOA)

Require: Fuzzy objectives $\tilde{f}_i(x)$, $i = 1, \ldots, p$; fuzzy constraints $\tilde{\Omega}$; step size $\Delta\omega$

Ensure: Set of Pareto-optimal solutions $\mathcal{X}$ and Pareto front $\mathcal{P}$

Step 1: Defuzzification by Embedding Theorem

for $i = 1$ **to** p **do**

  Extract α-cuts: $(f_i^L(x,\alpha), f_i^U(x,\alpha)) \leftarrow \tilde{f}_i(x)$

  $F_i^L(x) \leftarrow \int_0^1 f_i^L(x,\alpha)\, d\alpha$

  $F_i^U(x) \leftarrow \int_0^1 f_i^U(x,\alpha)\, d\alpha$

end for

Transform fuzzy constraints $\tilde{\Omega}$ into deterministic constraints set S

Step 2: Initialization

$\mathcal{X} \leftarrow \emptyset$

for $\omega_1 = 0$ **to** 1 **step** $\Delta\omega$ **do**

  $\omega_2 \leftarrow 1 - \omega_1$

  Step 3: Parametric Objectives

  for $i = 1$ **to** p **do**

    $F_i^\omega(x) \leftarrow \omega_1 F_i^L(x) + \omega_2 F_i^U(x)$

  end for

  Step 4: Weighted Constraints

  Build S^ω by weighting constraints in S with (ω_1, ω_2)

  if $S^\omega = \emptyset$ **then**

    Mark (ω_1, ω_2) as *infeasible*

    continue

  end if

  Step 5: Individual Maxima (Ideal Points)

  for $i = 1$ **to** p **do**

    $\phi_i \leftarrow \max\limits_{x \in S^\omega} F_i^\omega(x)$

  end for

  if $\exists i$ s.t. $\phi_i = 0$ **then**

    Mark as *infeasible*

    continue

  end if

  Step 6: Optimal Transformation Technique (O_{TT})

  $O_{TT} \leftarrow \min_i\{|\ \phi_i\ |\}$

  Define scalar objective:

$$F^\omega(x) \leftarrow \frac{1}{O_{TT}} \sum_{i=1}^{p} \left(\frac{F_i^\omega(x)}{\phi_i} \right)^2$$

  Step 7: Solve Scalarized Problem

  $x^*(\omega_1) \leftarrow \arg\max\limits_{x \in S^\omega} F^\omega(x)$

  Step 8: Archiving

  $\mathcal{X} \leftarrow \mathcal{X} \cup \big\{(x^*(\omega_1),\ \omega_1,\ \omega_2,\ O_{TT},\ (F_1^\omega(x^*), \ldots, F_p^\omega(x^*)))\big\}$

end for

Step 9: Return Pareto Set

$\mathcal{P} \leftarrow \{(F_1^\omega(x^*), \ldots, F_p^\omega(x^*)) \mid (x^*, \ldots) \in \mathcal{X}\}$

return $\mathcal{X}, \mathcal{P}$

3.3. **Didactic Example.** This section evaluates the performance of the RFMOOA through five test problems from the literature, integrating hybrid uncertainties (triangular and trapezoidal). We detail the application of the method on a first reference case, then we deduce the Pareto-optimal solutions for the other four examples as a function of each weight ω_1.

Example 3.10.

Original problem [10]: Consider the first multiobjective optimization problem with coefficient the triangular fuzzy numbers:

$$\begin{cases} \max \tilde{f}_1(x) \approx (5,7,9)x_1 \oplus (4,5,6)x_2 \oplus (1,2,3)x_3, \\ \max \tilde{f}_2(x) \approx (4,6,7)x_1 \oplus (2,4,7)x_2 \oplus (4,5,6)x_3, \\ \max \tilde{f}_3(x) \approx (2,4,6)x_1 \oplus (3,4,6)x_2 \oplus (2,3,5)x_3, \\ st \\ \qquad (2,5,7)x_1 \oplus (2,3,4)x_2 \oplus (1,2,3)x_3 \approx (8,16,24), \\ \qquad (1,2,3)x_1 \oplus (1,2,3)x_2 \oplus (1,3,4)x_3 \preceq (7,17,22), \\ \qquad (2,3,4)x_1 \oplus (1,2,4)x_2 \oplus (2,3,4)x_3 \succeq (12,18,25) \\ \qquad x_j \geq 0 \;\; j=1,2,3. \end{cases} \tag{3.15}$$

Defuzzification: By relying on the embedding theorem (see Theorem 2.6) and by performing the integration over the interval $[0,1]$, the fuzzy multi-objective optimization problem is initially transformed into a deterministic vector optimization problem:

$$\begin{cases} \max F_1(x) = (6x_1+4.5x_2+1.5x_3, 8x_1+5.5x_2+2.5x_3)\,, \\ \max F_2(x) = (5x_1+3x_2+4.5x_3, 6.5x_1+5.5x_2+5.5x_3)\,, \\ \max F_3(x) = (3x_1+3.5x_2+2.5x_3, 5x_1+5x_2+4x_3)\,, \\ st \\ \qquad (3.5x_1+2.5x_2+1.5x_3, 6x_1+3.5x_2+2.5x_3) = (12,20)\,, \\ \qquad (1.5x_1+1.5x_2+2x_3, 2.5x_1+2.5x_2+3.5x_3) \leq (12,19.5)\,, \\ \qquad (2.5x_1+1.5x_2+2.5x_3, 3.5x_1+3x_2+3.5x_3) \geq (15,21.5) \\ \qquad x_j \geq 0, \;\; j=1,2,3. \end{cases} \tag{3.16}$$

Using weighted scalarization, the vector multiobjective problem (3.16) is transformed into a multiobjective scalar optimization model defined as follows:

$$\begin{cases} \max F_1(x) = \omega_1(6x_1+4.5x_2+1.5x_3)+\omega_2(8x_1+5.5x_2+2.5x_3), \\ \max F_2(x) = \omega_1(5x_1+3x_2+4.5x_3)+\omega_2(6.5x_1+5.5x_2+5.5x_3), \\ \max F_3(x) = \omega_1(3x_1+3.5x_2+2.5x_3)+\omega_2(5x_1+5x_2+4x_3), \\ st \\ \qquad \omega_1(3.5x_1+2.5x_2+1.5x_3)+\omega_2(6x_1+3.5x_2+2.5x_3) = \omega_1(12)+\omega_2(20), \\ \qquad \omega_1(1.5x_1+1.5x_2+2x_3)+\omega_2(2.5x_1+2.5x_2+3.5x_3) \leq \omega_1(12)+\omega_2(19.5), \\ \qquad \omega_1(2.5x_1+1.5x_2+2.5x_3)+\omega_2(3.5x_1+3x_2+3.5x_3) \geq \omega_1(15)+\omega_2(21.5) \\ \qquad x_j \geq 0, \;\; j=1,2,3. \\ \qquad \omega_1 \in [0,1] \end{cases} \tag{3.17}$$

By setting the weighting vector to $\omega = (0.5, 0.5)$, the problem (3.15) is defuzzified into a deterministic multi-objective optimization problem defined by:

$$
\left\{
\begin{array}{l}
\max F_1^{(0.5,0.5)}(x) = 7x_1 + 5x_2 + 2x_3,\\
\max F_2^{(0.5,0.5)}(x) = 5.75x_1 + 4.25x_2 + 5x_3\\
\max F_3^{(0.5,0.5)}(x) = 4x_1 + 4.25x_2 + 3.25x_3\\
st\\
\qquad 4.75x_1 + 3x_2 + 2x_3 = 16,\\
\qquad 2x_1 + 2x_2 + 2.75x_3 \leq 15.75,\\
\qquad 3x_1 + 2.25x_2 + 3x_3 \geq 18.25,\\
\qquad x_j \geq 0,\ j = 1,2,3.
\end{array}
\right.
\tag{3.18}
$$

Optimal transformation: By solving each objective under the set of constraints and then calculating the maximum values of all the objective functions, we have:
$\max F_1^{(0.5,0.5)} = 19.28$ $\max F_2^{(0.5,0.5)} = 31.48$ $\max F_3^{(0.5,0.5)} = 21.08$ Now, we get, $O_{TT} = 19.28$, then using an optimal transformation technique, problem (3.18) is converted into a linear mono objective following

$$
\left\{
\begin{array}{l}
\max\limits_{x\in\mathbb{R}^3} \ F^{(0.5,0.5)}(x)\\
\\
\text{st}\\
\\
\qquad 4.75x_1 + 3x_2 + 2x_3 = 16,\\
\qquad 2x_1 + 2x_2 + 2.75x_3 \leq 15.75,\\
\qquad 3x_1 + 2.25x_2 + 3x_3 \geq 18.25,\\
\qquad x_j \geq 0,\ j = 1,2,3..
\end{array}
\right.
\tag{3.19}
$$

with $F^{(0.5,0.5)}(x) = \frac{1}{19.28}\left[\left(\frac{7x_1+5x_2+2x_3}{19.28}\right)^2 + \left(\frac{5.75x_1+4.25x_2+5x_3}{31.48}\right)^2 + \left(\frac{4x_1+4.25x_2+3.25x_3}{21.08}\right)^2\right]$

Resolution: By solving the problem using MATLAB, we obtain $x_1 = 1.2896,\ x_2 = 0.1913, x_3 = 4.6503$.

Validation: Based on the Theorem 3.8, the optimal solution of the fuzzy multiobjective problem (3.15) for a fixed weight ω is obtained as follows $x^{(0.5,0.5)} = (1.2896, 0.1913, 4.6503)$.

Thus, the set of Pareto-optimal solutions is generated by varying the weights ω_1 and ω_2, subject to the constraint $\omega_1 = 1 - \omega_2$, $(\omega_1, \omega_2) \in [0, 1]$. By adopting a discretization step of $\Delta\omega = 0.05$, the results are presented in Table 1.

TABLE 1. Pareto-optimal solutions for Problem (3.15)

ω_1	ω_2	x_1	x_2	x_3	O_{TT}
0.50	0.50	1.2896	0.1913	4.6503	19.2800
0.55	0.45	1.2016	0.3552	4.6129	19.0439
0.60	0.40	1.1253	0.4876	4.5906	18.7634
0.65	0.35	1.0570	0.5975	4.5799	18.4525
0.70	0.30	0.9939	0.6910	4.5785	18.1176
0.75	0.25	0.9341	0.7720	4.5849	17.7626
0.80	0.20	0.8763	0.8435	4.5984	17.3902
0.85	0.15	0.8192	0.9078	4.6182	17.0021
0.90	0.10	0.7618	0.9664	4.6441	16.5992
0.95	0.05	0.7033	1.0206	4.6761	16.1819
1.00	0.00	0.6429	1.0714	4.7143	15.7500

In order to visualize the trade-off between the different objectives, the Pareto front corresponding to the calculated solutions is shown in Figure 1.

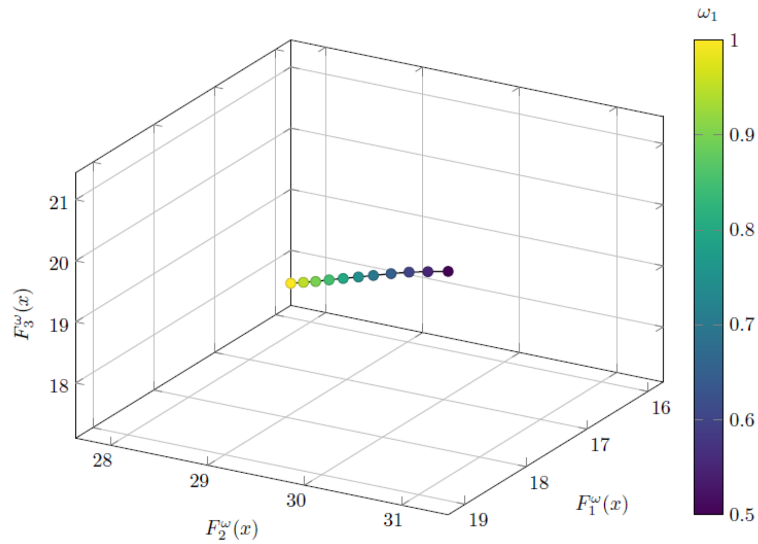

FIGURE 1. Pareto front of problem (3.15) for $\omega_1 \in [0, 1]$

Figure 1 presents the Pareto front in the space of weighted objectives after defuzzification $(F_1^\omega(x), F_2^\omega(x), F_3^\omega(x))$ for problem (3.15).

The front is continuous, regular, and monotonic for $\omega_1 \in (0.5, 1.0)$. The solutions form a relatively compact and slightly tilted cluster. A progressive decrease in $F_1^\omega(x)$ is observed, accompanied by comparable reductions in $F_2^\omega(x)$ and $F_3^\omega(x)$. This quasi-proportional evolution of the three objectives reflects their strong correlation following defuzzification by integration over $[0, 1]$.

The complete absence of feasible solutions for $\omega_1 < 0.5$ confirms the restrictive impact of weighted constraints: the embedding theorem clearly favors the lower bounds, strongly limiting compromises when too much importance is given to the upper bounds. The absence of discontinuities on the front validates the effectiveness of quadratic scalarization based on the Optimal Transformation Technique (O_{TT}) with $O_{TT} = \min_i \mid \phi_i(\omega) \mid$, which produces a dense and well-distributed set of Pareto-optimal solutions.

From the decision-maker's point of view, this relatively compact structure indicates that the trade-offs between the three objectives are fairly balanced. Values of ω_1 close to 0.5 achieve the best global performance, while increasing ω_1 leads to solutions that are more robust against lower-bound uncertainties, at the cost of a moderate and proportional decrease across all objectives.

Example 3.11. [11]Consider the second fuzzy multi-objective optimization problem with coefficients in trapezoidal fuzzy numbers:

$$\text{(3.20)} \qquad \begin{cases} \max \tilde{f}_1(x) \approx (2,3,4,5)x_1 \oplus (1,4,6,7)x_2 \oplus (3,4,5,6)x_3, \\ \max \tilde{f}_2(x) \approx (7,8,12,13)x_1 \oplus (13,14,16,17)x_2 \oplus (10,11,12,13)x_3, \\ st \\ \qquad (2,3,8,9)x_1 \oplus (6,7,8,9)x_2 \oplus (2,3,5,8)x_3 \preceq (16,18,20,22), \\ \qquad (7,9,12,14)x_1 \oplus (11,12,13,15)x_2 \oplus (14,18,19,23)x_3 \preceq (26,27,30,38), \\ \qquad x_j \geq 0 \;\; j = 1,2,3. \end{cases}$$

Similarly to Example 3.10, we generate the set of Pareto solutions for problem (3.20), which are presented in the following table:

TABLE 2. Pareto-optimal solutions for Problem (3.20)

w_1	w_2	x_1	x_2	x_3	O_{TT}
0.00	1.00	0.0000	2.4286	0.0000	15.786
0.05	0.95	0.0000	2.4234	0.0000	15.268
0.10	0.90	0.0000	2.4182	0.0000	14.751
0.15	0.85	0.0000	2.4128	0.0000	14.236
0.20	0.80	0.0000	2.4074	0.0000	13.722
0.25	0.75	0.0000	2.4019	0.0000	13.210
0.30	0.70	0.0000	2.3962	0.0000	12.700
0.35	0.65	0.0000	2.3905	0.0000	12.191
0.40	0.60	0.0000	2.3846	0.0000	11.685
0.45	0.55	0.0000	2.3786	0.0000	11.180
0.50	0.50	0.0000	2.3725	0.0000	10.676
0.55	0.45	0.0000	2.3663	0.0000	10.175
0.60	0.40	0.0000	2.3600	0.0000	9.735
0.65	0.35	0.0000	2.3535	0.0000	9.559
0.70	0.30	0.0000	2.3469	0.0000	9.382
0.75	0.25	0.0000	2.3402	0.0000	9.203
0.80	0.20	0.0000	2.3333	0.0000	9.022
0.85	0.15	0.0000	2.3263	0.0000	8.840
0.90	0.10	3.2059	0.0000	0.0000	8.656
0.95	0.05	3.2576	0.0000	0.0000	8.470
1.00	0.00	3.3125	0.0000	0.0000	8.281

The following figure illustrates the structure of the obtained Pareto front.

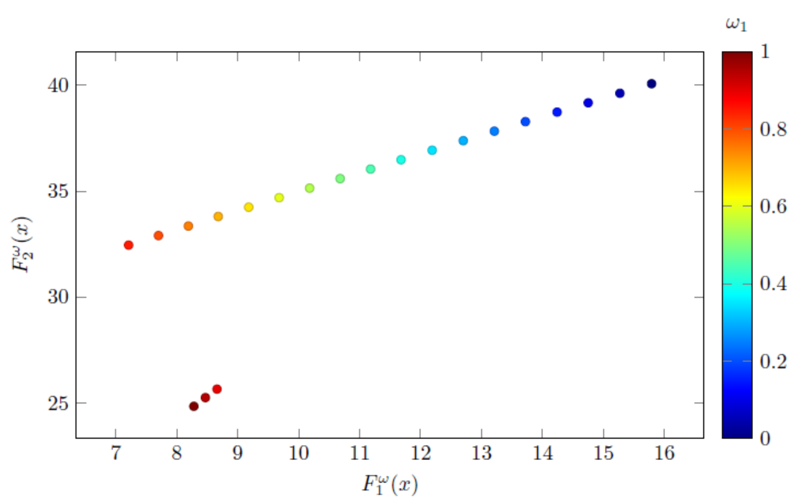

FIGURE 2. Pareto front of problem (3.20) for $\omega_1 \in [0, 1]$

The analysis of the figure shows that the weight ω_1 controls the transition between two distinct performance regions. For low to moderate values of ω_1 (blue and green points), the front progresses linearly, indicating a stable trade-off between the two objectives. However, as ω_1 approaches 1 (red points), a break occurs: instead of continuing to extend toward the upper-right, the solutions drop to an isolated cluster in the lower-left. This discontinuity highlights that prioritizing the lower bound, favoring F_2^{ω} over F_1^{ω}, results in a degradation of the latter. In maximization, the optimal compromise solutions are actually located in the upper-right region (blue/purple points), while the red points represent areas of underperformance for the system.

Example 3.12. [16] Consider the third fuzzy multi-objective optimization problem with coefficients in triangular fuzzy numbers

$$\left\{\begin{array}{l} \max \tilde{f}_1(x) \approx (7,10,14)x_1 \oplus (20,25,35)x_2, \\ \max \tilde{f}_2(x) \approx (10,14,25)x_1 \oplus (25,35,40)x_2, \\ st \\ \qquad (1,2,3)x_1 \oplus (1,4,6)x_2 \preceq (2,5,13), \\ \qquad (2,1,4)x_1 \oplus (4,5,6)x_2 \preceq (2,4,7), \\ \qquad x_j \geq 0 \;\; j=1,2. \end{array}\right. \tag{3.21}$$

Solving the problem (3.21), we obtain the results recorded in the following table for different configurations of the weight vector $\omega = (\omega_1, \omega_2)$:

TABLE 3. Pareto-optimal solutions for Problem (3.21)

w_1	w_2	x_1	x_2	O_{TT}
0.00	1.00	0.0000	1.0000	30.000
0.05	0.95	2.1939	0.0000	29.217
0.10	0.90	2.1875	0.0000	28.438
0.15	0.85	2.1809	0.0000	27.661
0.20	0.80	2.1739	0.0000	26.887
0.25	0.75	2.1667	0.0000	26.116
0.30	0.70	2.1591	0.0000	25.349
0.35	0.65	2.1512	0.0000	24.584
0.40	0.60	2.1429	0.0000	23.824
0.45	0.55	2.1341	0.0000	23.066
0.50	0.50	2.1250	0.0000	22.313
0.55	0.45	2.1154	0.0000	21.562
0.60	0.40	2.1053	0.0000	20.842
0.65	0.35	2.0946	0.0000	20.370
0.70	0.30	2.0833	0.0000	19.896
0.75	0.25	2.0714	0.0000	19.420
0.80	0.20	2.0588	0.0000	18.941
0.85	0.15	2.0455	0.0000	18.460
0.90	0.10	2.0312	0.0000	17.977
0.95	0.05	2.0161	0.0000	17.490
1.00	0.00	2.0000	0.0000	17.000

The evolution of the optimal solutions and the sensitivity of the model to variations in the ω_1 weights are illustrated by Figure 3.

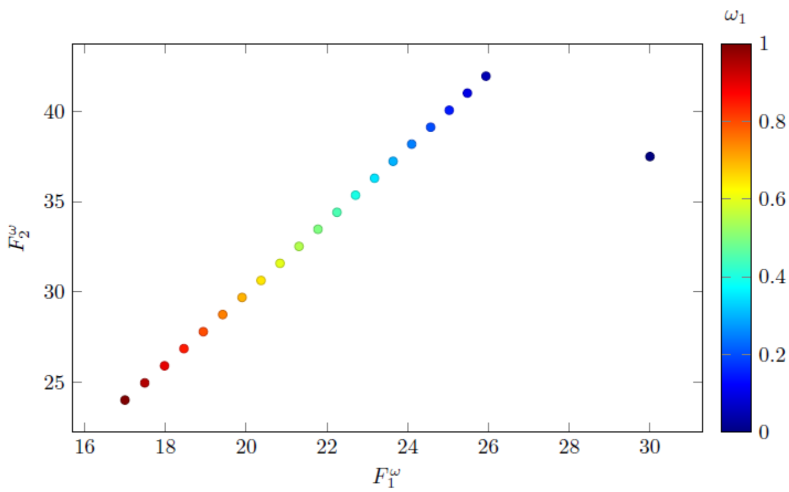

FIGURE 3. Pareto front of problem (3.21) for $\omega_1 \in [0, 1]$

Figure 3 depicts the Pareto front obtained after defuzzification of the fuzzy objective functions $\tilde{f}_1$ and $\tilde{f}_2$. Each point in this figure corresponds to a deterministic solution optimized under the influence of the parameter $\omega_1 \in [0, 1]$.

The analysis indicates that increasing ω_1 (points shifting toward red) leads to a simultaneous decrease in the deterministic values F_1^ω and F_2^ω.

A significant structural discontinuity is observed for $\omega_1 = 0$ (dark blue). This isolation arises because, at this extreme level, the optimization is restricted solely to the domain defined by the upper bounds of the constraints, resulting in a sharp rupture in the decision space topology. This singularity highlights the model's critical sensitivity to boundary conditions once the scenario moves away from the limit of uncertainty.

Conversely, the regularity of the front for $\omega_1 \in [0.1, 1]$ confirms the robustness of the proposed transformation method and its ability to generate reliable optimal solutions in the presence of fuzzy parameters.

Example 3.13. [17]Consider the four fuzzy multi-objective optimization problem with coefficients in trapezoidal fuzzy numbers:

$$(3.22)\qquad \begin{cases} \max \tilde{f}_1(x) \approx (2,3,6,8)x_1 \oplus (7,9,12)x_2 \oplus (5,8,9,11)x_3, \\ \max \tilde{f}_2(x) \approx (12,15,18)x_1 \oplus (9,11,12,14)x_2 \oplus (6,10,14,18)x_3, \\ \max \tilde{f}_3(x) \approx (7,9,13,15)x_1 \oplus (10,12,15,20)x_2 \oplus (6,9,12)x_3, \\ st \\ \qquad (14,16,20,25)x_1 \oplus (8,11,15)x_2 \oplus (10,15,19,22)x_3 \preceq (40,60,70,100), \\ \qquad (2,5,7)x_1 \oplus (3,4,7,10)x_2 \oplus (4,7,10,13)x_3 \preceq (25,30,50,60), \\ \qquad (30,35,45,50)x_1 \oplus (20,30,40,50)x_2 \oplus (10,20,25)x_3 \preceq (140,150,165,180), \\ \qquad (80,90,100,110)x_1 \oplus (40,60,70,90)x_2 \oplus (50,65,70,85)x_3 \preceq (400,500,600), \\ \qquad x_i \geq 0 \ \ (i = 1,2,3). \end{cases}$$

The set of Pareto-optimal solutions of problem (3.22) is given below:

TABLE 4. Pareto-optimal solutions for Problem (3.22)

w_1	w_2	x_1	x_2	x_3	O_{TT}
0.00	1.00	0.0000	2.5774	2.5119	52.182
0.05	0.95	0.0000	2.6600	2.4446	51.615
0.10	0.90	0.0000	2.7475	2.3728	51.059
0.15	0.85	0.0000	2.8403	2.2962	50.515
0.20	0.80	0.0000	2.9390	2.2143	49.983
0.25	0.75	0.0000	3.0442	2.1265	49.465
0.30	0.70	0.0000	3.1564	2.0321	48.962
0.35	0.65	0.0000	3.2765	1.9305	48.476
0.40	0.60	0.0000	3.4053	1.8207	48.009
0.45	0.55	0.0000	3.5438	1.7019	47.562
0.50	0.50	0.0000	3.6931	1.5729	47.137
0.55	0.45	0.0000	3.8546	1.4323	46.739
0.60	0.40	0.0000	4.0298	1.2787	46.370
0.65	0.35	0.0000	4.2206	1.1100	46.033
0.70	0.30	0.0000	4.4292	0.9243	45.734
0.75	0.25	0.0000	4.6583	0.7187	45.478
0.80	0.20	0.0000	4.9108	0.4900	45.270
0.85	0.15	0.0000	5.1909	0.2344	45.120
0.90	0.10	0.0000	5.4315	0.0000	44.810
0.95	0.05	0.0000	5.3488	0.0000	43.459
1.00	0.00	0.0000	5.2632	0.0000	42.105

The distribution of these solutions in the objective space is illustrated in the following figure:

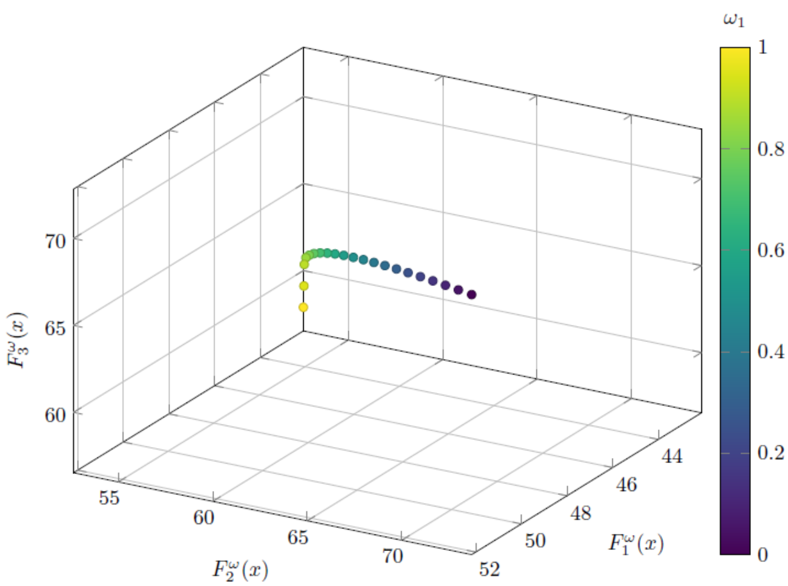

FIGURE 4. Pareto front of problem (3.22) for $\omega_1 \in [0, 1]$.

Figure 4 shows the three-dimensional Pareto front composed of the deterministic objectives F_1^ω, F_2^ω, and F_3^ω, calculated after defuzzification. In this space, each point represents an optimal solution where the parameter $\omega_1 \in [0,1]$ (color scale) controls the weighting of the bounds within the model. Observing the front reveals that an increase of ω_1 toward one induces a growth in the value of F_1^ω correlated with a gradual reduction of F_3^ω, while F_2^ω follows a nonlinear compensatory dynamic. The curvature of this efficiency trajectory highlights the complexity of trade-offs, showing that no improvement on a single objective can be achieved without impacting the other two. The continuous distribution of points confirms the feasibility of the model across the entire weighting spectrum and validates the stability of the embedding method for three-dimensional problems. In sum, this front provides the decision-maker with a precise mapping of the deterministic trade-offs available after processing the initial uncertainty.

Example 3.14. [10]Consider the five fuzzy multi-objective optimization problem with coefficients triangular fuzzy numbers and trapezoidal fuzzy numbers:

$$
\left\{
\begin{array}{l}
\max \tilde{f}_1(x) \approx (40,50,80)x_1 \oplus 100x_2 + 17.5x_3, \\
\max \tilde{f}_2(x) \approx 92x_1 \oplus (70,75,90)x_2 \oplus 50x_3, \\
\max \tilde{f}_3(x) \approx (10,20,70)x_1 \oplus 100x_2 + 75x_3, \\
st \\
\qquad (6,12,14)x_1 \oplus 17x_2 \preceq 1400, \\
\qquad 3x_1 + 9x_2 \oplus (3,8,10)x_3 \preceq 1000, \\
\qquad 10x_1 \oplus (7,13,15)x_2 \oplus 15x_3 \preceq 1750, \\
\qquad (4,6,8)x_1 \oplus 16x_3 \preceq 1325, \\
\qquad (7,12,19)x_1 \oplus 7x_3 \preceq 900, \\
\qquad 9.5x_1 \oplus (3.5,9.5,11.5)x_2 \oplus 4x_3 \preceq 1075, \\
\qquad x_i \geq 0 \ \ (i=1,2,3).
\end{array}
\right. \tag{3.23}
$$

The resolution of problem 3.23 leads to the following table:

The following figure highlights the distribution of Pareto-optimal solutions of problem 3.23 in the objective space, illustrating the trade-offs between the different functions.

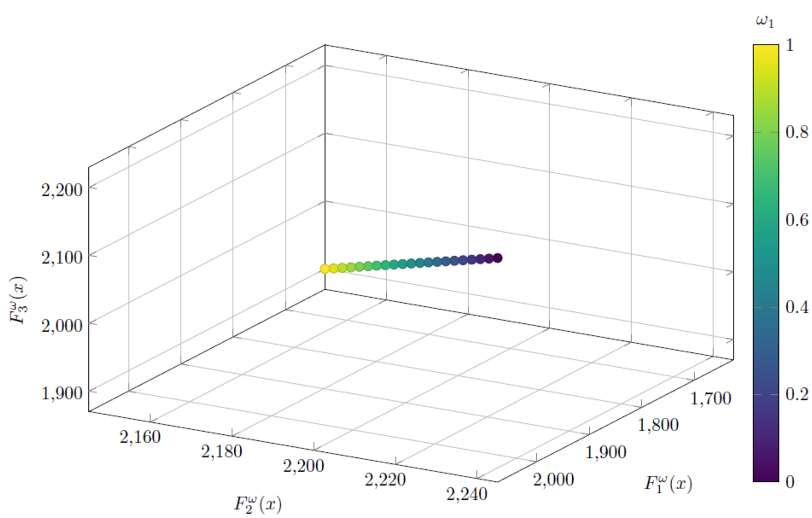

FIGURE 5. Pareto front of problem (3.23) for $\omega_1 \in [0,1]$.

Figure 5 illustrates the three-dimensional Pareto front consisting of the deterministic objectives F_1^ω, F_2^ω, and F_3^ω, calculated after defuzzification. The analysis reveals a major highlight: following the transformation, the numerical values of the deterministic objectives for the optimal solutions are numerically identical

w_1	w_2	x_1	x_2	x_3	O_{TT}
0.00	1.00	10.000	10.000	10.000	9447.3
0.05	0.95	10.000	10.000	10.000	9427.2
0.10	0.90	10.000	10.000	10.000	9406.6
0.15	0.85	10.000	10.000	10.000	9385.7
0.20	0.80	10.000	10.000	10.000	9366.0
0.25	0.75	10.000	10.000	10.000	9336.2
0.30	0.70	10.000	10.000	10.000	9299.7
0.35	0.65	10.000	10.000	10.000	9261.4
0.40	0.60	10.000	10.000	10.000	9220.9
0.45	0.55	10.000	10.000	10.000	9178.1
0.50	0.50	10.000	10.000	10.000	9132.9
0.55	0.45	10.000	10.000	10.000	9109.9
0.60	0.40	10.000	10.000	10.000	9091.5
0.65	0.35	10.000	10.000	10.000	9074.3
0.70	0.30	10.000	10.000	10.000	9058.3
0.75	0.25	10.000	10.000	10.000	9043.9
0.80	0.20	10.000	10.000	10.000	9031.2
0.85	0.15	10.000	10.000	10.000	9020.7
0.90	0.10	10.000	10.000	10.000	9012.5
0.95	0.05	10.000	10.000	10.000	9033.4
1.00	0.00	10.000	10.000	10.000	9058.8

TABLE 5. Pareto-optimal solutions for Problem (3.23)

across all considered weightings, resulting in a perfect superposition of points. The observed line of colored points is a graphical artifact due to the overlapping of solutions, as the variation of ω_1 (color gradient) induces no displacement within the objective space. This absolute invariance demonstrates that the topological structure of the deterministic front is entirely independent of the boundary weighting parameter for this specific problem. Consequently, no trade-off exists between the three deterministic indicators, and the system invariably converges toward a unique global optimum. In summary, the perfect superposition of points validates the structural stability of the embedding model against uncertainty variations in this limiting case.

3.4. **Global Analysis of Results.** The application of the RFMOOA algorithm to five benchmark problems drawn from the literature yields significant insights into its performance in the presence of triangular and trapezoidal fuzzy data.

- **Stability and sensitivity to boundaries:** For the first problem (Example 3.10), the Pareto front is continuous and smooth, illustrating well-balanced compensatory trade-offs. However, the absence of feasible solutions for $\omega_1 < 0.5$ reveals an increased sensitivity of the model to extreme weight configurations during the defuzzification phase. Conversely, in Examples3.11 and 3.12, marked discontinuities appear when ω_1 approaches 1 and 0, respectively. These abrupt transitions reflect sudden changes in the active constraint set, indicating a strong sensitivity of the model to the prioritization of lower or upper bounds. These structural ruptures highlight the need for the decision-maker to identify and avoid zones of high instability to ensure the robustness of the decision
- **Flexibility and invariance:** On the contrary, Examples 3.13 and 3.14 show particularly stable fronts. Example 3.13 exhibits a smooth curvature in the three-dimensional space, reflecting a high degree of

decision-making flexibility. Example 3.14, for its part, reveals an absolute invariance: the deterministic objectives are numerically identical across all values of ω_1, resulting in a perfect superposition of points. This case highlights the ability of RFMOOA to identify configurations of intrinsic robustness, where the system converges toward a stable equilibrium despite fluctuations in uncertainty. It illuminates situations where decision levers are structurally limited, thus preventing the search for non-existent trade-offs in favor of a unique, invariant global optimum.

Correlation analysis and limits: The visualization colored according to ω_1 allows us to appreciate the direct influence of the weighting of the bounds on the location of the solutions. In Example 3.10, a strong correlation between the objectives after defuzzification reduces major conflicts, resulting in a relatively flat front. Indeed, RFMOOA generates stable and reliable solutions in this context. However, due to the strong correlation between the defuzzified objectives, the diversity of the Pareto front is naturally limited, reducing its value in decision-making. In this case, ω_1 is used more to balance nominal performance and structural robustness, opening up an avenue for improvement for future work.

In summary, these results confirm the robustness of the proposed approach. Quadratic aggregation via the Optimal Transformation Technique (O_{TT}) allows the generation of dense and exploitable sets of optimal Pareto solutions. The approach successfully identifies areas of critical sensitivity as well as progressive trade-offs, demonstrating the applicability of RFMOOA as a decision support tool in a fuzzy context.

4. Conclusion

In this work, we have developed and validated a new approach, dedicated to solving fuzzy linear multi-objective optimization problems. The originality of this method lies in the exploitation of the embedding theorem in Banach spaces and the Riemann integral operator, allowing for the transformation of fuzzy objectives into deterministic equivalents through a rigorous integration over the interval $[0, 1]$. The analyses conducted on various test problems from the literature confirm the effectiveness of our approach, particularly through the integration of the Optimal Transformation Technique O_{TT} and the assignment of weights to the lower and upper bounds within the model. The quadratic aggregation technique not only ensures an effective normalization of the criteria, but also guarantees the generation of dense, continuous, and usable Pareto fronts.Our results highlight the algorithm's ability to identify areas of intrinsic robustness of the system, while alerting the decision-maker to areas of instability related to breaks in the active constraint set. From a theoretical standpoint, we have established the preservation of Pareto optimality when converting the fuzzy model into its deterministic counterpart, thus offering a rigorous mathematical framework for decision-making in fuzzy environments. Although the strong correlation observed between certain objectives in specific cases may limit the trade-off space, it highlights the importance of the weight ω_1 as a lever for adjusting between nominal performance and structural safety.

Competing interests. The authors declare no competing interests.

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