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SOMBOR INDEX OF GCD GRAPH FOR RING OF INTEGERS MODULO POWER OF PRIME

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ABSTRACT. The GCD graph of the group or ring is a simple undirected graph and connected, has its vertex set as all the elements of the corresponding algebraic structure. Two distinct vertices are adjacent if and only if the greatest common divisor (gcd) of the orders of the two vertices equals the order of their product. The Sombor index of the graph is the summation of the square root of the sum of the squares of the degrees of two distinct adjacent vertices. This article examines certain properties of the GCD graph for the ring of integers modulo a power of a prime, including the degree of every vertex, the number of vertices, the number of edges, and the observation of subgraphs. We also derive the formula of the Sombor index and investigate its upper and lower bounds.

1. INTRODUCTION

In the study of molecular structures, topological indices serve as powerful mathematical tools that capture the essence of chemical connectivity through the lens of graph theory. A molecular graph, in its simplest form, represents the framework of an organic molecule, where vertices correspond to non-hydrogen atoms and edges symbolize the covalent bonds that link them [21]. Over time, chemists and mathematicians have developed a variety of these indices to quantify subtle aspects of molecular structure, and among them, degree-based indices have emerged as particularly influential in chemical graph theory [9, 10].

Building on this foundation, researchers have explored how different indices reveal the hidden architecture of molecular systems. For instance, benzenoid graphs, which model aromatic hydrocarbons, have been extensively analyzed through a range of topological indices that illuminate their geometric and electronic properties [3]. Likewise, studies on vanadium carbide have shown how these indices can describe not only structural patterns but also compositional behaviors within complex chemical compounds [22].

Amid these advancements, a new index has emerged, the Sombor index, which offers a refined lens for capturing molecular structure. By weighting vertex degrees in a distinctive geometric form, the Sombor index enhances the sensitivity of traditional measures. In Quantitative Structure Property Relationship (QSPR) studies of monocarboxylic acids, graphs characterized by specific connectivity and minimal Sombor index values have yielded promising predictive power [13]. Moreover, contemporary research [6, 15] indicates that the Sombor index outperforms several classical descriptors in predicting physico-chemical properties, underscoring its potential as a bridge between mathematical theory and chemical application.

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The Sombor index, introduced by [11], is an intriguing topological index to study. This is evident from several published articles related to this index, including [1,2]. For instance, there are studies examining the relationships between indices, such as the relationship between the Sombor and Reduced Sombor indices [14]. Beyond the Sombor index, some studies investigate relationships between degree-based indices, such as the connection between the Sombor and Zagreb indices [4]. In addition, Shang [20] investigated the Sombor index in simplicial networks that model higher-order interactions in complex systems. Other topics of research on the Sombor index include upper and lower bounds, such as bounds for the Sombor index for certain graph parameters [4], trees [5], cycle and unicyclic graphs [16], and the Sombor and Modified Sombor indices [12]. In addition, researchers expand this topic to the Sombor energy of the non-commuting graph [17].

Recent years have seen significant scholarly interest in the examination of topological indices of graphs linked to algebraic structures. An example of a thoroughly examined graph defined on an algebraic structure is the GCD graph of a finite group (or ring) introduced by [19]. This interaction between group or ring theory and graph theory allows for deeper insights into the structure of groups and rings, such as connectivity, degree sequences, and spectral properties.

The GCD graph of the ring of integers modulo a power of a prime, $\mathbb{Z}_{p^k}$, is denoted as $\Gamma\left(\mathbb{Z}_{p^k}\right)$, where p is a prime number and integer $k \geq 2$. The vertex set of this graph consists of all elements of the ring $\mathbb{Z}_{p^k}$. The set of vertices and edges are denoted by $V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)$ and $E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)$, respectively. Two distinct vertices, $u \neq v \in V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)$, are said to be adjacent if and only if the greatest common divisor (gcd) of the orders of the two vertices equals the order of their product, $\gcd(o(u), o(v)) = o(u \cdot v)$, where " $\cdot$ " denotes multiplication modulo n [19]. The order of an element v in $\mathbb{Z}_{p^k}$ with respect to the addition operation (the additive group) is the smallest positive integer m such that $mv = 0$, where 0 is the identity element under addition. Moreover, since $\mathbb{Z}_{p^k}$ is a commutative ring, it follows that $o(v) = \dfrac{p^k}{\gcd\left(v, p^k\right)}$ for $v \in \mathbb{Z}_{p^k}$, and $\gcd(o(u), o(v)) = o(u \cdot v) = o(v \cdot u)$ [7]. In this paper, we refer to the underlying structure as a ring rather than a group, since two binary operations, addition and multiplication modulo p^k, are implicitly involved. This ensures that both additive and multiplicative relations within the modular arithmetic framework are properly represented in our analysis. Moreover, $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is a simple undirected graph and is connected. This is because vertex 0 is adjacent to all other vertices, while no vertex is adjacent to itself, and multiple edges are not allowed.

Motivated by the need to understand how the Sombor index behaves within algebraic structures, this study addresses the research gap concerning its application to the GCD graph defined over a ring. It is also necessary to study the unique patterns or properties specific to this index on the graph. Therefore, this paper focuses on the Sombor index and discusses the upper and lower bounds for the Sombor index of the GCD graph of $\mathbb{Z}_{p^k}$, where p is a prime number and $k \geq 2$ is an integer. This paper presents an original theoretical contribution rather than a survey, focusing on establishing the upper and lower bounds of the Sombor index and deriving explicit analytical formulas for it. By characterizing vertex degrees through number-theoretic properties and expressing the index as a function of p and k, the study extends prior research on algebraic graphs and degree-based topological indices. To the best of our knowledge, this is the first systematic investigation of the Sombor index for GCD graphs, offering new insights into their combinatorial and algebraic structure.

The paper has been organized as follows: The main results of this research are presented in Sections 2 and 3. Section 2 describes the Properties of GCD graph for the ring of Integers Modulo Power of Prime such as the degree of each vertex, the number of vertices and edges, the observation of connectivity, and the subgraph. Meanwhile, in Section 3, we find the Sombor index and its bounds. In the last section, we summarize the potential for future research that we believe could lead to novel theoretical results and practical applications.

2. Properties of GCD graph for Ring of Integers Modulo Power of Prime

This section presents the properties of the GCD graph for $\mathbb{Z}_{p^k}$, where p is a prime number and $k \geq 2$ is an integer. The properties are related to the degree of every vertex, the number of vertices, the number of edges, and the subgraph observation. For the sake of clarity, the notation used throughout this paper and its corresponding meanings are summarized in the table below.

Symbol	Definition
p^k	power of prime
$\mathbb{Z}_{p^k}$	ring of integers modulo power of prime
$\Gamma\left(\mathbb{Z}_{p^k}\right)$	GCD graph of $\mathbb{Z}_{p^k}$
$V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)$	vertex set of $\Gamma\left(\mathbb{Z}_{p^k}\right)$
$E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)$	edge set of $\Gamma\left(\mathbb{Z}_{p^k}\right)$
$o(v)$	order of v in $\mathbb{Z}_{p^k}$
$\deg(v)$	degree of vertex v
δ	minimum degree, i.e. $p^k - p^{k-1} + 1$
Δ	maximum degree, i.e. $p^k - 1$
$SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)$	Sombor index of $\Gamma\left(\mathbb{Z}_{p^k}\right)$

Table 1. Notation and its Definition

Furthermore, before delving into the specifics, we will first present the structure of $\Gamma\left(\mathbb{Z}_{p^k}\right)$ as an illustration.

Example 2.1. The discussion of this structure begins with the investigation of $\mathbb{Z}_{2^3}$ where $p = 2$ and $k = 3$. In particular, the element 0 has order 1, that is, $o(0) = 1$. Note that $o(1) = \dfrac{2^3}{\gcd(1, 2^3)} = 8$, $o(2) = \dfrac{2^3}{\gcd(2, 2^3)} = 4$, and so on. In a similar way, we obtain the orders of the elements in $\mathbb{Z}_{2^3}$ are as follows: $o(0) = 1$, $o(1) = 8$, $o(2) = 4$, $o(3) = 8$, $o(4) = 2$, $o(5) = 8$, $o(6) = 4$, and $o(7) = 8$. Using the above information, we have the following result about gcd values for pairs of elements in Z_{2^3} as given in Table 2. Moreover, we also derive the orders of the products of two distinct ring elements in Table 3.

gcd	$o(0)$	$o(1)$	$o(2)$	$o(3)$	$o(4)$	$o(5)$	$o(6)$	$o(7)$
$o(0)$	-	1	1	1	1	1	1	1
$o(1)$	1	-	4	8	2	8	4	8
$o(2)$	1	4	-	4	2	4	4	4
$o(3)$	1	8	4	-	2	8	4	8
$o(4)$	1	2	2	2	-	2	2	2
$o(5)$	1	8	4	8	2	-	4	8
$o(6)$	1	4	4	4	2	4	-	4
$o(7)$	1	8	4	8	2	8	4	-

Table 2. GCD for any element of $\mathbb{Z}_{2^3}$

Using the same formula, we obtain $o(u \cdot v) = \dfrac{p^k}{\gcd(u \cdot v, p^k)}$. Consequently, for every $u \in \mathbb{Z}_{2^3}$, it holds that $o(0 \cdot u) = o(0) = 1$. Observe that $o(1 \cdot 2) = \dfrac{2^3}{\gcd(1 \cdot 2, 2^3)} = 4$. Similarly, we obtain $o(1 \cdot 6) = o(2 \cdot 3) = o(2 \cdot 5) = o(2 \cdot 6) = o(3 \cdot 6) = o(5 \cdot 6) = o(2) = 4$. Furthermore, we obtain $o(1 \cdot 4) = \dfrac{2^3}{\gcd(1 \cdot 4, 2^3)} = 2$. In the same way, we obtain $o(3 \cdot 4) = o(4 \cdot 5) = o(4 \cdot 7) = o(4) = 2$. For all other combinations of the multiplication of two elements, the order of resulting product is 8. This also holds for the reverse order of multiplication since the ring $\mathbb{Z}_{2^3}$ is commutative.

$o(u \cdot v)$	0	1	2	3	4	5	6	7
0	-	1	1	1	1	1	1	1
1	1	-	4	8	2	8	4	8
2	1	4	-	4	1	4	2	4
3	1	8	4	-	2	8	4	8
4	1	2	1	2	-	2	1	2
5	1	8	4	8	2	-	4	8
6	1	4	2	4	1	4	-	4
7	1	8	4	8	2	8	4	-

TABLE 3. Order for two distinct element of $\mathbb{Z}_{2^3}$

From Table 2 and Table 3, based on the definition of the GCD graph, there is no edge between vertices $u, v \in \{2, 4, 6\}$ since $\gcd(o(u), o(v)) \neq o(u \cdot v) = o(v \cdot u)$. Hence, the GCD graph of $\mathbb{Z}_{2^3}$ is obtained as shown in Figure 1.

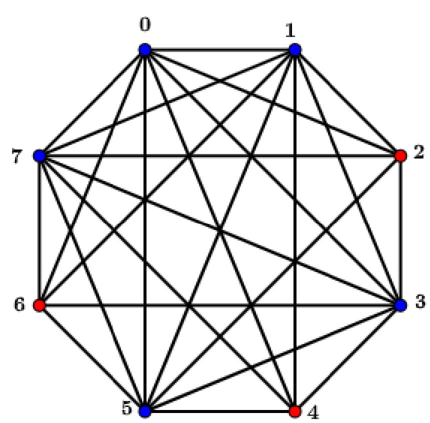

FIGURE 1. GCD Graph for $\mathbb{Z}_8$

Example 2.2. Similarly, we can construct the GCD graph of $\mathbb{Z}_{3^2}$. The procedure follows the same steps as in the previous example, where each vertex and its adjacency are determined based on the GCD graph definition. The resulting GCD graph of $\mathbb{Z}_{3^2}$ is shown in Figure 2.

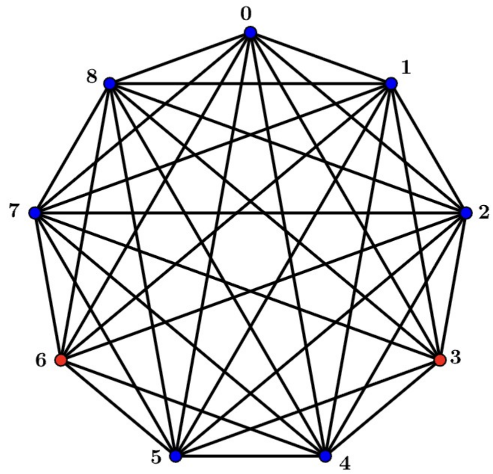

FIGURE 2. GCD Graph for $\mathbb{Z}_9$

Based on Example 2.1, the order of an element can be divided into two cases, namely those of the form 1, mp (i.e., 2 and 4), and p^k (i.e., 8) for some $m \in \{1,2,3\}$. Thus, the vertex set of GCD graph of $\mathbb{Z}_{2^3}$ can be written as $V_0 = \{0\}$, $V_1 = \{2,4,6\}$, and $V_3 = \{1,3,5,7\}$. Furthermore, for Example 2.2, we obtain the vertex sets $V_0 = \{0\}$, $V_1 = \{3,6\}$, and $V_3 = \{1,2,4,5,7,8\}$. From these observations, it can be concluded that the vertex set of the GCD graph is partitioned into two groups, namely the set of multiples of the prime number and the set consisting of elements that are not multiples of that prime.

Note that the vertex set of the graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is $V\left(\mathbb{Z}_{p^k}\right) = \{0,1,2,\ldots,p^k-1\}$. The elements in this set that are multiples of p are $\left\{0,p,2p,3p,\ldots,(p^{k-1}-1)p\right\}$. These elements can be written in the form mp, where $m \in \{0,1,2,\ldots,p^{k-1}-1\}$. Hence, there are p^{k-1} multiples of p in the set $\left\{0,1,\ldots,p^k-1\right\}$. Since the vertex 0 is treated separately in the graph, the multiples of p other than 0 form the set $V_1 = \{p,2p,3p,\ldots,(p^{k-1}-1)p\}$. Therefore, we obtain $|V_1| = p^{k-1}-1$. Furthermore, define $V_2 = V\left(\mathbb{Z}_{p^k}\right) \setminus V_1$, so that $V\left(\mathbb{Z}_{p^k}\right) = V_1 \cup V_2$.

We now state the result on the degree of every vertex in $\Gamma\left(\mathbb{Z}_{p^k}\right)$ in the following theorem. We break down $V\left(\mathbb{Z}_{p^k}\right)$ into two disjoint sets $V_1 = \left\{p,2p,3p,\ldots,\left(p^{k-1}-1\right)p\right\}$ and $V_2 = V\left(\mathbb{Z}_{p^k}\right)\setminus V_1$ such that $V\left(\mathbb{Z}_{p^k}\right) = V_1 \cup V_2$. The elements in the set V_1 are multiples of the prime number less than p^k. Therefore, $|V_1| = p^{k-1}-1$.

Theorem 2.3. *Let $\Gamma\left(\mathbb{Z}_{p^k}\right)$ be the GCD graph of the integer ring of modulo p^k, where p is a prime number and $k \geq 2$ is an integer. Let $v \in V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)$, then*

$$\deg(v) = \begin{cases} p^k - p^{k-1} + 1 & \text{, for } v \in V_1 \\ p^k - 1 & \text{, for } v \in V_2, \end{cases}$$

where $V_1 = \left\{p,2p,3p,\ldots,\left(p^{k-1}-1\right)p\right\}$ and $V_2 = V\left(\mathbb{Z}_{p^k}\right)\setminus V_1$.

Proof. According to [7], the order of an element a in $\mathbb{Z}_n$ can be expressed as $o(a) = \dfrac{n}{\gcd(a,n)}$. Furthermore, since the identity $\operatorname{lcm}(a,n)\cdot\gcd(a,n) = a\cdot n$ holds, the order of the element can equivalently be written as $o(a) = \dfrac{\operatorname{lcm}(a,n)}{a}$, referring to the work of [18].Therefore, in this proof the expression $o(a) = \dfrac{\operatorname{lcm}(a,n)}{a}$ is employed because it facilitates a clearer presentation and simplifies the subsequent steps of the proof. We consider 3 cases in this proof based on the definition of the GCD graph, and it is obtained the following cases:

Cases 1: For u and v in V_1.
Let $u, v \in V_1$. By the definition of V_1, each element is a multiple p that is less than p^k. hence, there exist integers $\alpha, \beta \in \{1, 2, \dots, k-1\}$ and a, b integers relatively prime to p, i.e. $\gcd(a, p) = \gcd(b, p) = 1$, such that $u = p^\alpha a, \quad v = p^\beta b$. Then, the order of u is given by $o(u) = \dfrac{p^k}{\gcd\left(p^\alpha a, p^k\right)} = \dfrac{p^k}{p^\alpha} = p^{k-\alpha}$, and similarly $o(v) = p^{k-\beta}$. Therefore, $\gcd(o(u), o(v)) = \gcd(p^{k-\alpha}, p^{k-\beta}) = p^{\min(k-\alpha, k-\beta)} = p^{k-\max(\alpha,\beta)}$. Moreover, we obtain $u \cdot v = p^\alpha a \cdot p^\beta b = p^{\alpha+\beta} \cdot ab$. Consider the following cases:

- If $\alpha + \beta < k$, in this case, $o(u \cdot v) = \dfrac{p^k}{p^{\alpha+\beta}} = p^{k-(\alpha+\beta)}$. On the other hand, $\gcd(o(u), o(v)) = p^{k-\max(\alpha,\beta)}$. Since $\alpha + \beta > \max(\alpha, \beta)$, it follows that $p^{k-(\alpha+\beta)} < p^{k-\max(\alpha,\beta)}$. Hence, $\gcd(o(u), o(v)) \neq o(u \cdot v)$.
- For $\alpha + \beta \geq k$, in this case, $u \cdot v = p^k \cdot \left(p^{\alpha+\beta-k} ab\right)$. Thus, $p^k \mid u \cdot v$, and hence $u \cdot v \equiv 0 \pmod{p^k}$. Therefore, $o(u \cdot v) = o(0) = 1$. Meanwhile, $\gcd(o(u), o(v)) = p^{k-\max(\alpha,\beta)} \geq p$. Thus, $\gcd(o(u), o(v)) \neq o(u \cdot v)$.

Consequently, for all $u, v \in V_1$, we have $\gcd(o(u), o(v)) \neq o(u \cdot v)$. This implies that u and v are not adjacent.

Case 2: For u and v in V_2.

In the second case, two sub-cases are considered, namely $u \neq 0$ and $v \neq 0$ or $u = 0$ and $v \neq 0$. First, since $0 \neq u, v \in V_2$, it follow $\gcd(u, p^k) = 1$ and $\gcd(v, p^k) = 1$. Therefore, $o(u) = \dfrac{lcm(u, p^k)}{u} = \dfrac{u \cdot p^k}{u} = p^k$ and $o(v) = \dfrac{lcm(v, p^k)}{v} = p^k$. Additionally, since $u \cdot v \in V_2$ for the same reason, we obtain $o(u \cdot v) = p^k$. Hence, $\gcd(o(u), o(v)) = o(u \cdot v)$. Second, for $u = 0 \in V_2$, we have $\gcd(o(0), o(v)) = 1 = o(0) = o(0 \cdot v)$. In other words, u and v are adjacent.

Case 3: For $u \in V_1$ and $v \in V_2$.
Let $u \in V_1$ and $v \in V_2$ be arbitrary. By the definition of V_1, we may express $u = ap$ for some $a \in \left\{1, 2, \dots, p^{k-1} - 1\right\}$. Hence, it follows that $\gcd(u, p^k) = p$. Consequently, the order of u is given by $o(u) = \dfrac{p^k}{\gcd(u, p^k)} = \dfrac{p^k}{p} = p^{k-1}$. Next, let $v \in V_2$ (i.e. v is not a multiple of p) with $v \neq 0$. Then $\gcd(v, p^k) = 1$, and hence $o(v) = \dfrac{p^k}{\gcd(v, p^k)} = p^k$. Therefore, $\gcd(o(u), o(v)) = \gcd(p^{k-1}, p^k) = p^{k-1}$. Observe that $u \cdot v = ap \cdot v = (av)p \pmod{p^k}$. Since v is not divisible by p, it follows that av is also not divisible by p. Consequently, $\gcd\left(u \cdot v, p^k\right) = p$, and thus $o(u \cdot v) = \dfrac{p^k}{p} = p^{k-1}$. Hence, we obtain $\gcd(o(u), o(v)) = o(u \cdot v)$, which implies u and v are adjacent. For the special case $v = 0$, we have $o(0) = 1$ and $u \cdot 0 = 0$. Therefore, $\gcd(o(u), o(0)) = 1 = o(0) = o(u \cdot 0)$, and thus u and 0 are also adjacent.

From the three cases above, it can be concluded that any $u \in V_2$ is adjacent to all vertices. Meanwhile, v and w are not adjacent for any $v, w \in V_1$. Furthermore, the number of elements in V_1, is found to be $|V_1| = p^{k-1} - 1$. Since each $u \in V_2$ is adjacent to all vertices, it follows that $\deg(u) = p^k - 1$. Next, since each $v \in V_1$ is only adjacent to $u \in V_2$, the degree of v is equal to $|\mathbb{Z}_{p^k}|$ minus $|V_1|$, which is $\deg(v) = p^k - p^{k-1} + 1$. $\square$

The number of vertices in the graph is $n = p^k$. Based on Theorem 2.3, the minimum and maximum degrees of the GCD graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$ are, respectively, $\delta = p^k - p^{k-1} + 1$ and $\Delta = p^k - 1$. Henceforth, the notations δ and Δ will be used more frequently than the symbol p. Furthermore, from the theorem, the number of vertices in each vertex set can also be determined.

Proposition 2.4. *Let $\Gamma\left(\mathbb{Z}_{p^k}\right)$ be the GCD graph of the integer ring of modulo p^k, where p is a prime and $k \geq 2$ is an integer. If $u \in V_1$ and $v \in V_2$, then $|V_2| = \delta$ and $|V_1| = \Delta - \delta + 1$.*

Proof. Let $u \in V_1$ and $v \in V_2$, then $\deg(u) = \delta$ and $n = \Delta + 1$. Based on Theorem 2.3, it is clear that the number of vertices in V_2 equal to the degree of u, which is δ. Furthermore, the number of vertices in V_1 is the total number of vertices minus the degree of u, which $\Delta + 1 - \delta$. Thus, we conclude $|V_2| = \delta$ and $|V_1| = \Delta - \delta + 1$. $\square$

It is worth noting that, by the well-known Handshaking Lemma (see [8]), the following result holds.

Lemma 2.5. *For any graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$, the sum of the degrees of all vertices are equals twice the number of edges. Thai is,*

$$\sum_{v \in V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)} \deg(v) = 2\left|E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right|.$$

The next property that we need to show is the number of edges in $\Gamma\left(\mathbb{Z}_{p^k}\right)$ as follows.

Theorem 2.6. *Let $\Gamma\left(\mathbb{Z}_{p^k}\right)$ be the GCD graph of the integer ring of modulo p^k, where p is a prime and $k \geq 2$ is an integer. The number of edges in the graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is $\left|E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right| = \frac{1}{2} \cdot \delta\left(2\Delta - \delta + 1\right).$*

Proof. Since the number of edges in the graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is equal to half the sum of the degrees of all vertices, it follows from Lemma 2.5 that, together with Theorem 2.3 and Proposition 2.4, we obtain

$$\begin{aligned}
\left|E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right| &= \frac{1}{2} \sum_{v \in V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)} \deg(v) \\
&= \frac{1}{2}\left(\sum_{v_1 \in V_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)} \deg(v_1) + \sum_{v_2 \notin V_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)} \deg(v_2)\right) \\
&= \frac{1}{2}\left(\left(\Delta - \delta + 1\right) \cdot \delta + \delta \cdot \Delta\right) = \frac{1}{2} \cdot \delta\left(2\Delta - \delta + 1\right).
\end{aligned}$$

$\square$

Example 2.7. Consider the vertex in red color in Figure 1. If the edges adjacent to this vertex are removed, a complete subgraph K_5 is formed, as shown in Figure 3.

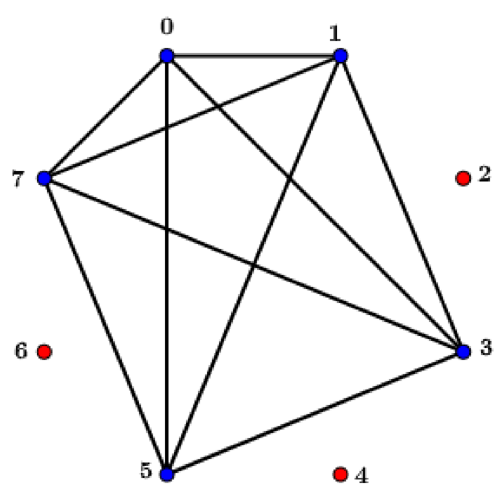

FIGURE 3. Complete Subgraph of GCD Graph $\mathbb{Z}_8$

However, if the edges that form the complete graph are removed from the graph $\Gamma(\mathbb{Z}_8)$ in Figure 1, the resulting subgraph is a complete bipartite subgraph, $K_{3,5}$ as shown in Figure 4.

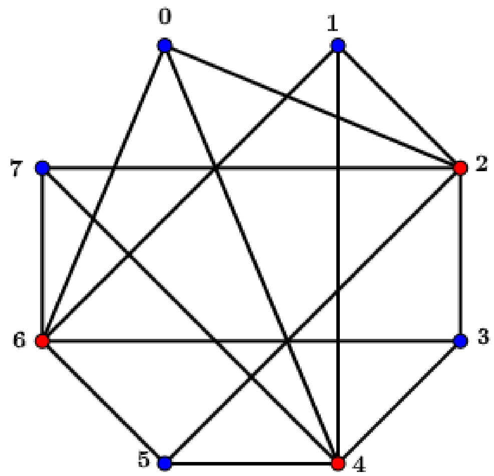

FIGURE 4. Complete Bipartite Subgraph of GCD Graph $\mathbb{Z}_8$

Therefore, we obtain the following theorem regarding the subgraph of the GCD graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$.

Theorem 2.8. *Let $\Gamma\left(\mathbb{Z}_{p^k}\right)$ be the GCD graph of the integer ring of modulo p^k, where p is a prime and $k \geq 2$ is an integer. $\Gamma\left(\mathbb{Z}_{p^k}\right)$ consists of complete subgraph K_δ and complete bipartite subgraph $K_{\Delta-\delta+1,\delta}$. Moreover, $K_{\delta+1}$ is also a complete subgraph.*

Proof. From Theorem 2.3, it is clear that for any two distinct vertices $v_1, v_2 \in V_2$, they are adjacent. Therefore, the subgraph formed by the vertex set V_2 is a complete graph. Furthermore, based on Proposition 2.4, we obtain $|V_2| = \delta$. Hence, one of the subgraphs of the GCD graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is the complete subgraph K_δ. Based on Theorem 2.3, every two distinct vertices $v_1, v_2 \in V_2$ are adjacent, hence, the subgraph induced by V_2 is a complete graph K_δ. Furthermore, every vertex $u \in V_1$ is adjacent to all vertices in V_2. Therefore, the vertex set $V_2 \cup \{u\}$ also forms a complete graph with $\delta + 1$ vertices, namely $K_{\delta+1}$. However, any two distinct vertices in V_1 are not adjacent. Consequently, it is impossible to form a complete subgraph of size greater than $\delta + 1$. Moreover, it is clear that the complete subgraph $K_{\delta+1}$ is not unique.

Note that when the edges that form the complete subgraph are removed from the graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$, it is clear that the subgraph formed by any vertex $u_1 \in V_1$ and $v_1 \in V_2$ is adjacent. Based on the explanation in Theorem 2.3, any two vertices $u_1, u_2 \in V_1$ are not adjacent. Therefore, a complete bipartite subgraph is formed with two partitions of sets, V_1 and V_2. Based on Proposition 2.4, we obtain the complete bipartite subgraph $K_{\Delta-\delta+1,\delta}$.

Every vertex in V_1 is adjacent to all vertices in V_2, while any two distinct vertices in V_1 are not adjacent. On the other hand, every pair of vertices in V_2 is adjacent in $\Gamma\left(\mathbb{Z}_{p^k}\right)$. Therefore, all vertices in V_2 must belong to the same partition, while all vertices in V_1 belong to the other partition. Moreover, no additional vertex can be included in either partition without violating the property of a complete bipartite graph. Consequently, the complete bipartite subgraph $K_{\Delta-\delta+1,\delta}$ is a maximal and unique subgraph of $\Gamma\left(\mathbb{Z}_{p^k}\right)$. Hence, the complete bipartite subgraph obtained from $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is maximal and unique. □

It should be noted that K_δ is also a maximal complete subgraph. Furthermore, we need to evaluate each subgraph in Theorem 2.8 to obtain the number of edges.

Theorem 2.9. *Let $\Gamma\left(\mathbb{Z}_{p^k}\right)$ be the GCD graph of the integer ring of modulo p^k, where p is a prime number and $k \geq 2$ is an integer. The number of edges in the complete subgraph K_δ and the complete bipartite subgraph $K_{\Delta-\delta+1,\delta}$ are, respectively $|E\left(\Gamma\left(K_\delta\right)\right)| = \frac{1}{2}\cdot\delta(\delta-1)$ and $|E\left(\Gamma\left(K_{\Delta-\delta+1,\delta}\right)\right)| = (\Delta-\delta+1)\,\delta$.*

Proof. Since K_δ is a complete subgraph, the number of vertices in the subgraph is δ, and the degree of each vertex is $\delta-1$. Therefore, the following can be concluded $|E\left(\Gamma\left(K_\delta\right)\right)| = \frac{1}{2}\cdot\delta(\delta-1)$. The complete bipartite graph is partitioned into two sets of vertices, V_1 and V_2. Based on Proposition 2.4, $|V_1| = \Delta-\delta+1$ and $|V_2| = \delta$. The total number of edges in a complete bipartite graph $\left|E\left(\Gamma\left(K_{p^{k-1}-1,p^k-p^{k-1}+1}\right)\right)\right| = (\Delta-\delta+1)\,\delta$. □

3. The Sombor Index of GCD graph for Ring of Integers Modulo Power of Prime

This section presents a formula for the Sombor index of the GCD graph for the ring of the integers modulo a power of primes.

The Sombor index is motivated by the function $f(x,y) = \sqrt{x^2+y^2}$, a function that has not previously been used in degree-based topological index theory [4]. The Sombor index is defined as $SO(G) = \sum_{v_iv_j\in E(G)}\sqrt{(\deg(v_i))^2+(\deg(v_j))^2}$, where $\deg(v_i)$ and $\deg(v_j)$ are the degrees of the vertex v_i and v_j, respectively. This index can also be viewed as a geometric interpretation of the degree-radius of an edge $e = v_iv_j$, which represents the distance from the origin to the ordered pair $(\deg(v_i), \deg(v_j))$ [5].

Let δ and Δ represent the minimum and maximum degrees of vertices in $\Gamma\left(\mathbb{Z}_{p^k}\right)$. We state the Sombor index of $\Gamma\left(\mathbb{Z}_{p^k}\right)$ as the main result in the following theorem.

Theorem 3.1. *Let $\Gamma\left(\mathbb{Z}_{p^k}\right)$ be GCD graph of the integer ring of modulo p^k, where p is a prime number and $k \geq 2$ is an integer, then the Sombor index is $SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = (\Delta-\delta+1)\,\delta\sqrt{\delta^2+\Delta^2} + \frac{1}{2}\cdot\delta(\delta-1)\Delta\sqrt{2}$.*

Proof. It is known that the GCD graph $\Gamma(\mathbb{Z}_{p^k})$ can be partitioned into two vertex sets V_1 and V_2. Furthermore, based on Theorem 2.8, the set V_2 induces a complete subgraph K_δ. Each vertex $u_1 \in V_1$ is adjacent to all vertices in V_2, forming a complete bipartite subgraph $K_{|V_1|,\delta}$, while any two distinct vertices in V_1 are not adjacent. According to Theorem 2.3, the vertex degrees are given by $\deg(u_1) = \delta$ for $u_1 \in V_1$ and $\deg(v_1) = \Delta$ for $v_1 \in V_2$.

Moreover, based on Theorem 2.9, the number of edges between V_1 and V_2 (the complete bipartite subgraph) is $|E(V_1,V_2)| = |V_1|\cdot|V_2| = (\Delta-\delta+1)\delta$ and the number of edges within V_2 (the complete subgraph) is $|E(V_2)| = \frac{1}{2}\delta(\delta-1)$. Furthermore, there are two types of edges in this graph: edges connecting $u_1 \in V_1$ with $v_1 \in V_2$ and edges connecting vertices $v_1, v_2 \in V_2$. The numbers of such edges are, respectively, $(\Delta-\delta+1)\,\delta$ for edges $u_1v_1 \in E$, and $\frac{1}{2}\delta(\delta-1)$ for edges $v_1v_2 \in E$. Observe that,

$$
\begin{aligned}
SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &= \sum_{uv\in E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)}\sqrt{(\deg(u))^2+(\deg(v))^2} \\
&= \sum_{u_1v_1\in E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)}\sqrt{(\deg(u_1))^2+(\deg(v_1))^2}+ \\
&\quad \sum_{v_1v_2\in E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)}\sqrt{(\deg(v_1))^2+(\deg(v_2))^2}\,(\Delta-\delta+1)\,\delta\sqrt{\delta^2+\Delta^2} \\
&\quad + \frac{1}{2}\cdot\delta(\delta-1)\sqrt{\Delta^2+\Delta^2} \\
SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &= (\Delta-\delta+1)\,\delta\sqrt{\delta^2+\Delta^2} + \frac{1}{2}\cdot\delta(\delta-1)\Delta\sqrt{2}.
\end{aligned}
$$

Therefore, the general formula for the Sombor index of the GCD graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is $SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = (\Delta-\delta+1)\,\delta\sqrt{\delta^2+\Delta^2}+\frac{1}{2}\cdot\delta(\delta-1)\Delta\sqrt{2}$. $\square$

Example 3.2. From Figure 1, the graph is partitioned into two vertex sets, $V_1 = \{2,4,6\}$ and $V_2 = \{0,1,3,5,7\}$. Furthermore, we have $\deg(u) = 5$ and $\deg(v) = 7$ for $u \in V_1$ and $v \in V_2$. Since any two vertices in V_2 are adjacent, the number of edges formed within V_2 is $C(5,2) = 10$. Each vertex in V_1 is adjacent only to vertices in V_2, resulting in $3\cdot 5 = 15$ edges. Therefore, the Sombor index of the GCD Graph for $\mathbb{Z}_8$ is given as follows:

$$\begin{aligned}
SO\left(\Gamma\left(\mathbb{Z}_8\right)\right) &= \sum_{uv\in E(\Gamma(\mathbb{Z}_8))} \sqrt{(\deg(u))^2+(\deg(v))^2} \\
&= \sum_{\substack{v_1v_2\in E(\Gamma(\mathbb{Z}_8))\\ v_1 \text{ and } v_2\in V_2}} \sqrt{(\deg(v_1))^2+(\deg(v_2))^2} \\
&\quad + \sum_{\substack{u_1v_1\in E(\Gamma(\mathbb{Z}_8))\\ u_1\in V_1, v_1\in V_2}} \sqrt{(\deg(u_1))^2+(\deg(v_1))^2} \\
&= 10\sqrt{7^2+7^2}+15\sqrt{5^2+7^2} \\
SO\left(\Gamma\left(\mathbb{Z}_8\right)\right) &= 70\sqrt{2}+15\sqrt{74}.
\end{aligned}$$

Observe that the minimum and maximum degrees of the GCD graph $\Gamma\left(\mathbb{Z}_{2^3}\right)$ are obtained as $\delta = 5$ and $\Delta = 7$, respectively. Next, by applying Theorem 3.1, we obtain

$$\begin{aligned}
SO\left(\Gamma\left(\mathbb{Z}_{2^3}\right)\right) &= (\Delta-\delta+1)\,\delta\sqrt{\delta^2+\Delta^2}+\frac{1}{2}\cdot\delta(\delta-1)\Delta\sqrt{2} \\
&= (7-5+1)\cdot 5\sqrt{5^2+7^2}+\frac{1}{2}\cdot 5(5-1)\cdot 7\sqrt{2} \\
SO\left(\Gamma\left(\mathbb{Z}_{2^3}\right)\right) &= 15\sqrt{74}+70\sqrt{2}.
\end{aligned}$$

From the calculations above, both using the definition of the Sombor index and Theorem 3.1, the value of the Sombor index is the same. Next, the upper and lower bounds of the Sombor index are discussed.

Theorem 3.3. *Let $\Gamma\left(\mathbb{Z}_{p^k}\right)$ be the GCD graph of the integer ring of modulo p^k, where p is a prime and $k \geq 2$ is an integer. The lower and upper bounds for the Sombor index of the GCD graph are $\frac{1}{2}\sqrt{2}\cdot\delta^2\cdot(\delta-1) \leq SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) \leq \frac{1}{2}\sqrt{2}\cdot\Delta^2\cdot(3\Delta-1)$.*

Proof. Since $p^{k-1}-1 = \Delta-\delta+1 \leq \Delta$ and by Theorem 3.1, we have

$$SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = (\Delta-\delta+1)\,\delta\sqrt{\delta^2+\Delta^2}+\frac{1}{2}\sqrt{2}\cdot\delta(\delta-1)\Delta.$$

Because $\Delta-\delta+1 \leq \Delta$, it follows that

$$\begin{aligned}
SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &\leq \Delta\cdot\delta\sqrt{\delta^2+\Delta^2}+\frac{1}{2}\sqrt{2}\cdot\delta\cdot(\delta-1)\cdot\Delta \\
&= \Delta\cdot\delta\left(\sqrt{\delta^2+\Delta^2}+\frac{1}{2}\sqrt{2}\cdot(\delta-1)\right).
\end{aligned}$$

Since $\delta \leq \Delta$, we obtain

$$\begin{aligned}
SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &\leq \Delta\cdot\Delta\left(\sqrt{\Delta^2+\Delta^2}+\frac{1}{2}\sqrt{2}\cdot(\Delta-1)\right) \\
&= \Delta^2\left(\Delta\cdot\sqrt{2}+\frac{1}{2}\sqrt{2}\cdot(\Delta-1)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \Delta^2 \cdot \sqrt{2}\left(\Delta + \frac{1}{2}(\Delta - 1)\right) \\
&= \frac{1}{2}\sqrt{2} \cdot \Delta^2 \cdot (3\Delta - 1)\,.
\end{aligned}
$$

On the other hand, since $\Delta - \delta + 1 \geq 1$, we obtain the lower bound

$$SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) \geq \delta\sqrt{\delta^2 + \Delta^2} + \frac{1}{2}\sqrt{2} \cdot \delta \cdot (\delta - 1) \cdot \Delta.$$

Using $\Delta \geq \delta$, we have

$$
\begin{aligned}
SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &\geq \delta\left(\sqrt{\delta^2 + \delta^2} + \frac{1}{2}\sqrt{2} \cdot (\delta - 1)\delta\right) \\
&= \sqrt{2} \cdot \delta^2\left(1 + \frac{1}{2}\sqrt{2} \cdot (\delta - 1)\right) \\
&\geq \sqrt{2} \cdot \delta^2\left(\frac{1}{2}\sqrt{2} \cdot (\delta - 1)\right) \\
&\geq \frac{1}{2}\sqrt{2} \cdot \delta^2 \cdot (\delta - 1).
\end{aligned}
$$

Therefore, the Sombor index satisfies the bounds

$$\frac{1}{2}\sqrt{2} \cdot \delta^2 \cdot (\delta - 1) \leq SO\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) \leq \frac{1}{2}\sqrt{2} \cdot \Delta^2 \cdot (3\Delta - 1)\,.$$

□

According to the last theorem, the lower and upper bounds of the Sombor index are, respectively, $\frac{1}{2}\sqrt{2} \cdot \delta^2 \cdot (\delta - 1)$ and $\frac{1}{2}\sqrt{2} \cdot \Delta^2 \cdot (3\Delta - 1)$ where δ and Δ represent the minimum and maximum degrees of vertices.

Example 3.4. Referring to Figure 1, we obtain the maximum degree $\Delta = 7$ and the minimum degree $\delta = 5$, consequently, the following inequality holds.

$$
\begin{aligned}
\delta^2 + \Delta^2 &= 7^2 + 5^2 \\
&= \sqrt{74} \cdot \sqrt{74} \\
&< \sqrt{625} \cdot \sqrt{74} \\
&= 5^2 \cdot \sqrt{74} \\
&= \delta^2\sqrt{\delta^2 + \Delta^2}.
\end{aligned}
$$

4. Conclusion

In this research, we examined the Sombor index of the GCD graph defined over the ring of integers modulo a power of a prime. We first established the fundamental graph-theoretic properties of this structure and, based on these characteristics, derived explicit analytical expressions for the Sombor index together with its upper and lower bounds. This study represents a new contribution to the intersection of algebraic graph theory and degree-based topological indices. Moreover, since the GCD graph defined over the ring of integers modulo a power of a prime is connected, future studies may also consider distance-based topological indices and spectral properties, potentially yielding new theoretical insights and practical applications across algebraic combinatorics, network analysis, and mathematical chemistry.

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