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SOME RESULTS ON PACHPATTE-TYPE OF OPIAL INEQUALITY

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ABSTRACT. This paper applies the modified Jensen inequality to generalize some cases of Pachpatte results of Opial-type inequalities on time scales. These inequalities further generalize some existing results.

1. PRELIMINARIES

The famous Opial inequality was first obtained by Z. Opial [7] in the sixties as follows:

Theorem 1.1. *([7]) Suppose $g \in C^1[0,\Lambda]$ satisfies $g(0) = g(\Lambda) = 0$ and $g(y) > 0$ for all $y \in (0,\Lambda)$. Then, the following integral inequality holds:*

$$\int_0^\Lambda |g(y)g'(y)|dy \le \frac{\Lambda}{4}\int_0^\Lambda (g'(y))^2 dy, \tag{1.1}$$

where $\Lambda/4$ is the best possible constant.

Inequality (1.1) and some of its generalizations have different applications in the theories of differential, difference and integro-differential equations. The inequality is accurately discussed in books exclusively devoted to Opial-type inequalities, see [1,8] for instance.

In the past few decades, several results have appeared in the literature on various generalizations and refinements of Opial type inequality. Of specific interest is the new integral inequalities involving two functions and their first order derivatives proved by Pachpatte [9], which is a special case of Opial's inequality. Specifically, Pachpatte stated this:

Theorem 1.2. *([9, Theorem 2]). Let $p(\varpi)$ and $q(\varpi)$ be positive and continuous with $q(\varpi)$ being a bounded and nonincreasing function on $[\alpha,\tau]$ and $\int_\alpha^\tau p^{-1}(\varpi)d\varpi < \infty$. Suppose $\eta_1(\varpi)$ and $\eta_2(\varpi)$ are absolutely continuous on $[\alpha,\tau]$ and $\eta_1(\alpha) = \eta_2(\alpha) = 0$. Then, the following inequality*

$$\begin{aligned}\int_\alpha^\tau q(\varpi)\left(|\eta_1'(\varpi)\eta_2(\varpi)| + |\eta_1(\varpi)\eta_2'(\varpi)|\right)d\varpi \\ \le \frac{1}{2}\int_\alpha^\tau \frac{d\varpi}{p(\varpi)}\int_\alpha^\tau p(\varpi)q(\varpi)\left[|\eta_1'(\varpi)|^2 + |\eta_2'(\varpi)|^2\right]d\varpi\end{aligned} \tag{1.2}$$

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holds with equality if and only if $q(\varpi) =$ constant and $\eta_1(\varpi) = \eta_2(\varpi) = C\int_\alpha^\varpi p^{-1}(s)ds$ for $\varpi \in [\alpha, \tau]$ and C is a constant.

In a related development, Lin and Yang [6] established the following inequality for two functions:

Theorem 1.3. *([6]) Let $p(t)$ and $q(t)$ be positive and continuous with $q(t)$ being a nonincreasing functions. Suppose $x_1(t)$ and $x_2(t)$ are absolutely continuous on $[\alpha, \tau]$, and $x_1(\alpha) = x_2(\alpha) = 0$. If $l \geq 0$ and $m \geq 1$, then, the following inequality holds:*

$$\begin{aligned}\int_\alpha^\tau q(t)\left|x_1(t)x_2(t)\right|^l\left[\left|x_1(t)x_2'(t)\right|^m + \left|x_1'(t)x_2(t)\right|^m\right]dt \\ \leq \frac{m}{2(l+m)}(\tau-\alpha)^{2l+m}\int_\alpha^\tau p(t)\left[\left|x_1'(t)\right|^{2(l+m)} + \left|x_2'(t)\right|^{2(l+m)}\right]dt.\end{aligned} \tag{1.3}$$

This paper therefore presents some generalization of special cases of Pachpatte results of Opial-type inequalities on time scales to the setting of several functions. The inequalities obtained here further generalize some existing results involving many functions as will be discussed in the next sections.

It is in order at this point to recall some necessary background facts that will be useful in the statement and proofs of our main results.

Time scale: A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real line $\mathbb{R}$ with topology of the subspace $\mathbb{R}$. Examples include $\mathbb{R}$, $\mathbb{Z}$, and $q^{\mathbb{N}_0} = \left\{q^k : k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, q > 0\right\}$. Since a time scale $\mathbb{T}$ may or may not be connected, the concept of jump operators are required. Let $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

while the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, then the t is called right-scattered and if $\rho(t) < t$ then t is left-scattered. The points that are both right-scattered and left-scattered are called isolated. If $\sigma(t) = t$ then t is said to be right-dense, and if $\rho(t) = t$ then t is left-dense. The points that are simultaneously right-dense and left-dense are called dense. The mapping $\mu : \mathbb{T} \longrightarrow [0, \infty)$ denoted by $\mu(t) = \sigma(t) - t$ is called the graininess function. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T}\backslash M$; otherwise $\mathbb{T}^k = \mathbb{T}$. Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be a function. Then, the function $f^\sigma : \mathbb{T} = \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$. Also, for a function $f : \mathbb{T} \longrightarrow \mathbb{R}$, the delta derivative is define by

$$f^\Delta(t) = \lim_{s\longrightarrow t, \sigma(s)\neq t}\frac{f^\sigma(s) - f(t)}{\sigma(s) - t}.$$

A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limit exists at all left-dense points in $\mathbb{T}$. See [2], [5] and [10] for detail discussion on the calculus of time scale.

Jensen's inequality. The Jensen inequality for convex functions in several variables on time scale and its highlited properties for multivariate convex functions on an arbitrary time scale as established in [4] will be required in this paper.

Lemma 1.4. *(Jensen's inequality [4]).*
Suppose that $U \subset \mathbb{R}^n$ is a closed convex set, $\phi \in C(U, \mathbb{R})$ is convex and $\mathbf{f}(\Omega_2) \subset U$. Moreover, let $k : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ be nonnegative such that $k(x, .)$ is λ_Δ-integrable. Then one has

$$\phi\left(\frac{\int_{\Omega_2} k(x,y)\mathbf{f}(y)\Delta y}{\int_{\Omega_2} k(x,y)\Delta y}\right) \leq \frac{\int_{\Omega_2} k(x,y)\phi(\mathbf{f}(y))\Delta y}{\int_{\Omega_2} k(x,y)\Delta y} \tag{1.4}$$

for all functions $\mathbf{f}:\Omega_2 \longrightarrow U$, *where* $f_j(y)$ *are* μ_Δ*-integrable for all* $j\in\{1,2.3...m\}$ *and* $\mathbf{f}(y)=(f_1(y),...,f_m(y))$

The required modified Jensen inequality as used in [3] is the following: Let $\psi,\varphi\in C([\alpha,\beta])$. Suppose φ is convex, ψ nonnegative and $\lambda(s)$ is nondecreasing. Then Let $\psi,\varphi\in C([\alpha,\beta])$. Suppose φ is convex, ψ nonnegative and $\lambda(s)$ is nondecreasing. Then

$$\left(\int_\epsilon^t d\lambda(s)\right)^{-\varsigma} \le \left(\int_\epsilon^t \varphi(\psi(s))^{\frac{1}{\varsigma}} d\lambda(s)\right)^{\zeta} \left(\int_\epsilon^t \psi(t) d\lambda(s)\right)^{-\varsigma} \left(\int_\epsilon^t d\lambda(s)\right)^{-\zeta}. \tag{1.5}$$

Throughout this paper, the left-hand sides of the inequalities exist if the right-hand sides exist.

2. Extension of Pachpatte Result on time scales

The main results of this paper are presented and proved in this section.

Theorem 2.1. *Let* $\mathbb{T}$ *be a time scale with* $s,\hbar\in\mathbb{T}$. *Let* ς, ζ *be real numbers, let* $\aleph,\chi\in C_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R})$ *where* h *and* χ *are positive rd-continuous functions on* $[\alpha,\beta]_{\mathbb{T}}$ *such that* $\int_{[0,t]} h(s)\Delta(s)<\infty$. *Define* φ *as convex function and if* $\wp_{\gamma_1},\wp_{\gamma_2}:[\alpha,\beta]_{\mathbb{T}}\longrightarrow\mathbb{R}$ *are delta differentiable with* $\alpha(0)=0$, *then*

$$\begin{aligned}&\int_{[\alpha,\beta]} (\sqrt{\chi(\hbar)})^{\varsigma+1} \wp_{\gamma_1}^{\Delta}(\hbar)\wp_{\gamma_2}^{\varsigma}(\hbar)\Delta(\hbar)\\ &\le \frac{1}{1+\zeta}(\beta-\alpha)^{\varsigma-\zeta}\left(\int_{[\alpha,\beta]}\frac{\Delta(\hbar)}{(\sqrt{\aleph(\hbar)})^{1+\zeta}}\right)\int_{[\alpha,\beta]}(\sqrt{\aleph(\hbar)\chi(\hbar)})^{1+\zeta}\wp_{\gamma_1}^{\Delta}(\hbar)^{1+\zeta}\Delta(\hbar).\end{aligned} \tag{2.1}$$

Proof. Using the modified Jensen's inequality

$$\left(\int_\epsilon^t d\lambda(s)\right)^{-\varsigma} \le \left(\int_\epsilon^t \varphi(\psi(s))^{\frac{1}{\varsigma}} d\lambda(s)\right)^{\zeta} \left(\int_\epsilon^t \psi(s) d\lambda(s)\right)^{-\varsigma} \left(\int_\epsilon^t d\lambda(s)\right)^{-\zeta}.$$

with

$$\begin{cases} \psi(s)=\sqrt{\chi(s)} \\ \text{and } d\lambda(s)=\wp_{\gamma_1}^{\Delta}\Delta(s). \end{cases}$$

Therefore

$$\begin{aligned}(\beta-\alpha)^{-\varsigma} &\le \left(\int_{[\alpha,\beta]}\varphi(\psi(s))^{\frac{1}{\varsigma}}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{\zeta}\left(\int_{[\alpha,\beta]}\psi(s)\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}\\
&=\left(\int_{[\alpha,\beta]}\varphi\left(\sqrt{\chi(s)}\right)^{\frac{1}{\varsigma}}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{\zeta}\left(\int_{[\alpha,\beta]}\sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}\\
&=\left(\int_{[\alpha,\beta]}\left(\sqrt{\chi(s)}\right)^{\frac{\varsigma}{\zeta}}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{\zeta}\left(\int_{[\alpha,\beta]}\sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}\\
&\le\left(\int_{[\alpha,\beta]}\sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}\Delta(s)\right)^{\zeta}\left(\int_{[\alpha,\beta]}\sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}.\end{aligned}$$

Then

$$\begin{aligned}&(\beta-\alpha)^{-\varsigma}\\ &\le\left(\int_{[\alpha,\beta]}\sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{\zeta}\left(\int_{[\alpha,\beta]}\sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}\end{aligned}$$

and

$$\left(\int_{[\alpha,\beta]}\sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}\Delta(s)\right)^{\varsigma}\le\left(\int_{[\alpha,\beta]}\sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{\zeta}(\beta-\alpha)^{\varsigma-\zeta}. \tag{2.2}$$

Now

$$\gamma_1(\hbar) = \int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)$$

$$\gamma_1^{\Delta}(\hbar) = \sqrt{\chi(\hbar)}\wp_{\gamma_1}^{\Delta}(\hbar). \tag{2.3}$$

In view of (2.2) and (2.3), we obtain

$$\wp_{\gamma_1}^{\Delta}(\hbar)\left(\int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{\varsigma} \leq (\beta-\alpha)^{\varsigma-\zeta}\,\frac{\gamma_1^{\Delta}(\hbar)}{\sqrt{\chi(\hbar)}}\gamma_1(\hbar)^{\zeta}$$

$$(\sqrt{\chi(\hbar)})^{\varsigma+1}\wp_{\gamma_1}^{\Delta}(\hbar)\left(\int_{[\alpha,\beta]} \wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{\varsigma} \leq (\beta-\alpha)^{\varsigma-\zeta}\,\gamma_1^{\Delta}(\hbar)\gamma_1(\hbar)^{\zeta}. \tag{2.4}$$

□

Remark 2.2. We observed that if $\mathbb{T}=\mathbb{R}, \wp_{\gamma_1}(s) = \int_{[\alpha,\beta]}\wp_{\gamma_1}^{\Delta}(s)\Delta(s), \varsigma = \zeta$ and integrate with respect to $\hbar$ then, the last inequality becomes

$$\int_{[\alpha,\beta]} \chi(\hbar)\wp_{\gamma_1}\wp_{\gamma_1}^{\Delta}(\hbar)\Delta(\hbar) \leq \int_{[\alpha,\beta]} \gamma_1^{\Delta}(\hbar)\gamma_1(\hbar)\Delta(\hbar) = \frac{1}{2}{\gamma_1}^2(\beta) \tag{2.5}$$

by the definition of γ_1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{[\alpha,\beta]} \chi(\hbar)\wp_{\gamma_1}\wp_{\gamma_1}^{\Delta}(\hbar)\Delta(\hbar) &\leq \frac{1}{2}{\gamma_1}^2(\beta) = \frac{1}{2}\left(\int_{[\alpha,\beta]} \frac{1}{\aleph(\hbar)}\sqrt{\aleph(\hbar)\chi(\hbar)}\gamma_1^{\Delta}(\hbar)\right)^2 \\ &\leq \frac{1}{2}\int_{[\alpha,\beta]} \frac{\Delta(\hbar)}{\aleph(\hbar)}\int_{[\alpha,\beta]} \aleph(\hbar)\chi(\hbar)(\gamma_1^{\Delta}(\hbar))^2\Delta(\hbar),\end{aligned} \tag{2.6}$$

which is Yang [6] inequality of Opial-type.

Furthermore, the proof of Theorem 2.1, can be refined as $\psi(s) \longrightarrow \sqrt{\chi(s)}$ and $d\lambda(s) \longrightarrow \wp_{\gamma_2}^{\Delta}(s)$ in the above modified Jensen inequality, then

$$\begin{aligned}&\left(\int_{[\alpha,\beta]} \varphi\left(\sqrt{\chi(s)}\right)^{\frac{1}{\zeta}}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{\zeta}\left(\int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta} \\ &\leq \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi(s)}\right)^{\frac{\varsigma}{\zeta}}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{\zeta}\left(\int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_1}^{\Delta}\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta} \\ &\leq \left(\int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_2}^{\Delta}\Delta(s)\right)^{\zeta}\left(\int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}\end{aligned}$$

implying that

$$\begin{aligned}&(\beta-\alpha)^{-\varsigma} \\ &\leq \left(\int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{\zeta}\left(\int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}.\end{aligned}$$

Then

$$\left(\int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_2}^{\Delta}\Delta(s)\right)^{\varsigma} \leq \left(\int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{\zeta}(\beta-\alpha)^{\varsigma-\zeta} \tag{2.7}$$

and

$$\gamma_2(t) = \int_{[\alpha,\beta]} \sqrt{\chi(s)}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)$$

$$\gamma_2^{\Delta}(\hbar) = \sqrt{\chi(\hbar)}\wp_{\gamma_2}^{\Delta}(\hbar) \tag{2.8}$$

In view of (2.7) and (2.8), we get

$$(\sqrt{\chi(\hbar)})^{\varsigma+1}\wp_{\gamma_2}^{\Delta}(\hbar)\left(\int_{[\alpha,\beta]}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{\varsigma} \leq (\beta-\alpha)^{\varsigma-\zeta}\,\gamma_2^{\Delta}(\hbar)\gamma_2(\hbar)^{\zeta}. \tag{2.9}$$

Adding both side of (2.4) and (2.9) yields

$$\begin{aligned}&(\sqrt{\chi(\hbar)})^{\varsigma+1}\wp_{\gamma_1}^{\Delta}(\hbar)\left(\int_{[\alpha,\beta]}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{\varsigma} + (\sqrt{\chi(\hbar)})^{\varsigma+1}\wp_{\gamma_2}^{\Delta}(\hbar)\left(\int_{[\alpha,\beta]}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{\varsigma}\\ &\leq (\beta-\alpha)^{\varsigma-\zeta}\,\gamma_1^{\Delta}(\hbar)\gamma_1(\hbar)^{\zeta} + (\beta-\alpha)^{\varsigma-\zeta}\,\gamma_2^{\Delta}(\hbar)\gamma_2(\hbar)^{\zeta}.\end{aligned} \tag{2.10}$$

Integrating both side of (2.10) with respect to delta derivative gives

$$\begin{aligned}&\int_{[\alpha,\beta]}(\sqrt{\chi(\hbar)})^{\varsigma+1}\left(\wp_{\gamma_2}^{\Delta}(\hbar)\wp_{\gamma_1}^{\varsigma}(\hbar) + \wp_{\gamma_1}^{\Delta}(\hbar)\wp_{\gamma_2}^{\varsigma}(\hbar)\right)\Delta(\hbar)\\ &\qquad\leq \frac{1}{1+\zeta}(\beta-\alpha)^{\varsigma-\zeta}\left(\int_{[\alpha,\beta]}\frac{\Delta(\hbar)}{(\sqrt{\aleph(\hbar)})^{1+\zeta}}\right)\\ &\qquad\times\int_{[\alpha,\beta]}(\sqrt{\aleph(\hbar)\chi(\hbar)})^{1+\zeta}\left(\wp_{\gamma_1}^{\Delta}(\hbar)^{1+\zeta} + \wp_{\gamma_2}^{\Delta}(\hbar)^{1+\zeta}\right)\Delta(\hbar).\end{aligned} \tag{2.11}$$

□

Lemma 2.3. *Let $\wp_{\gamma_1(s)}$ and $\wp_{\gamma_2(s)}$ are absolutely continuous functions. Then, for $\theta \geq 0$ the following inequality holds:*

$$\left(\wp_{\gamma_1(s)}\wp_{\gamma_2(s)}\right)^{\vartheta} \leq \left(\gamma_1(s)\gamma_2(s)\right)^{\vartheta}. \tag{2.12}$$

The proof of the last Lemma is trivial and it is not included here.

In (2.3), if $\gamma_1(s) = \gamma_2(s),\ \wp_{\gamma_1}(s) = \left(\int_{[\alpha,\beta]}\wp_{\gamma_2}^{\Delta}(s)\Delta(s)\right)^{\varsigma}$ and similarly, in (2.8) $\gamma_2(s) = \gamma_1(s),\ \wp_{\gamma_2}(s) = \left(\int_{[\alpha,\beta]}\wp_{\gamma_1}^{\Delta}(s)\Delta(s)\right)^{\varsigma}$. Therefore, (2.4) and (2.9) becomes

$$(\chi(\hbar))\wp_{\gamma_2}(\hbar)\wp_{\gamma_1}^{\Delta}(\hbar) \leq \gamma_2^{\Delta}(\hbar)\gamma_1(\hbar) \tag{2.13}$$

and

$$(\chi(\hbar))\wp_{\gamma_1}(\hbar)\wp_{\gamma_2}^{\Delta}(\hbar) \leq \gamma_1^{\Delta}(\hbar)\gamma_2(\hbar) \tag{2.14}$$

respectively, if $\chi(\hbar) = \varsigma = \zeta = 1$. Taking the addition of both (2.13) and (2.14) yields

$$\left[\wp_{\gamma_2}(\hbar)\wp_{\gamma_1}^{\Delta}(\hbar) + \wp_{\gamma_1}(\hbar)\wp_{\gamma_2}^{\Delta}(\hbar)\right] \leq \gamma_1(\hbar)\gamma_2^{\Delta}(\hbar) + \gamma_2(\hbar)\gamma_1^{\Delta}(\hbar). \tag{2.15}$$

On combining (2.12) and (2.15) and then integrate both side of (2.15) with respect to delta derivative, we get

$$\begin{aligned}&\int_{[\alpha,\beta]}\wp_{\gamma_1(\hbar)}^{\vartheta}\wp_{\gamma_2(\hbar)}^{\vartheta}\left[\wp_{\gamma_2}(\hbar)\wp_{\gamma_1}^{\Delta}(\hbar) + \wp_{\gamma_1}(\hbar)\wp_{\gamma_2}^{\Delta}(\hbar)\right]\Delta(\hbar)\\ &\qquad\leq \int_{[\alpha,\beta]}\gamma_1(\hbar)^{\vartheta}\gamma_2(\hbar)^{\vartheta}\left[\gamma_1(\hbar)\gamma_2^{\Delta}(\hbar) + \gamma_2(\hbar)\gamma_1^{\Delta}(\hbar)\right]\Delta(\hbar)\end{aligned} \tag{2.16}$$

that is

$$\begin{aligned}&\int_{[\alpha,\beta]}\wp_{\gamma_1(\hbar)}^{\vartheta}\wp_{\gamma_2}^{\vartheta}(\hbar)\left[\wp_{\gamma_2}(\hbar)\wp_{\gamma_1}^{\Delta}(\hbar) + \wp_{\gamma_1}(\hbar)\wp_{\gamma_2}^{\Delta}(\hbar)\right]\Delta(\hbar)\\ &\qquad\leq \int_{[\alpha,\beta]}\frac{\Delta}{\Delta(\hbar)}\left(\frac{1}{\vartheta+1}\gamma_1^{\vartheta+1}(\beta)\gamma_2^{\vartheta+1}(\beta)\right)\Delta(\hbar)\end{aligned} \tag{2.17}$$

which is

$$\begin{aligned}\int_{[\alpha,\beta]} &\wp^{\vartheta}_{\gamma_1(\hbar)}\wp^{\vartheta}_{\gamma_2}(\hbar)\left[\wp_{\gamma_2}(\hbar)\wp^{\Delta}_{\gamma_1}(\hbar)+\wp_{\gamma_1}(\hbar)\wp^{\Delta}_{\gamma_2}(\hbar)\right]\Delta(\hbar)\\ &\leq \frac{1}{2(\vartheta+1)}\left(\frac{1}{\vartheta+1}\gamma_1^{2(\vartheta+1)}(\beta)+\gamma_2^{2(\vartheta+1)}(\beta)\right).\end{aligned} \tag{2.18}$$

Applying Hölder's inequality with indices $2(\vartheta+1)$ and $\frac{2(\vartheta+1)}{(2\vartheta+1)}$ yields

$$\gamma_1^{2(\vartheta+1)}(\beta)=\left(\int_{[\alpha,\beta]}\wp^{\Delta}_{\gamma_2}(\hbar)\Delta(\hbar)\right)^{2(\vartheta+1)}\leq(\beta-\alpha)^{2(\vartheta+1)}\int_{[\alpha,\beta]}\wp^{\Delta}_{\gamma_2}(\hbar)^{2(\vartheta+1)}\Delta(\hbar)$$

and

$$\gamma_2^{2(\vartheta+1)}(\beta)=\left(\int_{[\alpha,\beta]}\wp^{\Delta}_{\gamma_1}(\hbar)\Delta(\hbar)\right)^{2(\vartheta+1)}\leq(\beta-\alpha)^{2(\vartheta+1)}\int_{[\alpha,\beta]}\wp^{\Delta}_{\gamma_1}(\hbar)^{2(\vartheta+1)}\Delta(\hbar).$$

Hence, we have Pachpatte's result [9] as follows:

$$\begin{aligned}&\int_{[\alpha,\beta]}\wp^{\vartheta}_{\gamma_1(\hbar)}\wp^{\vartheta}_{\gamma_2}(\hbar)\left[\wp_{\gamma_2}(\hbar)\wp^{\Delta}_{\gamma_1}(\hbar)+\wp_{\gamma_1}(\hbar)\wp^{\Delta}_{\gamma_2}(\hbar)\right]\Delta(\hbar)\\ &\leq\frac{1}{2(\vartheta+1)}(\beta-\alpha)^{2(\vartheta+1)}\int_{[\alpha,\beta]}\left[\wp^{2(\vartheta+1)}_{\gamma_1}(\hbar)+\wp^{2(\vartheta+1)}_{\gamma_2}(\hbar)\right]\Delta(\hbar).\end{aligned} \tag{2.19}$$

□

Theorem 2.4. *Let $\mathbb{T}$ be a time scale with $s,\hbar\in\mathbb{T}$. Let ς, ζ be real numbers, let $\gamma_1,\gamma_2,\hbar,\vartheta,\varpi,\chi\in C_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R})$ where h and $\chi(s)$ are positive rd-continuous functions on $[\alpha,\beta]_{\mathbb{T}}$ such that $\int_{[0,t]}h(s)\Delta(s)<\infty$. Define φ as convex function and if $\wp_{\gamma_1},\wp_{\gamma_2}:[\alpha,\beta]_{\mathbb{T}}\longrightarrow\mathbb{R}$ is delta differentiable such that $\wp_{\gamma_1}(\alpha)=\wp_{\gamma_2}(\alpha)=0$. For $\vartheta\geq0$ and $\varpi\geq1$, then*

$$\chi(\hbar)^{\frac{\varpi(\varsigma+1)}{2(\vartheta+\varpi)}}\wp^{\varpi\varsigma}_{\gamma_1}(s)\Delta(s)\wp^{\varpi\Delta}_{\gamma_2}(\hbar)\leq\gamma_1^{\zeta}(\hbar)\gamma_2^{\zeta\Delta}(\hbar)\left(\beta-\alpha\right)^{\varsigma-\zeta}.$$

Proof. By the modified Jensen's inequality of the form:

$$\left(\int_{[\alpha,t]}\sqrt{\chi(s)}\wp^{\Delta}_{\gamma_2}(s)\Delta(s)\right)^{\varsigma}\leq\left(\int_{[\alpha,t]}\sqrt{\chi(s)}\wp^{\Delta}_{\gamma_2}(s)\Delta(s)\right)^{\zeta}(\beta-\alpha)^{\varsigma-\zeta} \tag{2.20}$$

with

$$\begin{cases}\psi(s)=\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}\\ \text{and}\\ d\lambda(s)=(\wp^{\Delta}_{\gamma_1}(s))^{\varpi}\Delta(s)\end{cases}$$

then

$$\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_1}(s))^{\varpi}\Delta(s)\right)^{\varsigma}\leq\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_1}(s))^{\varpi}\Delta(s)\right)^{\zeta}(\beta-\alpha)^{\varsigma-\zeta} \tag{2.21}$$

$$\gamma_1^{\zeta}(s)=\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_1}(s))^{\varpi}\Delta(s)\right)^{\zeta}$$

$$\frac{(\gamma_2^{\Delta}(\hbar))^{\zeta}}{\chi(\hbar)^{\frac{\varpi}{2(\vartheta+\varpi)}}}=(\wp^{\Delta}_{\gamma_1}(\hbar))^{\varpi}. \tag{2.22}$$

In view of (2.21) and (2.22), we get

$$\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_1}(s))^{\varpi}\Delta(s)\right)^{\varsigma}\leq\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_1}(s))^{\varpi}\Delta(s)\right)^{\zeta}(\beta-\alpha)^{\varsigma-\zeta}$$

$$\chi(\hbar)^{\frac{\varpi(\varsigma+1)}{2(\vartheta+\varpi)}}\left(\int_{[\alpha,t]}\left((\wp^{\Delta}_{\gamma_1}(s)\right)^{\varpi}\Delta(s)\right)^{\varsigma}(\wp^{\Delta}_{\gamma_1}(\hbar))^{\varpi}\leq\gamma_1^{\zeta}(\hbar)\gamma_2^{\zeta\Delta}(\hbar)\left(\beta-\alpha\right)^{\varsigma-\zeta},$$

that is,

$$\chi(\hbar)^{\frac{\varpi(\varsigma+1)}{2(\vartheta+\varpi)}}\wp^{\varpi\varsigma}_{\gamma_1}(s)\Delta(s)\wp^{\varpi\Delta}_{\gamma_2}(\hbar)\leq\gamma_1^{\zeta}(\hbar)\gamma_2^{\zeta\Delta}(\hbar)\left(\beta-\alpha\right)^{\varsigma-\zeta}. \tag{2.23}$$

□

Theorem 2.5. *Let $\mathbb{T}$ be a time scale with $s,\hbar\in\mathbb{T}$, let ς, ζ real numbers, $\gamma_1,\gamma_2,\hbar,\vartheta,\varpi,\chi\in C_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R})$ where h and $\chi(s)$ are positive rd-continuous functions on $[\alpha,\beta]_{\mathbb{T}}$ such that $\int_{[0,t]}h(s)\Delta(s)<\infty$. We define φ as convex function and if $\wp_{\gamma_2},\wp_{\gamma_1}:[\alpha,\beta]_{\mathbb{T}}\longrightarrow\mathbb{R}$ is delta differentiable with $\wp_{\gamma_1}(\alpha)=\wp_{\gamma_2}(\alpha)=0$. For, $\vartheta\geq 0$ and $\varpi\geq 1$, then*

$$\chi(\hbar)^{\frac{\varpi(\varsigma+1)}{2(\vartheta+\varpi)}}\wp^{\varpi\varsigma}_{\gamma_2}(s)\Delta(s)\wp^{\varpi\Delta}_{\gamma_1}(\hbar)\leq\gamma_2^{\zeta}(\hbar)\gamma_1^{\zeta\Delta}(\hbar)\left(\beta-\alpha\right)^{\varsigma-\zeta}.$$

The proof of Theorem 2.5 is similar to the proof of Theorem 2.4.

Proof. The modified Jensen inequality can also be used as follows:

$$\left(\int_{[\alpha,t]}\sqrt{\chi(s)}\wp^{\Delta}_{\gamma_2}(s)\Delta(s)\right)^{\varsigma}\leq\left(\int_{[\alpha,t]}\sqrt{\chi(s)}\wp^{\Delta}_{\gamma_1}(s)\Delta(s)\right)^{\zeta}(\beta-\alpha)^{\varsigma-\zeta} \tag{2.24}$$

with

$$\begin{cases}\psi(s)=\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}\\ \text{and}\\ d\lambda(s)=(\wp^{\Delta}_{\gamma_2}(s))^{\varpi}\Delta(s)\end{cases}$$

then

$$\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_2}(s))^{\varpi}\Delta(s)\right)^{\varsigma}\leq\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_2}(s))^{\varpi}\Delta(s)\right)^{\zeta}(\beta-\alpha)^{\varsigma-\zeta} \tag{2.25}$$

$$\gamma_2^{\zeta}(s)=\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_2}(s))^{\varpi}\Delta(s)\right)^{\zeta}$$

$$\frac{(\gamma_1^{\Delta}(\hbar))^{\zeta}}{\chi(\hbar)^{\frac{\varpi}{2(\vartheta+\varpi)}}}=(\wp^{\Delta}_{\gamma_2}(\hbar))^{\varpi}. \tag{2.26}$$

In view of (2.25) and (2.26), we get

$$\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_2}(s))^{\varpi}\Delta(s)\right)^{\varsigma}\leq\left(\int_{[\alpha,t]}\chi(s)^{\frac{\varpi}{2(\vartheta+\varpi)}}(\wp^{\Delta}_{\gamma_2}(s))^{\varpi}\Delta(s)\right)^{\zeta}(\beta-\alpha)^{\varsigma-\zeta}$$

$$\chi(\hbar)^{\frac{\varpi(\varsigma+1)}{2(\vartheta+\varpi)}}\left(\int_{[\alpha,t]}\left((\wp^{\Delta}_{\gamma_2}(s)\right)^{\varpi}\Delta(s)\right)^{\varsigma}(\wp^{\Delta}_{\gamma_1}(\hbar))^{\varpi}\leq\gamma_2^{\zeta}(\hbar)\gamma_1^{\zeta\Delta}(\hbar)\left(\beta-\alpha\right)^{\varsigma-\zeta}$$

that is

$$\chi(\hbar)^{\frac{\varpi(\varsigma+1)}{2(\vartheta+\varpi)}}\wp^{\varpi\varsigma}_{\gamma_2}(s)\Delta(s)\wp^{\varpi\Delta}_{\gamma_1}(\hbar)\leq\gamma_2^{\zeta}(\hbar)\gamma_1^{\zeta\Delta}(\hbar)\left(\beta-\alpha\right)^{\varsigma-\zeta}. \tag{2.27}$$

On combining (2.12), (2.23) and (2.27) then integrate with respect to $\hbar$ from α to β yields the following inequality

$$\begin{aligned}
&\int_{[\alpha,\beta]} \chi(\hbar)\left(\wp_{\gamma_1}(\hbar)\wp_{\gamma_2}(\hbar)\right)^{\vartheta}\left[\wp_{\gamma_2}^{\varpi\varsigma}(\hbar)\Delta(\hbar)\wp_{\gamma_1}^{\varpi\Delta}(\hbar)+\chi(\hbar)\wp_{\gamma_1}^{\varpi\varsigma}(\hbar)\Delta(\hbar)\wp_{\gamma_2}^{\varpi\Delta}(\hbar)\right]\Delta(\hbar)\\
&\leq (\beta-\alpha)^{\frac{(\varpi-1)(2\vartheta+\varpi)+\varpi(\varsigma-\zeta)}{\varpi}}\int_{[\alpha,\beta]}\left(\gamma_1(\hbar)\gamma_2(\hbar)\right)^{\frac{\vartheta}{\varpi}}\left[\gamma_2^{\zeta}(\hbar)\gamma_1^{\zeta\Delta}(\hbar)+\gamma_1^{\zeta}(\hbar)\gamma_2^{\zeta\Delta}(\hbar)\right]\Delta(\hbar)\\
&=\frac{\varpi}{\vartheta+\varpi}(\beta-\alpha)^{\frac{(\varpi-1)(2\vartheta+\varpi)+\varpi(\varsigma-\zeta)}{\varpi}}\left(\gamma_1^{\zeta}(\beta)\gamma_2^{\zeta}(\beta)\right)^{\frac{\vartheta+\varpi}{\varpi}}\\
&\leq \frac{\varpi}{2(\vartheta+\varpi)}(\beta-\alpha)^{\frac{(\varpi-1)(2\vartheta+\varpi)+\varpi(\varsigma-\zeta)}{\varpi}}\left(\gamma_1^{\frac{2\zeta(\vartheta+\varpi)}{\varpi}}(\beta)+\gamma_2^{\frac{2\zeta(\vartheta+\varpi)}{\varpi}}(\beta)\right).
\end{aligned} \tag{2.28}$$

Applying Hölder's inequality with indices $\frac{2(\vartheta+\varpi)}{\varpi}$ and $\frac{2(\vartheta+\varpi)}{2\vartheta+\varpi}$ yields

$$\gamma_1^{\frac{2\zeta(\vartheta+\varpi)}{\varpi}}(\beta)=\int_{[\alpha,\beta]}\chi(s)\wp_{\gamma_1}^{\frac{2\zeta(\vartheta+\varpi)}{\varpi}}\Delta(s) \tag{2.29}$$

$$\gamma_2^{\frac{2\zeta(\vartheta+\varpi)}{\varpi}}(\beta)=\int_{[\alpha,\beta]}\chi(s)\wp_{\gamma_2}^{\frac{2\zeta(\vartheta+\varpi)}{\varpi}}\Delta(s). \tag{2.30}$$

We therefore substitute (2.29) and (2.30) to obtain the following inequality

$$\begin{aligned}
&\int_{[\alpha,\beta]} \chi(\hbar)\left(\wp_{\gamma_1}(\hbar)\wp_{\gamma_2}(\hbar)\right)^{\vartheta}\left[\wp_{\gamma_2}^{\varpi\varsigma}(\hbar)\Delta(\hbar)\wp_{\gamma_1}^{\varpi\Delta}(\hbar)+\chi(\hbar)\wp_{\gamma_1}^{\varpi\varsigma}(\hbar)\Delta(\hbar)\wp_{\gamma_2}^{\varpi\Delta}(\hbar)\right]\Delta(\hbar)\\
&\leq \frac{\varpi}{2(\vartheta+\varpi)}(\beta-\alpha)^{\frac{(2\vartheta+\varpi)+\varpi(\varsigma-\zeta)}{\varpi}}\int_{[\alpha,\beta]}\chi(\hbar)\left[\gamma_1^{\zeta\Delta}(\hbar)+\gamma_2^{\zeta\Delta}(\hbar)\right]\Delta(\hbar),
\end{aligned} \tag{2.31}$$

which is Lin and Yang's [6] result.

2.1. **Remark.** We see that if $\mathbb{T}=\mathbb{R}$ and $q(t)=\chi(\hbar),\ \zeta=\varsigma=1,\ \wp_{\gamma_1}(\hbar)=\eta_1(\varpi),\eta_2(\varpi)=\wp_{\gamma_2}(\hbar),\ \aleph(\hbar)=p(\varpi)$ and $\Delta(\hbar)=d\varpi$ (2.11) reduces to (1.2).

3. On Opial-type Inequality of Pachpatte's Result of Many Functions

The modified Jensen's inequality would be used to obtain an extension and refinement of Opial-type inequalities on time scales to several functions. The results are as follows.

Theorem 3.1. *Let $\mathbb{T}$ be a time scale with $s,\hbar\in\mathbb{T}$. Let ς, ζ be real numbers , let $\aleph_i,\chi_1...\chi_n\in C_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R})$ where h and χ are positive rd-continuous functions on $[\alpha,\beta]_{\mathbb{T}}$ such that $\int_{[0,t]} r(t)\Delta(s)<\infty$. Define φ as a convex function and if $\wp_{\gamma_1},\wp_{\gamma_2},...\wp_{\gamma_n}:[\alpha,\beta]_{\mathbb{T}}\longrightarrow\mathbb{R}$ is delta differentiable with $\alpha(0)=0$, then we have*

$$\begin{aligned}
&\int_{[\alpha,\beta]}\left(\sqrt{\chi_1(\hbar)},...,\sqrt{\chi_n(\hbar)}\right)^{\varsigma+1}\left(\wp_{\gamma_1}^{\Delta}(\hbar),\wp_{\gamma_2}^{\Delta}(\hbar),...,\wp_{\gamma_n}^{\Delta}(\hbar)\right)\times\left(\wp_{\gamma_2}(\hbar),\wp_{\gamma_3}(\hbar),...,\wp_{\gamma_{n+1}}(\hbar)\right)^{\varsigma}\Delta(\hbar)\\
&\leq\frac{1}{1+\zeta}(\beta-\alpha)^{\varsigma-\zeta}\left(\int_{[\alpha,\beta]}\frac{\Delta(\hbar)}{(\sqrt{\aleph_i(\hbar)})^{1+\zeta}}\right)\\
&\times\int_{[\alpha,\beta]}\left((\sqrt{\aleph_i(\hbar)})(\chi_1(\hbar),...,\chi_n(\hbar))\right)^{1+\zeta}(\wp_{\gamma_1}^{\Delta})^{1+\zeta}(\hbar),...,(\wp_{\gamma_n}^{\Delta})^{1+\zeta}(\hbar),(\wp_{\gamma_{n+1}}^{\Delta})^{1+\zeta}(\hbar)\Delta(\hbar).
\end{aligned} \tag{3.1}$$

Proof. By the Modified Jensen's inequality, letting $\psi(s)=\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)},\sqrt{\chi_{n+1}(s)}$ and $d_n\lambda(s)=\wp_{\gamma_1}^{\Delta}(s),\wp_{\gamma_2}^{\Delta}(s),...,\wp_{\gamma_n}^{\Delta}(s)\Delta(s)$.

$$
\begin{aligned}
&(\beta-\alpha)^{-\varsigma} \leq \left(\int_{[\alpha,\beta]} \varphi(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)})^{\frac{1}{\zeta}} \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{\zeta} \\
&\times \left(\int_{[\alpha,\beta]} \psi(s) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{-\varsigma} (\beta-\alpha)^{-\zeta} \\
&= \left(\int_{[\alpha,\beta]} \varphi\left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n}(s)\right)^{\frac{1}{\zeta}} \times \wp^{\Delta}_{\gamma_1}(\hbar), \wp^{\Delta}_{\gamma_2}(\hbar), ..., \wp^{\Delta}_{\gamma_n}(\hbar)\Delta(s)\right)^{\zeta} \\
&\times \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\right) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{-\varsigma} (\beta-\alpha)^{-\zeta} \\
&\leq \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n}(s)\right)^{\frac{\varsigma}{\zeta}} \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{\zeta} \\
&\times \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n}(s)\right) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{-\varsigma} (\beta-\alpha)^{-\zeta} \\
&= \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n}(s)\right) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{\zeta} \\
&\times \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n}(s)\right) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{-\varsigma} (\beta-\alpha)^{-\zeta},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&(\beta-\alpha)^{-\varsigma} \leq \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n}(s)\Delta(s)\right) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{\zeta} \\
&\times \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\right) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{-\varsigma} (\beta-\alpha)^{-\zeta}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\right) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{\varsigma} \\
&\leq \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n}(s)\right) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{\zeta} (\beta-\alpha)^{\varsigma-\zeta}.
\end{aligned}
\tag{3.2}
$$

Now

$$
\gamma_i(\hbar) = \int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\right) \times \wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\Delta(s)
$$

$$
\gamma_i^{\Delta}(\hbar) = \left(\sqrt{\chi_1(\hbar)},...,\sqrt{\chi_n}(\hbar)\right) \times \wp^{\Delta}_{\gamma_1}(\hbar), \wp^{\Delta}_{\gamma_2}(\hbar), ..., \wp^{\Delta}_{\gamma_n}(\hbar). \tag{3.3}
$$

Combining (3.2) and (3.3), we obtain

$$
\begin{aligned}
&\wp^{\Delta}_{\gamma_1}(\hbar), \wp^{\Delta}_{\gamma_2}(\hbar), ..., \wp^{\Delta}_{\gamma_n}(\hbar) \times \left(\int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\right) \left[\wp^{\Delta}_{\gamma_1}(s), \wp^{\Delta}_{\gamma_2}(s), ..., \wp^{\Delta}_{\gamma_n}(s)\right] \Delta(s)\right)^{\varsigma} \\
&\leq (\beta-\alpha)^{\varsigma-\zeta} \frac{\gamma_i^{\Delta}(\hbar)\gamma_i(\hbar)^{\zeta}}{\sqrt{\chi_1(\hbar)},...,\sqrt{\chi_n(\hbar)}},
\end{aligned}
$$

that is

$$\left(\sqrt{\chi_1(\hbar)},...,\sqrt{\chi_n(\hbar)}\right)^{\varsigma+1}\times\wp^{\Delta}_{\gamma_1}(\hbar),\wp^{\Delta}_{\gamma_2}(\hbar),...,\wp^{\Delta}_{\gamma_n}(\hbar)\left(\int_{[\alpha,\beta]}\wp^{\Delta}_{\gamma_1}(s),\wp^{\Delta}_{\gamma_2}(s),...,\wp^{\Delta}_{\gamma_n}(s)\Delta(s)\right)^{\varsigma}$$
$$\leq(\beta-\alpha)^{\varsigma-\zeta}\,\gamma_i^{\Delta}(\hbar)\gamma_i(\hbar)^{\zeta}.$$

Integrating both side of the last inequality with respect to delta derivative, we have the following inequality

$$\int_{[\alpha,\beta]}\left(\sqrt{\chi_1(\hbar)},...,\sqrt{\chi_n(\hbar)}\right)^{\varsigma+1}\left(\wp^{\Delta}_{\gamma_1}(\hbar),\wp^{\Delta}_{\gamma_2}(\hbar),...,\wp^{\Delta}_{\gamma_n}(\hbar)\right)\times\left(\wp_{\gamma_2}(\hbar),\wp_{\gamma_3}(\hbar),\wp_{\gamma_{n+1}}(\hbar)\right)^{\varsigma}\Delta(\hbar)$$
$$\leq(\beta-\alpha)^{\varsigma-\zeta}\int_{[\alpha,\beta]}\gamma_i^{\Delta}(\hbar)\gamma_i(\hbar)^{\zeta}\Delta(\hbar)=\frac{1}{1+\zeta}\,(\beta-\alpha)^{\varsigma-\zeta}\,\gamma_i(\beta)^{1+\zeta}$$
$$\leq\frac{1}{1+\zeta}\,(\beta-\alpha)^{\varsigma-\zeta}\left(\int_{[\alpha,\beta]}\frac{1}{\sqrt{\aleph_i(\hbar)^{1+\zeta}}}\int_{[\alpha,\beta]}\sqrt{(\aleph_i(\hbar))(\chi_1(\hbar),...,\chi_n(\hbar))}\times\wp^{\Delta}_{\gamma_1}(\hbar),...,\wp^{\Delta}_{\gamma_n}(\hbar),\wp^{\Delta}_{\gamma_{n+1}}(\hbar)\right)^{1+\zeta}$$

which implies

$$\int_{[\alpha,\beta]}\left(\sqrt{\chi_1(\hbar)},...,\sqrt{\chi_n(\hbar)}\right)^{\varsigma+1}\left(\wp^{\Delta}_{\gamma_1}(\hbar),\wp^{\Delta}_{\gamma_2}(\hbar),...,\wp^{\Delta}_{\gamma_n}(\hbar)\right)\times\left(\wp_{\gamma_2}(\hbar),\wp_{\gamma_3}(\hbar),...,\wp_{\gamma_{n+1}}(\hbar)\right)^{\varsigma}\Delta(\hbar)$$
$$\leq\frac{1}{1+\zeta}\,(\beta-\alpha)^{\varsigma-\zeta}\left(\int_{[\alpha,\beta]}\frac{\Delta(\hbar)}{(\sqrt{\aleph_i(\hbar)})^{1+\zeta}}\right)\tag{3.4}$$
$$\times\int_{[\alpha,\beta]}\left((\sqrt{\aleph_i(\hbar))(\chi_1(\hbar),...,\chi_n(\hbar)})\right)^{1+\zeta}\,(\wp^{\Delta}_{\gamma_1})^{1+\zeta}(\hbar),...,(\wp^{\Delta}_{\gamma_n})^{1+\zeta}(\hbar),(\wp^{\Delta}_{\gamma_{n+1}})^{1+\zeta}(\hbar)\Delta(\hbar).$$

Furthermore, letting $\psi(s)\longrightarrow\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}$ and $d_i\lambda(s)\longrightarrow\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)$ in the proof of Theorem 3.1, then we have

$$(\beta-\alpha)^{-\varsigma}\leq\left(\int_{[\alpha,\beta]}\varphi(\psi(s))^{\frac{2}{\zeta}}\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{\zeta}$$
$$\times\left(\int_{[\alpha,\beta]}\psi(s)\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}$$

$$\leq\left(\int_{[\alpha,\beta]}\varphi\left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\right)^{\frac{1}{\zeta}}\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{\zeta}$$
$$\left(\int_{[\alpha,\beta]}\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\times\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}$$

$$\leq\left(\int_{[\alpha,\beta]}\left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\right)^{\frac{\varsigma}{\zeta}}\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{\zeta}$$
$$\left(\int_{[\alpha,\beta]}\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\times\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}$$

$$\leq\left(\int_{[\alpha,\beta]}\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{\zeta}$$
$$\left(\int_{[\alpha,\beta]}\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\times\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta},$$

which is

$$(\beta-\alpha)^{-\varsigma} \leq \left(\int_{[\alpha,\beta]} \sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)} \times \wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{\zeta}$$

$$\left(\int_{[\alpha,\beta]} \sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{-\varsigma}(\beta-\alpha)^{-\zeta}.$$

Hence

$$\begin{aligned}&\left(\int_{[\alpha,\beta]} \sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)} \times \wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{\varsigma}\\ &\leq \left(\int_{[\alpha,\beta]} \sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{\zeta}(\beta-\alpha)^{\varsigma-\zeta}.\end{aligned} \tag{3.5}$$

Now

$$\gamma_j(\hbar) = \int_{[\alpha,\beta]} \left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\right)\left(\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\right)\Delta(s)$$

$$\gamma^{\Delta}_j(\hbar) = \sqrt{\chi_1(\hbar)},...,\sqrt{\chi_n(\hbar)} \times \wp^{\Delta}_{\gamma_2}(\hbar),\wp^{\Delta}_{\gamma_3}(\hbar),...,\wp^{\Delta}_{\gamma_{n+1}}(\hbar) \tag{3.6}$$

In view of (2.6) and (2.7), we obtain

$$\begin{aligned}&\wp^{\Delta}_{\gamma_2}(\hbar),\wp^{\Delta}_{\gamma_3}(\hbar),...,\wp^{\Delta}_{\gamma_{n+1}}(\hbar)\left(\int_{[\alpha,\beta]}\left(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}\right)\times\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{\varsigma}\\ &\leq (\beta-\alpha)^{\varsigma-\zeta}\,\frac{\gamma^{\Delta}_j(\hbar)\gamma_j(\hbar)^{\zeta}}{\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)}},\end{aligned}$$

which implies

$$\begin{aligned}&(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)})^{\varsigma+1}\left(\wp^{\Delta}_{\gamma_2}(\hbar),\wp^{\Delta}_{\gamma_3}(\hbar),...,\wp^{\Delta}_{\gamma_{n+1}}(\hbar)\right)\left(\int_{[\alpha,\beta]}\wp^{\Delta}_{\gamma_2}(s),\wp^{\Delta}_{\gamma_3}(s),...,\wp^{\Delta}_{\gamma_{n+1}}(s)\Delta(s)\right)^{\varsigma}\\ &\leq (\beta-\alpha)^{\varsigma-\zeta}\,\gamma^{\Delta}_j(\hbar)\gamma_j(\hbar)^{\zeta}.\end{aligned}$$

Integrating both side of the latter inequality with respect to delta derivative, we have the following inequality:

$$\begin{aligned}&\int_{[\alpha,\beta]}(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)})^{\varsigma+1}\left(\wp^{\Delta}_{\gamma_2}(\hbar),\wp^{\Delta}_{\gamma_3}(\hbar),...,\wp^{\Delta}_{\gamma_{n+1}}(\hbar)\right)\times\wp_{\gamma_1}(\hbar),\wp_{\gamma_2}(\hbar),...,\wp_{\gamma_n}(\hbar)\Delta(\hbar)\\ &\leq (\beta-\alpha)^{\varsigma-\zeta}\int_{[\alpha,\beta]}\gamma^{\Delta}_j(\hbar)\gamma_j(\hbar)^{\zeta}\Delta(\hbar) = \frac{1}{1+\zeta}(\beta-\alpha)^{\varsigma-\zeta}\,\gamma_j(\beta)^{1+\zeta}\\ &\leq \frac{1}{1+\zeta}(\beta-\alpha)^{\varsigma-\zeta}\left(\int_{[\alpha,\beta]}\frac{\Delta(\hbar)}{\sqrt{\aleph_i(\hbar)}}\int_{[\alpha,\beta]}\sqrt{\aleph_i(s)(\chi_1(s),...,\chi_n(s))}\wp^{\Delta}_{\gamma_2}(\hbar)\Delta(\hbar)\right)^{1+\zeta},\end{aligned}$$

which implies

$$\begin{aligned}&\int_{[\alpha,\beta]}(\sqrt{\chi_1(s)},...,\sqrt{\chi_n(s)})^{\varsigma+1}\left(\wp^{\Delta}_{\gamma_2}(\hbar),\wp^{\Delta}_{\gamma_3}(\hbar),...,\wp^{\Delta}_{\gamma_{n+1}}(\hbar)\right)\times\wp_{\gamma_1}(\hbar),\wp_{\gamma_2}(\hbar),...,\wp_{\gamma_n}(\hbar)\Delta(\hbar)\\ &\leq \frac{1}{1+\zeta}(\beta-\alpha)^{\varsigma-\zeta}\left(\int_{[\alpha,\beta]}\frac{\Delta(\hbar)}{(\sqrt{\aleph_i(\hbar)})^{1+\zeta}}\right)\int_{[\alpha,\beta]}(\sqrt{\aleph_i(\hbar)(\chi_1(s),...,\chi_n(s))})^{1+\zeta}\wp^{\Delta}_{\gamma_1}(\hbar)^{1+\zeta}\Delta(\hbar).\end{aligned}$$

Adding both side of (3.5) and (3.6) yields

$$\begin{aligned}
&\int_{[\alpha,\beta]} (\sqrt{\chi_1(s)}, ..., \sqrt{\chi_n(s)})^{\varsigma+1} \left(\wp^{\Delta}_{\gamma_2}(\hbar)\wp^{\varsigma}_{\gamma_1}(\hbar) + \wp^{\Delta}_{\gamma_1}(\hbar)\wp^{\varsigma}_{\gamma_2}(\hbar)\right) \\
&+ \left(\wp^{\Delta}_{\gamma_4}(\hbar)\wp^{\varsigma}_{\gamma_3}(\hbar) + \wp^{\Delta}_{\gamma_3}(\hbar)\wp^{\varsigma}_{\gamma_4}(\hbar)\right) + ... + \left(\wp^{\Delta}_{\gamma_{n+1}}(\hbar)\wp^{\varsigma}_{\gamma_n}(\hbar) + \wp^{\Delta}_{\gamma_n}(\hbar)\wp^{\varsigma}_{\gamma_{n+1}}(\hbar)\right) \Delta(\hbar) \\
&\leq \frac{1}{1+\zeta} (\beta-\alpha)^{\varsigma-\zeta} \left(\int_{[\alpha,\beta]} \frac{\Delta(\hbar)}{(\sqrt{\aleph(\hbar)})^{1+\zeta}}\right) \int_{[\alpha,\beta]} (\sqrt{\aleph(\hbar)(\chi_1(s), ..., \chi_n(s))})^{1+\zeta} \\
&\times \left(\wp^{\Delta}_{\gamma_1}(\hbar)^{1+\zeta} + \wp^{\Delta}_{\gamma_2}(\hbar)^{1+\zeta}\right) + \left(\wp^{\Delta}_{\gamma_3}(\hbar)^{1+\zeta} + \wp^{\Delta}_{\gamma_4}(\hbar)^{1+\zeta}\right) \\
&+ .... + \left(\wp^{\Delta}_{\gamma_n}(\hbar)^{1+\zeta} + \wp^{\Delta}_{\gamma_{n+1}}(\hbar)^{1+\zeta}\right) \Delta(\hbar).
\end{aligned}$$

□

4. Conclusion

This paper has established some new Opial inequalities of Pachpatte-type on time scale through the application of modified Jesen's inequality. The results obtained extend and generalize some known results in literature. Indeed, in special cases the results in [6] and [9] are derived. Considerable applications of a class of these inequalities are abound in the theories of differential, difference and integro-differential equations as well boundary value problems, see [1, 8] for instance. The inequalities obtained here may be useful for future research.

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