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PRIME AND ODD PRIME LABELINGS OF BROOM GRAPHS AND SOME RELATED GRAPHS

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ABSTRACT. Let G be a simple graph with a finite vertex set. A prime labeling of G is a bijection assignment of natural numbers to its vertices such that every pair of adjacent vertices receives relatively prime labels. The conjecture that every tree is a prime graph remains an interesting open problem. An odd prime labeling is defined as a mapping from each vertex of G to the odd integers from 1 to $2|V(G)|-1$ under the same coprime condition. Furthermore, it has been conjectured that every prime graph also satisfies the odd prime labeling property. In this paper, we investigate prime and odd prime labelings of broom graphs and some related graphs. Broom graphs are considered because their structure combines paths and stars, forming a non-uniform tree with diverse vertex degree characteristics. This structure makes broom graphs a representative model for testing both conjectures on a class of trees that is more complex than paths or stars individually. Future work may extend this study to more complex classes of trees and attempt to establish both conjectures in a more general setting.

1. INTRODUCTION

The concept of graph labeling was first introduced by Sedlacek in 1964 and has since become an important topic in graph theory due to its wide range of applications [1]. Graph labeling has been widely applied in computer science and networking systems, including frequency allocation [2], coding theory [3], channel numbering in data transmission systems [4], communication network design [5], and cryptography [6, 7]. In general, graph labeling is defined as a function that assigns integers to the vertices, edges, or both of a graph according to certain specified rules [8]. Graph labeling can be classified based on its domain into vertex labeling, edge labeling, and total labeling [9]. A particular type of vertex labeling, known as prime labeling, has been extensively studied in graph theory. This labeling concept was introduced by Entringer in 1980, who conjectured that every tree admits a prime labeling [10]. Furthermore, Tout et al. (1982) formally defined prime labeling on a graph G as a bijection $\delta : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that every pair of adjacent vertices receives relatively prime labels. A graph that satisfies this condition is called a prime graph [11].

Numerous studies have addressed Entringer's conjecture. Several classes of trees have been shown to admit prime labeling, including paths, stars, caterpillars with maximum degree five [11], olive trees, spiders [12], trees of order up to 50 [13], banana trees, palm trees, binomial trees and several families of spider graphs [14], bistars [15], comb graphs [16], and H-graphs [17]. Comprehensive surveys on prime

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labeling on tree structures are presented in [18–20]. Despite these results, the existence of prime labeling for arbitrary trees remains an open problem. Consequently, the study of new families of tree graphs continues to be an active and significant research area in graph labeling theory.

The concept of prime labeling has been extended to odd prime labeling, defined as a bijection $\delta : V(G) \to \{1, 3, \ldots, 2|V(G)| - 1\}$ such that $\gcd(\delta(\alpha), \delta(\beta)) = 1$ for every edge $\alpha\beta \in E(G)$ [21]. Graphs admitting such a labeling are called odd prime graphs. Prajapati and Shah (2018) conjectured that every prime graph is also an odd prime graph. The conjecture has been verified for numerous graph families, including paths, cycles, wheels, ladders, fans, complete bipartite graphs, friendship graphs, flower graphs, gear graphs, helm graphs, generalized Petersen graphs, and several graph compositions such as path amalgamations [22], snake, spider, and firecracker graphs, unions of even cycles $\bigcup_{i=1}^{n} C_{k_i}$ with $k \equiv 0 \pmod 2$, binary trees, prism graphs, second powers of paths and cycles (P_n^2 and C_n^2), prism $GP_{(n,1)}$ [23], circular ladder graphs [24], volcano graphs, $C_3 \odot_{(x_1y_0)} F_n$, $C_3 \circledcirc \overline{K}_n$, tadpole graphs, palm trees, and $C_l \odot_{(x_1y_0)} mP_{n+1}$ [25]. Nevertheless, the conjecture remains open in its general form and continues to be an active research problem in graph labeling theory.

Motivated by previous studies, this work investigates prime and odd prime labelings of broom graphs and some related graph classes, including $Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m$, $P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}$, and $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$. These graph families are selected because they represent natural generalizations of broom graph structures via graph operations and parameter extensions, yielding non-uniform tree configurations with more complex vertex degree distributions. Such structures provide a suitable framework for examining labeling properties on broader classes of combinatorially complex trees. The main contribution of this study is the construction of prime and odd prime labelings for these graph classes, thereby extending existing results on tree labeling to more structurally complex graph families.

2. PRELIMINARIES

We first present several definitions and lemmas that form the theoretical foundation of this study.

Definition 2.1. [26] Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two simple graphs, as well as $\alpha \in V(G_1)$ and $\beta \in V(G_2)$. The vertex identification operation on vertices α and β, denoted by $G = G_1 \circledcirc_{\alpha\beta} G_2$, is a graph obtained by combining G_1 and G_2 as well as attaching vertices α and β into a single vertex, while all edges adjacent to either α or β are preserved. Formally, graph G satisfies $|V(G)| = |V(G_1)| + |V(G_2)| - 1$ and $|E(G)| = |E(G_1)| + |E(G_2)|$.

Definition 2.2. [27] Let $G = (V(G), E(G))$ be a graph and $\alpha \in V(G)$. The amalgamation of m copies of graph G at vertex α, denoted by $\mathrm{Amal}(G, \alpha, m)$, is a graph obtained from m identical copies of G, specifically $G_1, G_2, \ldots, G_m$, with each copy contains a vertex corresponding to α. All such vertices are then identified into one single vertex. Consequently, the resulting amalgamation graph possesses one central vertex formed by this identification, while all edges and other vertices from each copy of graph are preserved.

Definition 2.3. [28] Broom graph $Br_{(n,m)}$ is a graph of order $m + n$ and size $m + n - 1$, obtained by the identification result of a graph path P_n and a graph star S_m. This is achieved by amalgamating the central vertex of the star graph (vertex of maximum degree) with one of the end-vertices of graph path P_m.

Definition 2.4. [29] A superstar graph of order $xm + 1$ and size xm, denoted by $S_{(m,x)}$, is a graph that is isomorphic to $\mathrm{Amal}(P_{x+1}, \alpha, m)$, which is constructed by duplicating the path graph P_{x+1} into m copies. Furthermore, one of the end vertices of each path graph is identified as a single central vertex.

Lemma 2.5. *[30] Let x and y be consecutive positive integers, i.e., $y = x + 1$. Then $\gcd(x, y) = 1$, which implies that x and y are relatively prime.*

Lemma 2.6. *[31] Let $x \in \mathbb{Z}$ be an odd integer and $y \in \mathbb{N}$ be a positive integer that has no odd factors other than 1, then* $\gcd(x, x+y) = 1$.

3. RESULTS AND DISCUSSION

In this section we demonstrate that broom graphs and its related graphs, such as $Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m$, $P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}$, and $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$ satisfy the characteristics of prime and odd prime labeling. We present the appropriate functions construction and formally demonstrate that every pair of adjacent vertices on these graphs possesses relatively prime labels. Consequently, all investigated graphs can be classified as prime and odd prime graphs.

The vertex set and edge set of the broom graph $(Br_{(n,m)})$ are defined as follows:

$$V(Br_{(n,m)}) = \{\alpha_\rho \mid \rho = 1, 2, \ldots, n\} \cup \{\beta_\tau \mid \tau = 1, 2, \ldots, m\}.$$

$$E(Br_{(n,m)}) = \{\alpha_\rho\alpha_{\rho+1} \mid \rho = 1, 2, \ldots, n-1\} \cup \{\alpha_1\beta_\tau \mid \tau = 1, 2, \ldots, m\}.$$

Consequently, the vertices α_ρ form a path P_n, while vertex α_1 serves as the center connected to each vertex β_τ, thereby forming the broom graph structure. An illustration of the vertex and edge notation of the broom graph $(Br_{(n,m)})$ is presented in Figure 1.

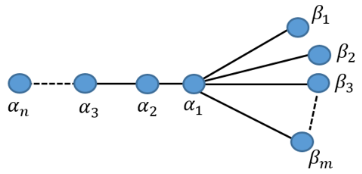


FIGURE 1. The Vertices and Edges Notation of Broom Graph $Br_{n,m}$

Theorem 3.1. *For every $n \geq 2$ and $m \geq 2$, broom graph $(Br_{(n,m)})$ is a prime graph.*

Proof. In order to prove that the broom graph $(Br_{(n,m)})$ is a prime graph, we establish a prime labeling construction for this graph. Specifically, we define a bijection $\delta : V(Br_{(n,m)}) \to \{1, 2, \ldots, n+m\}$ as follows.

$$\delta(\alpha_\rho) = \rho, \qquad \text{for } \rho = 1, 2, \ldots, n.$$

$$\delta(\beta_\tau) = n + \tau, \qquad \text{for } \tau = 1, 2, \ldots, m.$$

It is evident that the labels of the vertices α_ρ and $\alpha_{\rho+1}$ are consecutive integers. According to Lemma 2.5, we obtain that $\gcd(\delta(\alpha_\rho), \delta(\alpha_{\rho+1})) = 1$ for every $\rho = 1, 2, \ldots, n-1$. In addition, we observe that $\delta(\alpha_1) = 1$. Consequently, the label of vertex α_1 is relatively prime to the labels of all vertices β_τ for every $\tau = 1, 2, \ldots, m$. Thus, the function δ satisfies the prime labeling condition, namely that any two adjacent vertices have relatively prime labels. Based on these results, we conclude that the broom graph $Br_{(n,m)}$ is a prime graph. □

Theorem 3.2. *For every $n \geq 2$ and $m \geq 2$, broom graph $(Br_{(n,m)})$ is an odd prime graph.*

Proof. In a similar manner, we demonstrate that the broom graph is an odd prime graph by constructing a bijection $\delta : V(Br_{(n,m)}) \to \{1, 3, \ldots, 2(m+n)-1\}$ as follows.

$$\delta(\alpha_\rho) = 2\rho - 1, \qquad \text{for } \rho = 1, 2, \ldots, n.$$

$$\delta(\beta_\tau) = 2(n+\tau) - 1, \qquad \text{for } \tau = 1, 2, \ldots, m.$$

It is evident that $|\delta(\alpha_\rho) - \delta(\alpha_{\rho+1})| = 2$. Since 2 has no odd factors other than 1, it follows from Lemma 2.6 that $\gcd(\delta(\alpha_\rho), \delta(\alpha_{\rho+1})) = 1$. Consequently, every pair of adjacent vertices has relatively prime labels. Therefore, it can be concluded that the broom graph $Br_{(n,m)}$ is an odd prime graph. □

After proving that the broom graph admits both prime and odd prime labelings, we extend the study to a related graph constructed using the vertex identification operation described in Definition 2.1. The following theorems show that this graph also admits prime labeling and odd prime labeling.

It has been demonstrated that $|V(Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m)| = 2m + n$ with the vertex and edge sets defined as follows.

$$V(Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m) = \{\alpha_\rho \mid \rho = 1, 2, \dots, n\} \cup \{\beta_\tau, \gamma_\mu \mid \tau, \mu = 1, 2, \dots, m\}.$$

$$E(Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m) = \{\alpha_\rho\alpha_{\rho+1} \mid \rho = 1, 2, \dots, n-1\} \cup \{\alpha_1\beta_\tau, \alpha_n\gamma_\mu \mid \tau, \mu = 1, 2, \dots, m\}.$$

Theorem 3.3. *For every $n \geq 2$ and $m \geq 2$, the graph $Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m$ is a prime graph.*

Proof. The prime labeling of the graph $Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m$ is defined by the function $\delta : V(Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m) \to \{1, 2, \dots, 2m + n\}$ as follows.

$$\delta(\alpha_\rho) = \begin{cases} 1, & \text{for } \rho = 1, \\ n - \rho + 2, & \text{for } \rho = 2, 3, \dots, n-1, \\ 2, & \text{for } \rho = n. \end{cases}$$

$$\delta(\beta_\tau) = \begin{cases} n + 2\tau, & \text{for } \tau = 1, 2, \dots, m \text{ and } n \text{ is even}, \\ n + 2\tau - 1, & \text{for } \tau = 1, 2, \dots, m \text{ and } n \text{ is odd}. \end{cases}$$

$$\delta(\gamma_\mu) = \begin{cases} n + 2\mu - 1, & \text{for } \mu = 1, 2, \dots, m \text{ and } n \text{ is even}, \\ n + 2\mu, & \text{for } \mu = 1, 2, \dots, m \text{ and } n \text{ is odd}. \end{cases}$$

Since $\delta(\alpha_1) = 1$, it follows that $\gcd(\delta(\alpha_1), \delta(\alpha_2)) = 1$ and $\gcd(\delta(\alpha_1), \delta(\beta_\tau)) = 1$. According to Lemma 2.5, the labels of vertices α_ρ and $\alpha_{\rho+1}$ for $\rho = 2, 3, \dots, n-1$ are relatively prime because they are consecutive integers. Furthermore, $\delta(\alpha_n) = 2$ is relatively prime to every label of vertex γ_μ, since $\delta(\gamma_\mu)$ is an odd integer and 2 is relatively prime to all odd numbers. Hence, the function δ satisfies the prime labeling criteria, thereby demonstrating that graph $Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m$ is a prime graph. □

Theorem 3.4. *For every $n \geq 2$ and $m \geq 2$, the graph $Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m$ is an odd prime graph.*

Proof. In a similar manner, we demonstrate that the graph $Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m$ is an odd prime graph by constructing an appropriate function. Specifically, we define a function $\delta : V(Br_{(n,m)} \odot_{(\alpha_n\gamma_0)} S_m) \to \{1, 3, \dots, 4m + 2n - 1\}$ as follows.

$$\delta(\alpha_\rho) = \begin{cases} 1, & \text{for } \rho = 1, \\ 2(n - \rho) + 3, & \text{for } \rho = 2, 3, \dots, n. \end{cases}$$

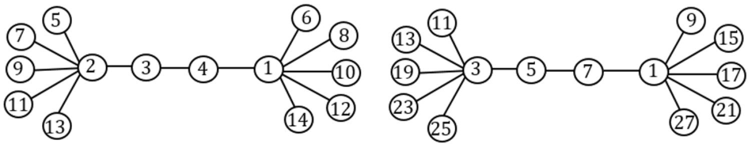
Suppose that $r \equiv n + 1 \pmod 3$ with $r \in \{0, 1, 2\}$. It follows that:

$$\delta(\beta_\tau) = \begin{cases} 2n + 4\tau - 1, & \text{for } \tau \equiv r \pmod 3, \\ 2n + 4\tau - 3, & \text{for } \tau \not\equiv r \pmod 3. \end{cases}$$

$$\delta(\gamma_\mu) = \begin{cases} 2n + 4\mu - 3, & \text{for } \mu \equiv r \pmod 3, \\ 2n + 4\mu - 1, & \text{for } \mu \not\equiv r \pmod 3. \end{cases}$$

Notably, $\delta(\alpha_1) = 1$ is relatively prime to $\delta(\alpha_2)$ and $\delta(\beta_\tau)$. Subsequently, for each $\rho = 2, 3, \ldots, n-1$, we obtain $|\delta(\alpha_\rho) - \delta(\alpha_{\rho+1})| = 2$. According to Lemma 2.6, $\gcd(\delta(\alpha_\rho), \delta(\alpha_{\rho+1})) = 1$ since 2 has no odd factors other than 1. Furthermore, observe that $\delta(\alpha_n) \not\equiv 0 \pmod 3$, which implies that $\gcd(\delta(\alpha_n), \delta(\gamma_\mu)) = 1$. Therefore, every pair of adjacent vertices has relatively prime labels. Hence, the graph $Br_{(n,m)} \odot_{(\alpha_n \gamma_0)} S_m$ is an odd prime graph. □

For instance, Figure 2 illustrates the prime and odd prime labeling of the graph $Br_{4,5} \odot_{(\alpha_4 \gamma_0)} S_5$.



(a) (b)

FIGURE 2. (a) Prime Labeling of the Graph $Br_{4,5} \odot_{(\alpha_4 \gamma_0)} S_5$; (b) Odd Prime Labeling of the Graph $Br_{4,5} \odot_{(\alpha_4 \gamma_0)} S_5$

Figure 2 illustrates the labeling construction for the graph $Br_{4,5} \odot_{(\alpha_4 \gamma_0)} S_5$. We next investigate a related graph structure obtained by identifying a vertex of a path graph with the central vertex of a superstar graph. The graph $P_n \odot_{(\alpha_1 \beta_0)} S_{(m,x)}$ has $mx + n$ vertices, and its vertex and edge sets are defined as follows.

$$V(P_n \odot_{(\alpha_1 \beta_0)} S_{(m,x)}) = \{\alpha_\rho \mid \rho = 1, 2, \ldots, n\} \cup \{\beta_\tau^\mu \mid \tau = 1, 2, \ldots, x;\ \mu = 1, 2, \ldots, m\}.$$

$$E(P_n \odot_{(\alpha_1 \beta_0)} S_{(m,x)}) = \{\alpha_\rho \alpha_{\rho+1} \mid \rho = 1, 2, \ldots, n-1\} \quad \cup \{\alpha_1 \beta_1^\mu \mid \mu = 1, 2, \ldots, m\}$$
$$\cup \{\beta_\tau^\mu \beta_{\tau+1}^\mu \mid \tau = 1, 2, \ldots, x-1;\ \mu = 1, 2, \ldots, m\}.$$

Figure 3 illustrates the vertex notation of the graph $P_n \odot_{(\alpha_1 \beta_0)} S_{(m,x)}$.

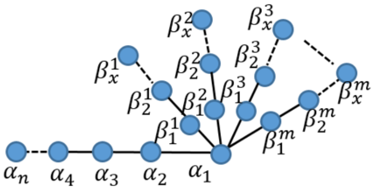


FIGURE 3. The Vertices and Edges Notation of the Graph $P_n \odot_{(\alpha_1 \beta_0)} S_{(m,x)}$

Theorem 3.5. *For every $n, m, x \geq 2$, the graph $P_n \odot_{(\alpha_1 \beta_0)} S_{(m,x)}$ is a prime graph.*

Proof. The prime labeling of the graph $P_n \odot_{(\alpha_1 \beta_0)} S_{(m,x)}$ is defined by a function $\delta : V(P_n \odot_{(\alpha_1 \beta_0)} S_{(m,x)}) \to \{1, 2, \ldots, mx + n\}$ as follows.

$$\delta(\alpha_\rho) = \rho, \quad \text{for } \rho = 1, 2, \ldots, n.$$

$$\delta(\beta_\tau^\mu) = n + (\mu - 1)x + \tau, \quad \text{for } \tau = 1, 2, \ldots, x;\ \mu = 1, 2, \ldots, m.$$

According to the definition of the function, we obtain $\delta(\alpha_1) = 1$. Since 1 is relatively prime to every positive integer, it follows that $\gcd(\delta(\alpha_1), \delta(\beta_\tau^\mu)) = 1$. Furthermore, for every $\rho = 1, 2, \ldots, n-1$, we obtain $|\delta(\alpha_\rho) - \delta(\alpha_{\rho+1})| = 1$, which implies that $\delta(\alpha_\rho)$ and $\delta(\alpha_{\rho+1})$ are consecutive integers. Since any two consecutive integers are relatively prime, it follows that $\gcd(\delta(\alpha_\rho), \delta(\alpha_{\rho+1})) = 1$. Similarly, for every $\tau = 1, 2, \ldots, x-1$, the labels of vertices β_τ^μ and $\beta_{\tau+1}^\mu$ are also consecutive integers, implying that $\gcd(\delta(\beta_\tau^\mu), \delta(\beta_{\tau+1}^\mu)) = 1$. Consequently, every pair of adjacent vertices in the graph has relatively prime labels. Therefore, the defined function δ satisfies the prime labeling condition. Hence, the graph $P_n \odot_{(\alpha_1 \beta_0)} S_{(m,x)}$ is a prime graph. □

Theorem 3.6. *For every $n, m, x \geq 2$, the graph $P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}$ is an odd prime graph.*

Proof. To prove that the graph $P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}$ is an odd prime graph, we define an odd prime function $\delta : V(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \to \{1, 3, \ldots, 2(mx + n) - 1\}$ as follows.

$$\delta(\alpha_\rho) = 2\rho - 1, \qquad \text{for } \rho = 1, 2, \ldots, n.$$

$$\delta(\beta^\mu_\tau) = 2(n + (\mu - 1)x + \tau) - 1, \qquad \text{for } \tau = 1, 2, \ldots, x;\ \mu = 1, 2, \ldots, m.$$

It is evident that every pair of adjacent vertices has relatively prime labels. Therefore, the graph $P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}$ is an odd prime graph. ☐

Having established that the graph $P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}$ admits both prime labeling and odd prime labeling, we further extend the investigation to a more complex graph obtained by performing an additional vertex identification operation. The graph $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$ has $2mx + n$ vertices and $2mx + n - 1$ edges, which are denoted as follows.

$$\begin{aligned} V((P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}) =& \{\alpha_\rho \mid \rho = 1, 2, \ldots, n\} \\ & \cup \{\beta^\mu_\tau, \gamma^\mu_\tau \mid \tau = 1, 2, \ldots, x;\ \mu = 1, 2, \ldots, m\}. \end{aligned}$$

$$\begin{aligned} E\Big((P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}\Big) =& \{\alpha_\rho\alpha_{\rho+1} \mid \rho = 1, 2, \ldots, n - 1\} \\ & \cup \{\alpha_1\beta^\mu_1, \alpha_n\gamma^\mu_1 \mid \mu = 1, 2, \ldots, m\} \\ & \cup \{\beta^\mu_\tau\beta^\mu_{\tau+1}, \gamma^\mu_\tau\gamma^\mu_{\tau+1} \mid \tau = 1, 2, \ldots, x - 1; \\ & \mu = 1, 2, \ldots, m\}. \end{aligned}$$

Figure 4 illustrates the visual representation of the graph structure, along with the vertex notation of the graph $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$.

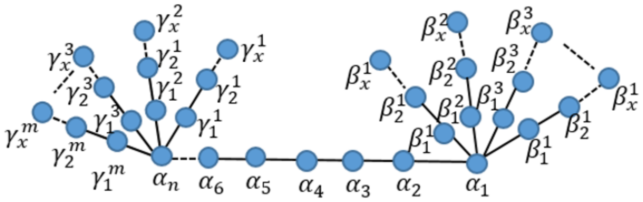


FIGURE 4. The Vertices and Edges Notation of the Graph $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$

Theorem 3.7. *For every $n, m, x \geq 2$, the graph $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$ is a prime graph.*

Proof. A prime labeling for the graph $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$ is defined via a function $\delta : V\big((P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}\big) \to \{1, 2, \ldots, 2mx + n\}$, specified as follows.

$$\delta(\alpha_\rho) = \begin{cases} 1, & \text{for } \rho = 1, \\ n - \rho + 2, & \text{for } \rho = 2, 3, \ldots, n. \end{cases}$$

For $\tau = 1, 2, \ldots, x$ and $\mu = 1, 2, \ldots, m$

$$\delta(\beta^\mu_\tau) = \begin{cases} n + (2\mu - 2)x + \tau, & \text{for } n \text{ and } x \text{ odd}, \\ n + (2\mu - 1)x + \tau, & \text{otherwise}, \end{cases}$$

$$\delta(\gamma_\tau^\mu) = \begin{cases} n + (2\mu - 1)x + \tau, & \text{for } n \text{ and } x \text{ odd,} \\ n + 2\mu x - x - \tau + 1, & \text{for } n \text{ odd and } x \text{ even,} \\ n + (2\mu - 2)x + \tau, & \text{otherwise.} \end{cases}$$

It can be easily shown that the labels of every pair of adjacent vertices are relatively prime. Thus, the graph $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$ is a prime graph. □

Theorem 3.8. *For every $n, m, x \geq 2$, the graph $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$ is an odd prime graph.*

Proof. Define the bijective function $\delta : V\big((P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}\big) \to \{1, 2, \dots, 4mx + 2n - 1\}$ as follows.

$$\delta(\alpha_\rho) = \begin{cases} 1, & \text{for } \rho = 1, \\ 2(n - \rho) + 3, & \text{for } \rho = 1, 2, \dots, n. \end{cases}$$

(i) Case $x \equiv 0 \pmod 3$

For $\tau = 1, 2, \dots, x$ and $\mu = 1, 2, \dots, m$

$$\delta(\beta_\tau^\mu) = 2(n + (2\mu - 2)x + \tau) - 1$$

$$\delta(\gamma_\tau^\mu) = \begin{cases} 2(n + (2\mu - 1)x + \tau) - 1, & \text{for } n \not\equiv 1 \pmod 3, \\ 2(n - \tau) + 4\mu x + 1, & \text{for } n \equiv 1 \pmod 3. \end{cases}$$

(ii) Case $x \not\equiv 0 \pmod 3$

For $\tau = 1, 2, \dots, x$ and $\mu = 1, 2, \dots, m$

$$\delta(\beta_\tau^\mu) = \begin{cases} 2\big(n + (2\mu - 1)x + \tau\big) - 1, & \text{for } \mu \not\equiv r \pmod 3, \\ 2\big(n + (2\mu - 2)x + \tau\big) - 1, & \text{for } \mu \equiv r \pmod 3, \end{cases}$$

$$\delta(\gamma_\tau^\mu) = \begin{cases} 2\big(n + (2\mu - 2)x + \tau\big) - 1, & \text{for } \mu \not\equiv r \pmod 3, \\ 2\big(n + (2\mu - 1)x + \tau\big) - 1, & \text{for } \mu \equiv r \pmod 3. \end{cases}$$

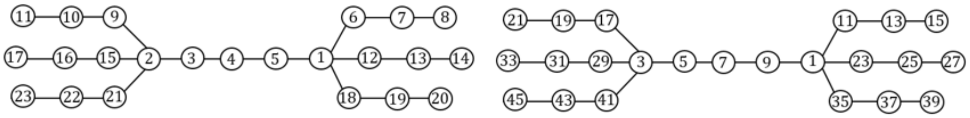
with

$$r = \begin{cases} 0, & \text{for } (n \equiv 0 \pmod 3 \text{ and } x \equiv 1 \pmod 3) \\ & \quad \text{or } (n \equiv 2 \pmod 3 \text{ and } x \equiv 2 \pmod 3), \\ 1, & \text{for } n \equiv 1 \pmod 3 \text{ and } x \not\equiv 0 \pmod 3, \\ 2, & \text{for } (n \equiv 0 \pmod 3 \text{ and } x \equiv 2 \pmod 3) \\ & \quad \text{or } (n \equiv 2 \pmod 3 \text{ and } x \equiv 1 \pmod 3). \end{cases}$$

Following a similar analysis, it can be shown that every pair of adjacent vertices has relatively prime labels. Therefore, the graph $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$ is an odd prime graph. □

To further illustrate Theorems 3.7 and 3.8, an example of the corresponding labeling is presented in Figure 5.

The findings of this study demonstrate that the broom graph and some related graphs, such as $Br_{n,m} \odot_{(\alpha_n\gamma_0)} S_m$, $P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}$, and $(P_n \odot_{(\alpha_1\beta_0)} S_{(m,x)}) \odot_{(\alpha_n\gamma_0)} S_{(m,x)}$, satisfy the properties of prime and odd prime labeling for every choice of parameters. These results extend previous studies on prime labeling of tree graphs, as presented in [13] and comprehensively reviewed in Gallian's survey [10], by introducing non-uniform tree classes that combine path and star structures. The local complexity arising from variations in vertex degrees within the broom graph shows that simple arithmetic patterns, particularly the use of consecutive integers and parity-based divisibility properties, are sufficiently stable to preserve the



(a) (b)

FIGURE 5. (a) Prime Labeling of the Graph $(P_5 \odot_{(\alpha_1\beta_0)} S_{3,3}) \odot_{(\alpha_5\gamma_0)} S_{3,3}$; (b) Odd Prime Labeling of the Graph $(P_5 \odot_{(\alpha_1\beta_0)} S_{3,3}) \odot_{(\alpha_5\gamma_0)} S_{3,3}$

relatively prime condition for every pair of adjacent vertices, even after performing graph operations such as vertex identification and edge subdivision.

In the context of odd prime labeling, these results provide further support for the conjecture of Prajapati and Shah (2018), which suggests that every prime graph may also admit an odd prime labeling. The construction obtained in this study demonstrates that the transformation from a prime labeling scheme to an odd prime labeling scheme can be carried out systematically without losing the coprimality property of adjacent vertices. Conceptually, these findings indicate that a pattern-based constructive approach is an effective strategy for extending the verification of both conjectures to more complex families of trees. Future research may focus on extending these results to broader classes of trees and developing a more general structural framework that can bridge the theories of prime labeling and odd prime labeling.

4. CONCLUSION

This study proves that broom graphs and some related graphs satisfy prime and odd prime labeling properties through explicit arithmetic pattern-based constructions. The findings demonstrate that trees with non-uniform structures still allow labeling schemes that preserve the relatively prime condition for every pair of adjacent vertices. These results extend the known classes of labeled tree graphs and open opportunities for further generalization to more complex graph structures. Furthermore, future research may focus on proving the conjecture that every prime graph is also an odd prime graph in general.

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