ISSN: 2832-4293

Pan-American Journal of Mathematics 5 (2026), 7
https://doi.org/10.28919/cpr-pajm/5-7

ON WEAKLY CYCLIC SCHOUTEN-SYMMETRIC MANIFOLDS

JAEMAN KIM

ABSTRACT. The purpose of the present paper is to study several geometric properties of a weakly cyclic Schouten-symmetric manifold. Also the existence of such a manifold is ensured by a non-trivial example.

1. INTRODUCTION

The Riemannian curvature tensor R on a Riemannian manifold (M^n, g) decomposes into a conformally invariant part, the Weyl tensor W and a non-conformally invariant part, the Schouten tensor P. More precisely,

$$R = P \odot g + W, \tag{1.1}$$

where the symbol $\odot$ is the Nomizu-Kulkarni product of symmetric (0,2)-tensors generating a curvature type tensor:

$$(h \odot k)(X, Y, Z, W) = h(X, Z)k(Y, W) + h(Y, W)k(X, Z)$$
$$-h(X, W)k(Y, Z) - h(Y, Z)k(X, W).$$

Since the Weyl tensor W is conformally invariant, to study the deformation of the conformal metric, one only need to understand the Schouten tensor P [1]. As a natural generalization of the notion of a space of constant curvature, the notion of a symmetric manifold was introduced by Cartan [3] who obtained a classification of such a manifold. During the last seven decades the notion of a symmetric manifold has been weakened by many authors in several ways to a different extent such as a weakly symmetric manifold by Binh and Tamassy [2]; a weakly Z-symmetric manifold by Mantica and Molinari [12]; a weakly cyclic Z-symmetric manifold by De, Mantica and Suh [6] and Kim [9]; a semisymmetric manifold by Lichnerowich [11], Szabo [14,15] and Kowalski [10].

A (0,2) symmetric tensor P on a Riemannian manifold (M^n, g) $(n > 2)$ is called a Schouten tensor if it is defined as

$$P(X, Y) := \frac{1}{n-2}(Ric(X, Y) - \frac{s}{2(n-1)}g(X, Y)), \tag{1.2}$$

where Ric and s are the Ricci tensor and scalar curvature tensor, respectively.

A Riemannian manifold (M^n, g) $(n > 2)$ is said to be weakly Schouten-symmetric if its Schouten tensor P

DEPARTMENT OF MATHEMATICS EDUCATION, KANGWON NATIONAL UNIVERSITY, CHUNCHON 200-701, KANGWON DO, KOREA

E-mail address: jaeman64@kangwon.ac.kr.

Submitted on Feb. 03, 2026.

2020 *Mathematics Subject Classification.* 53A55, 53B20.

Key words and phrases. weakly cyclic Schouten-symmetric manifolds; conformally flat; quasi Einstein; semisymmetric; conformal killing vector field; parallel vector field; geodesic.

*Corresponding author.

satisfies the following relation:

$$(\nabla_X P)(Y,Z) = A(X)P(Y,Z) + B(Y)P(Z,X) + C(Z)P(X,Y),$$

where A, B, C are 1-forms and ∇ denotes the covariant differentiation with respect to the metric tensor g. Furthermore, a Riemannian manifold (M^n, g) $(n > 2)$ is called a weakly cyclic Schouten-symmetric manifold if its Schouten tensor P fulfills the condition:

$$(\nabla_X P)(Y,Z) + (\nabla_Y P)(Z,X) + (\nabla_Z P)(X,Y)$$

$$= A(X)P(Y,Z) + B(Y)P(Z,X) + C(Z)P(X,Y), \tag{1.3}$$

where A, B, C are the associated 1-forms.

An n-dimensional manifold of this kind is denoted by $[WCSS]_n$. The aim of this paper is to study several geometric properties of $[WCSS]_n$ and provide an example of $[WCSS]_4$ with one parameter family of its associated 1-forms.

2. Main results

The Schouten tensor P of (M^n, g) is said to be of Codazzi type if it satisfies

$$(\nabla_X P)(Y,Z) = (\nabla_Y P)(X,Z). \tag{2.4}$$

At first we have

Theorem 2.1. *Let (M^n, g) be a $[WCSS]_n$. If its Schouten tensor P is of Codazzi type, then the manifold is weakly Schouten-symmetric.*

Proof. From (1.2), (1.3) and (2.4), it follows that

$$3(\nabla_X P)(Y,Z) = A(X)P(Y,Z) + B(Y)P(Z,X) + C(Z)P(X,Y),$$

which yields

$$(\nabla_X P)(Y,Z) = D(X)P(Y,Z) + E(Y)P(Z,X) + F(Z)P(X,Y).$$

Here $D(X) = \frac{1}{3}A(X), E(Y) = \frac{1}{3}B(Y), F(Z) = \frac{1}{3}C(Z)$.

This completes the proof. □

A Riemannian manifold $(M^n, g)(n > 2)$ is said to be quasi Einstein if there exists a nonzero 1-form T associated with a unit vector field such that its Ricci tensor satisfies the condition

$$Ric(X,Y) = ag(X,Y) + bT(X)T(Y),$$

where a, b are smooth functions.

In particular if $b = 0$, then the manifold is Einstein. The notion of quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi umbilical hypersurfaces. This manifold has received a great deal of attention and is studied in considerable detail by many authors [4,5,7,8,13].

Theorem 2.2. *Let (M^n, g) be a $[WCSS]_n$ with $B - C \neq 0$ in (1.3). Then the manifold is quasi Einstein.*

Proof. Interchanging Y, Z in (1.3) and then subtracting the relation obtained thus from (1.3), we get

$$0 = (B - C)(Y)P(Z, X) - (B - C)(Z)P(X, Y).$$

Let us define $D = B - C$. Then the last relation reduces to

$$D(Y)P(Z, X) = D(Z)P(X, Y). \tag{2.5}$$

Contracting (2.5) with respect to X and Z, we have from (1.2)

$$D(Y)(\frac{s}{2(n-1)}) = P(D^\sharp, Y), \tag{2.6}$$

where $D^\sharp$ is a vector field associated with the 1-form D, that is, $g(D^\sharp, X) = D(X)$.
On the other hand, if we replace Y by $D^\sharp$ in (2.5), we get

$$D(D^\sharp)P(Z, X) = D(Z)P(X, D^\sharp),$$

which yields from (2.6)

$$\begin{aligned} P(X, Z) &= \frac{s}{2(n-1)D(D^\sharp)} D(X)D(Z) \\ &= \frac{s}{2(n-1)} U(X)U(Z), \end{aligned} \tag{2.7}$$

where $U(X) = \frac{1}{\sqrt{D(D^\sharp)}} D(X)$.
By virtue of (1.2) and (2.7), we have

$$Ric(X, Z) = \frac{s}{2(n-1)} g(X, Z) + \frac{(n-2)s}{2(n-1)} U(X)U(Z), \tag{2.8}$$

showing that the manifold is quasi Einstein. This completes the proof. □

A Riemannian manifold $(M^n, g)(n > 2)$ is said to be Einstein if its Ricci tensor Ric satisfies

$$Ric(X, Y) = \frac{s}{n} g(X, Y).$$

Note that such a manifold has constant s [1] and hence

$$(\nabla_X Ric)(Y, Z) = 0.$$

Theorem 2.3. *Let (M^n, g) be a $[WCSS]_n$ $(n > 2)$. If the manifold is Einstein with $s \neq 0$, then the sum of the associated 1-forms in (1.3), $A + B + C$, is zero.*

Proof. From (1.2), (1.3) and the Einstein manifold, it follows that

$$0 = \frac{s}{2n(n-1)} (A(X)g(Y, Z) + B(Y)g(Z, X) + C(Z)g(X, Y)),$$

which implies for $X = Y = Z$

$$s(A(X) + B(X) + C(X))g(X, X) = 0.$$

Due to $s \neq 0$, the last equation tells us

$$A + B + C = 0.$$

This completes the proof. □

For a $(0,k)$-tensor field H on (M^n,g), we define a $(0,k+2)$-tensor field $R\cdot H$ [10,14,15] by

$$(R\cdot H)(X_1,\ldots,X_k;X,Y)=(R(X,Y)\cdot H)(X_1,\ldots,X_k)$$
$$=-H(R(X,Y)X_1,X_2,\ldots,X_k)-\ldots-H(X_1,\ldots,X_{k-1},R(X,Y)X_k).$$

If a Riemannian manifold (M^n,g) satisfies the conditions $R\cdot R=0$, $R\cdot Ric=0$ or $R\cdot W=0$, then (M^n,g) is said to be semisymmetric, Ricci-semisymmetric or Weyl-semisymmetric, respectively.

Theorem 2.4. *Let (M^n,g) be a $[WCSS]_n$ with $B-C\neq 0$ in (1.3). If the associated vector field $U^\sharp$ of 1-form U in (2.7) is parallel, then the manifold is Ricci-semisymmetric.*

Proof. From (2.8) it follows that

$$(R(X,Y)\cdot Ric)(Z,V)=-Ric(R(X,Y)Z,V)-Ric(Z,R(X,Y)V)$$
$$=\frac{-(n-2)s}{2(n-1)}(U(R(X,Y)Z)U(V)+U(R(X,Y)V)U(Z))$$
$$=\frac{-(n-2)s}{2(n-1)}(g(R(X,Y)Z,U^\sharp)U(V)+g(R(X,Y)V,U^\sharp)U(Z))=0$$

because of $g(R(X,Y)Z,V)=-g(R(X,Y)V,Z)$ and $\nabla U^\sharp=0$. This completes the proof. □

Theorem 2.5. *Let (M^n,g) be a $[WCSS]_n$ with $B-C\neq 0$ in (1.3). If the associated vector field $U^\sharp$ of 1-form U in (2.7) is parallel, then the semisymmetry and Weyl-semisymmetry are equivalent.*

Proof. Taking account of (1.2), theorem 2.4 and $R\cdot s=0$, we obtain

$$R\cdot P=0.$$

Hence we have from (1.1)

$$R\cdot R=R\cdot W.$$

This completes the proof. □

A vector field X is said to be conformally Killing on a Riemannian manifold (M^n,g) $(n>2)$ [1] if it satisfies the relation

$$\pounds_X g=fg,$$

where f and $\pounds$ denote a smooth function and Lie differentiation, respectively. In particular, if $f=0$, then the vector field X is said to be Killing.

Theorem 2.6. *Let (M^n,g) be a compact orientable $[WCSS]_n$ with $B-C\neq 0$ in (1.3). If the associated vector field $U^\sharp$ of 1-form U in (2.7) is conformally Killing and the scalar curvature s of (M^n,g) is nonpositive, then the vector field $U^\sharp$ is parallel.*

Proof. It is known from [16] that for a vector field X in a compact orientable Riemannian manifold (M^n,g),

$$\int_M[Ric(X,X)-||\nabla X||^2-\frac{n-2}{n}(divX)^2]dM\leq 0 \tag{2.9}$$

and equality holds if and only if X is a conformally Killing vector field. Here div denotes the divergence. Taking account of (2.8) and (2.9), we have for $X=U^\sharp$

$$\int_M[(\frac{s}{2(n-1)}+\frac{(n-2)s}{2(n-1)})-||\nabla U^\sharp||^2-\frac{n-2}{n}(divU^\sharp)^2]dM\leq 0.$$

By virtue of $s\leq 0$, the last inequality yields $\nabla U^\sharp=0$. This completes the proof. □

As a consequence we immediately obtain

Corollary 2.7. *Let (M^n, g) be a compact orientable $[WCSS]_n$ with $B - C \neq 0$ in (1.3). If the associated vector field $U^\sharp$ of 1-form U in (2.7) is Killing and the scalar curvature s of (M^n, g) is nonpositive, then the vector field $U^\sharp$ is parallel.*

The Weyl tensor W of type (0,4) of a Riemannian manifold $(M^n, g)(n > 3)$ is defined by

$$W(X,Y,Z,V) = R(X,Y,Z,V) - \frac{1}{n-2}[Ric(Y,Z)g(X,V) - Ric(X,Z)g(Y,V)$$

$$+g(Y,Z)Ric(X,V) - g(X,Z)Ric(Y,V)] + \frac{s}{(n-1)(n-2)}[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)]. \tag{2.10}$$

Theorem 2.8. *Let (M^n, g) be a $[WCSS]_n$ $(n > 3)$ with $B - C \neq 0$ in (1.3). If its Weyl tensor W vanishes and the scalar curvature s of (M^n, g) is constant, then the associated 1-form U in (2.7) is closed.*

Proof. Differentiating (2.10) covariantly and then contracting the relation obtained thus, we get

$$(divW)(X,Y,Z) = (divR)(X,Y,Z) - \frac{1}{n-2}[(\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z)$$

$$+\frac{1}{2}ds(X)g(Y,Z) - \frac{1}{2}ds(Y)g(X,Z)] + \frac{1}{(n-1)(n-2)}[ds(X)g(Y,Z) - ds(Y)g(X,Z)]. \tag{2.11}$$

Also it is well known [1] that the relation

$$(divR)(X,Y,Z) = (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z)$$

holds. Taking account of the last relation and (2.11) we have

$$(divW)(X,Y,Z) = \frac{n-3}{n-2}[(\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z)$$

$$-\frac{1}{2(n-1)}(ds(X)g(Y,Z) - ds(Y)g(X,Z))].$$

Due to $W = 0$ and $s =$ constant, the last relation becomes

$$(\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) = 0.$$

From (2.8) and $s =$ constant, it follows that the last equation yields

$$(\nabla_X U)(Y)U(Z) + U(Y)(\nabla_X U)(Z) - (\nabla_Y U)(X)U(Z) - U(X)(\nabla_Y U)(Z) = 0. \tag{2.12}$$

Putting $Z = U^\sharp$ in (2.12), we get

$$(\nabla_X U)(Y) - (\nabla_Y U)(X) = 0,$$

showing that U is closed. This completes the proof. □

Theorem 2.9. *Let (M^n, g) be a $[WCSS]_n$ $(n > 3)$ with $B - C \neq 0$ in (1.3). If its Weyl tensor W vanishes and the scalar curvature s of (M^n, g) is constant, then the integral curve of $U^\sharp$ is geodesic.*

Proof. Taking account of (2.12) and putting $Y = Z = U^\sharp$, we obtain

$$(\nabla_{U^\sharp} U)(X) = 0$$

or equivalently

$$\nabla_{U^\sharp} U^\sharp = 0,$$

showing that the integral curve of $U^\sharp$ is geodesic. This completes the proof. □

Theorem 2.10. *Let (M^n, g) be a $[WCSS]_n$ $(n > 3)$ with $B - C \neq 0$ in (1.3). If its Weyl tensor W vanishes and the scalar curvature s of (M^n, g) is constant, then the vector field $U^\sharp$ associated with 1-form U in (2.7) is parallel.*

Proof. Taking account of (2.12) and putting $Y = U^\sharp$, we get from theorem 2.9

$$(\nabla_X U)(Z) = 0$$

or equivalently

$$\nabla_X U^\sharp = 0,$$

showing that the vector field $U^\sharp$ is parallel. This completes the proof. □

3. An example of one parameter family of $[WCSS]_4$

Now we can show that there exists a $[WCSS]_4$ with one parameter family of its associated 1-forms. Let (R^4_+, g) be a Riemannian manifold given by

$$R^4_+ = \{(x_1, x_2, x_3, x_4) | x_4 > 0\}$$

and

$$g = (x_4)^{\frac{4}{3}}[(dx_1)^2 + (dx_2)^2 + (dx_3)^2] + (dx_4)^2.$$

This kind of metric was appeared in [6]. In the metric described as above, the only nonvanishing components for the Christoffel symbols Γ^k_{ij}, the curvature tensors R_{ijkl} and the Ricci tensors Ric_{jk} are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3}(x_4)^{-1},$$

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \frac{-2}{3}(x_4)^{\frac{1}{3}},$$

$$R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x_4)^{\frac{2}{3}},$$

$$R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9}(x_4)^{-\frac{2}{3}},$$

$$Ric_{11} = Ric_{22} = Ric_{33} = \frac{2}{3}(x_4)^{-\frac{2}{3}},$$

$$Ric_{44} = -\frac{2}{3}(x_4)^{-2}, s = \frac{4}{3}(x_4)^{-2}.$$

Therefore the non-vanishing components of the Schouten tensors P_{jk} and their covariant derivatives $P_{jk;l}$ are

$$P_{11} = P_{22} = P_{33} = \frac{2}{9}(x_4)^{-\frac{2}{3}}, P_{44} = -\frac{4}{9}(x_4)^{-2}, \tag{3.13}$$

$$P_{11;4} = P_{22;4} = P_{33;4} = -\frac{4}{27}(x_4)^{-\frac{5}{3}}, P_{44;4} = \frac{8}{9}(x_4)^{-3}. \tag{3.14}$$

Let us define the associated 1-forms A, B, C of (1.3) on (R^4_+, g) as follows:

$$A_i = -\frac{2}{3}(x_4)^{-1}$$

for $i = 4$ and 0 otherwise,

$$B_i = \frac{-(16+t)}{3}(x_4)^{-1}$$

for $i = 4$ and 0 otherwise,

$$C_i = \frac{t}{3}(x_4)^{-1}$$

for $i = 4$ and 0 otherwise. Here $t \in R$.
In the manifold (R^4_+, g), (1.3) reduces to the following equations:

$$P_{11;4} + P_{14;1} + P_{41;1} = A_4 P_{11} + B_1 P_{14} + C_1 P_{41},$$

$$P_{22;4} + P_{24;2} + P_{42;2} = A_4 P_{22} + B_2 P_{24} + C_2 P_{42},$$
$$P_{33;4} + P_{34;3} + P_{43;3} = A_4 P_{33} + B_3 P_{34} + C_3 P_{43},$$
$$P_{44;4} + P_{44;4} + P_{44;4} = A_4 P_{44} + B_4 P_{44} + C_4 P_{44}.$$

From (3.13), (3.14) and the definition of A, B, C, it is easy to see that the last equations hold true. Therefore the Riemannian manifold (R^4_+, g) with A, B, C mentioned in the above is a $[WCSS]_4$.

Competing interests. The author declares no competing interests.

References

[1] A.L. Besse, Einstein Manifolds, Springer, 1987. https://doi.org/10.1007/978-3-540-74311-8.

[2] L. Tamássy, T.Q. Binh, On Weakly Symmetric and Weakly Projectively Symmetric Riemannian Manifolds, Colloq. Math. Soc. Janos Bolyai 56 (1989), 663–670.

[3] E. Cartan, Sur une Classe Remarquable d'Espaces de Riemann, Bull. Soc. Math. Fr. 2 (1926), 214–264. https://doi.org/10.24033/bsmf.1105.

[4] M.C. Chaki, R.K. Maity, On Quasi Einstein Manifolds, Publ. Math. Debrecen 57 (2000), 297–306.

[5] U.C. De, G.C. Ghosh, On Quasi Einstein Manifolds, Period. Math. Hung. 48 (2004), 223–231. https://doi.org/10.1023/b:mahu.0000038977.94711.ab.

[6] U.C. De, C.A. Mantica, Y.J. Suh, On Weakly Cyclic Z Symmetric Manifolds, Acta Math. Hung. 146 (2014), 153–167. https://doi.org/10.1007/s10474-014-0462-9.

[7] J. Kim, Some Properties of N(k)-Quasi Einstein Manifolds, Int. J. Pure Appl. Math. 88 (2013), 351–362.

[8] J. Kim, Concircular Transformations and Quasi-Einstein Manifolds, Far East J. Math. Sci. 103 (2018), 45–52.

[9] J. Kim, Notes on Weakly Cyclic Z-Symmetric Manifolds, Bull. Korean Math. Soc. 55 (2018), 227–237.

[10] O. Kowalski, An Explicit Classification of 3-Dimensional Riemannian Spaces Satisfying $R(X,Y) \cdot R = 0$, Czechoslov. Math. J. 46 (1996), 427–474. https://doi.org/10.21136/cmj.1996.127308.

[11] A. Lichnerowich, Courbure Numberes de Betti el Riemannian, Bull. Soc. Math. France 54 (1950), 214–264.

[12] C.A. Mantica, L.G. Molinari, Weakly Z-Symmetric Manifolds, Acta Math. Hung. 135 (2011), 80–96. https://doi.org/10.1007/s10474-011-0166-3.

[13] A.A. Shaikh, On Pseudo Quasi-Einstein Manifolds, Period. Math. Hung. 59 (2009), 119–146. https://doi.org/10.1007/s10998-009-0119-6.

[14] Z.I. Szabo, Structure Theorems on Riemannian Spaces Satisfying $R(X,\ Y) \cdot R = 0$. I. The Local Version, J. Differ. Geom. 17 (1982), 531–582. https://doi.org/10.4310/jdg/1214437486.

[15] Z. Szabo, Structure Theorems on Riemannian Spaces Satisfying $R(X,\ Y) \cdot R = 0$. II. Global Versions, Geom. Dedicata 19 (1985), 65–108. https://doi.org/10.1007/bf00233102.

[16] Y. Watanabe, Integral Inequalities in Compact Orientable Manifolds, Riemannian or Kählerian, Kodai Math. J. 20 (1968), 264–271. https://doi.org/10.2996/kmj/1138845694.