ISSN: 2832-4293

Pan-American Journal of Mathematics 5 (2026), 4
https://doi.org/10.28919/cpr-pajm/5-4

SPECTRAL ANALYSIS OF ORDER GCD GRAPH ON INTEGERS MODULO RING

MAMIKA UJIANITA ROMDHINI[1,*], ABDURAHIM[1], FAISAL AL-SHARQI[2], AMIT SEHGAL[3], AND HOSSEIN RASHMANLOU[4]

ABSTRACT. Let n be a positive integer and let $\mathbb{Z}_n$ denote the ring of integers modulo $\mathbb{Z}_n$. We introduce the order GCD graph $\Theta_{\mathbb{Z}_n}$, whose vertex set is $\mathbb{Z}_n$, where two distinct vertices a and b are adjacent if and only if $gcd(|a|,|b|) = |a \cdot b|$, with $|a|$ denoting the multiplicative order of a in $\mathbb{Z}_n$. We investigate fundamental structural properties of $\Theta_{\mathbb{Z}_n}$, including a spectral analysis of the graph by studying the eigenvalues of its adjacency matrix and their relationship to the arithmetic structure of $\mathbb{Z}_n$. Several illustrative examples are provided to highlight the interplay between number-theoretic properties of n and the spectral characteristics of $\Theta_{\mathbb{Z}_n}$.

1. INTRODUCTION

Most studies on graphs defined over groups or rings primarily focus on computing various graph-theoretical parameters. However, beyond such computations, graphs can serve as insightful tools for uncovering new structural information about groups or rings and for establishing conditions or constraints on the graphs derived from them. The exploration of these properties often leads to the identification of elegant and mathematically intriguing graphs [4]. Over the years, graph-theoretic approaches have been widely employed in the study of algebraic structures, particularly groups, to extract both structural and spectral characteristics through graphical representations. More recently, researchers have broadened this scope by introducing and analyzing novel classes of group-based graphs, including prime coprime graphs, power graph [7,12], non-commuting graphs [10], identity graphs, and order gcd graphs, thereby deepening the dynamic interconnection between algebra and graph theory.

Spectral graph theory offers a profound connection between algebraic systems and combinatorial structures. Over the past few decades, significant research has focused on investigating graphs derived from algebraic objects such as rings, groups, and semigroups. Among these developments, a particularly noteworthy concept is the order GCD graph of a group or ring, which encapsulates the relationships between elements whose products, or, more generally, their interactions, satisfy specific divisibility or annihilation

---

[1] DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICS AND NATURAL SCIENCES, UNIVERSITY OF MATARAM, MATARAM 83125, INDONESIA; SPECIAL INTEREST GROUP ON GRAPH AND FUZZY THEORY (SIGGFT), FACULTY OF MATHEMATICS AND NATURAL SCIENCES, UNIVERSITY OF MATARAM, MATARAM 83125, INDONESIA

[2] DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION FOR PURE SCIENCES, UNIVERSITY OF ANBAR, RAMADI, ANBAR, IRAQ; COLLEGE OF ENGINEERING, UNIVERSITY OF SCIENCE AND TECHNOLOGY, DHI QAR, IRAQ; DEPARTMENT OF MATHEMATICS, COLLEGE OF EDUCATION, AL-AYEN IRAQI UNIVERSITY, AN NASIRIYAH, IRAQ

[3] DEPARTMENT OF MATHEMATICS, PANDIT NEKI RAM SHARMA GOVERNMENT COLLEGE, ROHTAK, HARYANA, INDIA

[4] SCHOOL OF PHYSICS, DAMGHAN UNIVERSITY, DAMGHAN 3671641167, IRAN

E-mail addresses: `mamika@unram.ac.id`, `abdurahim@staff.unram.ac.id`, `faisal.ghazi@uoanbar.edu.iq`, `amit_sehgal_iit@yahoo.com`, `h.rashmanlou@stu.umz.ac.ir`.

Submitted on Dec. 17, 2025.

2020 *Mathematics Subject Classification.* Primary 05C25; Secondary 05C09.

Key words and phrases. order GCD graph; $\mathbb{Z}_n$; energy of a graph; spectral radius; spectrum.

[*] Corresponding author.

conditions. In recent years, there has been growing scholarly interest in the study of topological indices of graphs associated with algebraic structures (see, for example, [1–3]). One of the most extensively investigated examples is the GCD graph of a finite group (or ring), first introduced by [11]. The interplay between group or ring theory and graph theory provides valuable insights into the underlying structure of these algebraic systems, including connectivity, degree distributions, and spectral properties.

Recent advancements in graph theory extend its scope from simple graphs to algebraic structures such as groups, rings, and fields [8], with applications ranging from cryptography and coding theory to communication networks and computer science. McLean [6] offers background relevant to subgroup-induced graph structure. A significant contribution in this direction is the GCD graph, introduced by Klotz and Sander in 2007 [5] in their study of unitary Cayley graphs. They later investigated its relation to graph products. More recently, Sarkar and Patra (2024) [11] introduced the order GCD graph on the integer modulo group $\mathbb{Z}_n$, denoted as $\Theta_{\mathbb{Z}_n}$. In this graph, the vertices correspond to the elements of $\mathbb{Z}_n$, written as $V\left(\Theta_{\mathbb{Z}_n}\right)$. Adjacency between two distinct vertices a and b occurs precisely when the greatest common divisor of their orders equals the order of their product, i.e., $gcd(|a|,|b|) = |a \cdot b|$ [11], where $|a|$ and $|b|$ are the order of a and b, respectively. The underlying structure is identified as a ring rather than a group, as it inherently involves two binary operations, addition and multiplication modulo p^k. This formulation ensures that our analysis consistently captures both additive and multiplicative relationships in modular arithmetic.

2. PRELIMINARIES

We now present the fundamental concepts and notations used in spectral graph theory. The symbols and their corresponding definitions are summarized in the table below.

TABLE 1. Notation and its Definition

Symbol	Definition
$\Theta(\mathbb{Z}_n)$	prime coprime graph of $\mathbb{Z}_n$
$\lvert u\rvert$	order of u in $\mathbb{Z}_n$
$\mathbb{Z}_n$	group of integers modulo n
$\Theta_{\mathbb{Z}_n}$	prime coprime graph of $\mathbb{Z}_n$
d_u	degree of vertex u
$A(\Theta_{\mathbb{Z}_n})$	adjacency matrix of $\Theta_{\mathbb{Z}_n}$
γ_i	eigenvalues of the matrix
$E_A(\Theta_{\mathbb{Z}_n})$	adjacency energy of $\Theta_{\mathbb{Z}_n}$
$Spec(\Theta_{\mathbb{Z}_n})$	spectrum of $\Theta_{\mathbb{Z}_n}$
$\rho_A(\Theta_{\mathbb{Z}_n})$	A-spectral radius of $\Theta_{\mathbb{Z}_n}$

The adjacency matrix $A_{n\times n}$ corresponding to $\Theta_{\mathbb{Z}_n}$ is given by $A\left(\Theta_{\mathbb{Z}_n}\right) = [a_{ij}]$ whose (i,j)-th entry is

$$a_{ij} = \begin{cases} 1, & \text{for adjacent vertices} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

The characteristic polynomial of $A\left(\Theta_{\mathbb{Z}_n}\right)$ is $P_{A(\Theta_{\mathbb{Z}_n})}(\gamma) = |\gamma I_n - A(\Theta_{\mathbb{Z}_n})|$. The roots of $P_{A(\Theta_{\mathbb{Z}_n})}(\gamma) = 0$ are the eigenvalues of $A(\Theta_{\mathbb{Z}_n})$ and denoted as $\gamma_1, \gamma_2, \ldots, \gamma_n$.

Moreover, the $A-$spectrum of $\Theta_{\mathbb{Z}_n}$ can be written as:

$$\sigma_A(\Theta_{\mathbb{Z}_n}) = \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_{p^k} \\ k_1, & k_2, & \ldots, & k_{p^k} \end{pmatrix},$$

where $k_1, k_2, \ldots, k_r$ are their respective multiplicities of eigenvalues. Therefore, the $A-$energy of $\Theta_{\mathbb{Z}_{p^k}}$ can be defined as follows:

$$E_A(\Theta_{\mathbb{Z}_n}) = \Sigma_{i=1}^{n} |\gamma_i|,$$

and $A-$spectral radius of $\Theta_{\mathbb{Z}_{p^k}}$ is defined as

$$\rho_A(\Theta_{\mathbb{Z}_n}) = max\{|\gamma| : \gamma \in \sigma_A(\Theta_{\mathbb{Z}_n}).$$

To facilitate the derivation of the characteristic polynomial of $\Theta_{\mathbb{Z}_n}$, we first recall the following theorem.

Theorem 2.1. *[9] If an $n \times n$ matrix*

$$T = \begin{pmatrix} 0 & J_{1\times(n_1-1)} & J_{1\times n_2} \\ J_{(n_1-1)\times 1} & 0_{n_1-1} & J_{(n_1-1)\times n_2} \\ J_{n_2\times 1} & J_{n_2\times(n_1-1)} & (J-I)_{n_2} \end{pmatrix},$$

where $n = n_1 + n_2$, then characteristic polynomial of T is

$$P_T(\gamma) = \gamma^{n_1-2}(\gamma+1)^{n_2}(\gamma^2 - n_2\gamma - (n_2+1)(n_1-1)).$$

3. MAIN RESULTS

This section presents the main results of the study. We begin by formulating the characteristic polynomial of a specific form of matrix.

Theorem 3.1. *If an $n \times n$ matrix*

$$T = \begin{pmatrix} (J-I)_2 & J_{2\times(n_1-2)} & J_{2\times n_2} \\ J_{(n_1-2)\times 2} & 0_{n_1-2} & J_{(n_1-2)\times n_2} \\ J_{n_2\times 2} & J_{n_2\times(n_1-2)} & (J-I)_{n_2} \end{pmatrix},$$

where $n = n_1 + n_2$, then characteristic polynomial of T is

$$P_T(\gamma) = \gamma^{n_1-3}(\gamma+1)^{n_2}\left(\gamma^3 - n_2\gamma^2 - (n_1 - 2 + (n_1-1)(n_2+1))\gamma - (n_2+2)(n_1-2)\right).$$

Proof. The characteristic formula of M is the determinant of $\gamma I_n - M$ as follows.

$$P_T(\mu) = \begin{vmatrix} (\gamma+1)I_2 - J_2 & -J_{2\times(n_1-2)} & -J_{2\times n_2} \\ -J_{(n_1-2)\times 2} & \gamma I_{n_1-2} & -J_{(n_1-2)\times n_2} \\ -J_{n_2\times 2} & -J_{n_2\times(n_1-2)} & (\gamma+1)I_{n_2} - J_{n_2} \end{vmatrix}.$$

Let R_i and C_i be the matrix's i-th row and column, respectively. Suppose now R'_i and C'_i are the new i-th row and column of the matrix acquired from R_i and C_i by the elementary row and column operations, respectively. These apply to $P_T(\gamma)$ as follows:

1. We replace the $n_1 + i$-rows with the row operation: $R_{n_1+i} \longrightarrow R_{n_1+i} - R_1$, for $i = 1, 2, \ldots, n_2$.
2. We replace the first column with the summation of the element of the first column and elements of $n_1+1, n_1+2, \ldots, n_1+n_2$ rows, or in other words, $C_1 \longrightarrow C_1 + C_{n_1+1} + C_{n_1+2} + \ldots + C_{n_1+n_2}$.
3. The next step is replacing the second column, $C_2 \longrightarrow C_2 + \left(\frac{1}{\gamma}\right) C_3 + \left(\frac{1}{\gamma}\right) C_4 + \ldots + \left(\frac{1}{\gamma}\right) C_{n_1}$.
4. The last step is replacing the first column, $C_1 \longrightarrow C_1 + \left(\frac{n_2+1}{\gamma}\right) C_3 + \left(\frac{n_2+1}{\gamma}\right) C_4 + \ldots + \left(\frac{n_2+1}{\gamma}\right) C_{n_1}$.

We can express $P_T(\gamma)$ as

$$P_T(\gamma) = \begin{vmatrix} \gamma - n_2 - (n_1-2)\left(\frac{n_2+1}{\gamma}\right) & -1 - \frac{(n_1-2)}{\gamma} & -J_{1\times(n_1-2)} & -J_{1\times n_2} \\ -(n_2+1)-(n_1-2)\left(\frac{n_2+1}{\gamma}\right) & \gamma - \frac{(n_1-2)}{\gamma} & -J_{1\times(n_1-2)} & -J_{1\times n_2} \\ 0_{(n_1-2)\times 1} & 0_{(n_1-2)\times 1} & \gamma I_{n_1-2} & -J_{(n_1-2)\times n_2} \\ 0_{n_2\times 1} & 0_{n_2\times 1} & 0_{n_2\times(n_1-2)} & (\gamma+1)I_{n_2} \end{vmatrix}.$$

It is obvious that the above determinant is in the form of an upper triangular matrix, then we can directly derive that

$$P_T(\gamma) = \gamma^{n_1-3}(\gamma+1)^{n_2}\left(\gamma^3 - n_2\gamma^2 - (n_1 - 2 + (n_1-1)(n_2+1))\gamma - (n_2+2)(n_1-2)\right).$$

□

3.1. **Spectral properties of $\Theta_{\mathbb{Z}_{p^k}}$.** For further discussion, the graph $\Theta_{\mathbb{Z}_n}$ considered in this study corresponds to the case where $n = p^k$. Hence, it is denoted by $\Theta\left(\mathbb{Z}_{p^k}\right)$.

To provide a clearer understanding of the structural properties, we present an explicit construction of the order GCD graph corresponding to the ring $\mathbb{Z}_{2^3}$ in the following example.

Example 3.2. The discussion begins with an examination of the ring $\mathbb{Z}2^3$, where $p = 2$ and $k = 3$. The element orders in $\mathbb{Z}2^3$ are determined as follows: $|0| = 1$, $|1| = 8$, $|2| = 4$, $|3| = 8$, $|4| = 2$, $|5| = 8$, $|6| = 4$, and $|7| = 8$. From these results, we obtain the gcd values for all possible element pairs in $\mathbb{Z}_{2^3}$, as shown in Table 2. In addition, the orders corresponding to the products of distinct elements in the ring are similar to Table 2.

gcd	$\|0\|$	$\|1\|$	$\|2\|$	$\|3\|$	$\|4\|$	$\|5\|$	$\|6\|$	$\|7\|$
$\|0\|$	-	1	1	1	1	1	1	1
$\|1\|$	1	-	4	8	2	8	4	8
$\|2\|$	1	4	-	4	2	4	4	4
$\|3\|$	1	8	4	-	2	8	4	8
$\|4\|$	1	2	2	2	-	2	2	2
$\|5\|$	1	8	4	8	2	-	4	8
$\|6\|$	1	4	4	4	2	4	-	4
$\|7\|$	1	8	4	8	2	8	4	-

TABLE 2. GCD for any element of $\mathbb{Z}_{2^3}$

Therefore, the order GCD graph of $\mathbb{Z}_{2^3}$, $\Theta_{\mathbb{Z}_{2^3}}$, as presented below.

Furthermore, based on the structure illustrated in Figure 1, the corresponding adjacency matrix of the ring $\mathbb{Z}_{2^3}$ can be constructed as follows:

$$A(\Theta_{\mathbb{Z}_{2^3}}) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & J_{1\times 3} & J_{1\times 4} \\ J_{3\times 1} & 0_3 & J_{3\times 4} \\ J_{4\times 1} & J_{4\times 3} & (J-I)_4 \end{pmatrix}_{8\times 8}.$$

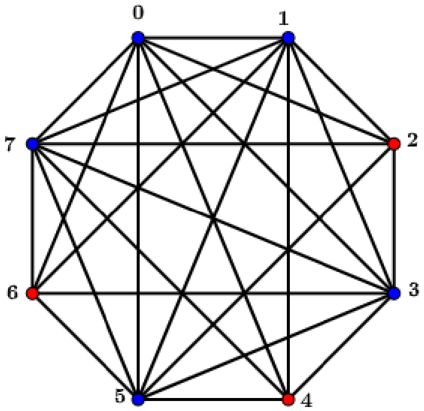

FIGURE 1. GCD Graph for $\mathbb{Z}_{2^3}$

From the above matrix, the eigenvalues of $A(\Theta_{\mathbb{Z}_{2^3}})$ are obtained as 0 (of multiplicity 2), -1 (of multiplicity 4), and $2 \pm \sqrt{19}$ (of each multiplicity 1). Hence, the spectral radius of $\Theta_{\mathbb{Z}_{2^3}}$ is $2+\sqrt{19}$. Consequently, the energy is computed as $\Theta_{\mathbb{Z}_{2^3}}$ as $4+2\sqrt{19}$, which equals twice the spectral radius, thereby confirming the result of the corollary.

The next theorem presents the explicit form of the characteristic polynomial corresponding to the adjacency matrix $A(\Theta_{\mathbb{Z}_{p^k}})$ of the order GCD graph over $\mathbb{Z}_{p^k}$.

Theorem 3.3. *For the order GCD graph $\Theta_{\mathbb{Z}_{p^k}}$ on $\mathbb{Z}_{p^k}$ (p prime, $k \in \mathbb{N}$), the characteristic polynomial of $A(\Theta_{\mathbb{Z}_{p^k}})$ is*

$$P_{A(\Theta_{\mathbb{Z}_{p^k}})}(\gamma) = \gamma^{p^{k-1}-2}(\gamma+1)^{p^k-p^{k-1}}(\gamma^2-(p^k-p^{k-1})\gamma-(p^k-p^{k-1}+1)(p^{k-1}-1)).$$

Proof. Let $W_1 = \{p, 2p, 3p, \dots, (p^{k-1}-1)\,p\}$ and $W_2 = \mathbb{Z}_{p^k} \backslash W_1$. According to the definition of the GCD graph, each vertex in W_1 is non-adjacent to every other vertex within W_1. However, it is adjacent to all vertices in W_2, along with two additional vertices from W_2. Hence, by Equation 2.1, the $p^k \times p^k$ adjacency matrix of $\Theta_{\mathbb{Z}_{p^k}}$ must take the form

$$A(\Theta_{\mathbb{Z}_{p^k}}) = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

Equivalently, the adjacency matrix can be represented in the form of block matrices as follows:

$$A(\Theta_{\mathbb{Z}_{p^k}}) = \begin{pmatrix} 0 & J_{1\times(p^{k-1}-1)} & J_{1\times(p^k-p^{k-1})} \\ J_{(p^{k-1}-1)\times 1} & 0_{p^{k-1}-1} & J_{(p^{k-1}-1)\times(p^k-p^{k-1})} \\ J_{(p^k-p^{k-1})\times 1} & J_{(p^k-p^{k-1})\times(p^{k-1}-1)} & (J-I)_{p^k-p^{k-1}} \end{pmatrix}.$$

Based on Theorem 2.1, with $n_1 = p^{k-1}$, $n_2 = p^k - p^{k-1}$, then we get

$$P_{A(\Theta_{\mathbb{Z}_{p^k}})}(\gamma) = \gamma^{p^{k-1}-2}(\gamma+1)^{p^k-p^{k-1}}(\gamma^2 - (p^k - p^{k-1})\gamma - (p^k - p^{k-1}+1)(p^{k-1}-1)).$$

□

As Theorem 3.3 provides the characteristic polynomial of $\Theta_{\mathbb{Z}_{p^k}}$, we can consequently determine the spectral radius and energy of $\Theta_{\mathbb{Z}_{p^k}}$.

Theorem 3.4. *For the order GCD graph $\Theta_{\mathbb{Z}_{p^k}}$ on $\mathbb{Z}_{p^k}$ (p prime, $k \in \mathbb{N}$, then the spectral radius for $\Theta_{\mathbb{Z}_{p^k}}$ is*

$$\rho(\Theta_{\mathbb{Z}_{p^k}}) = \frac{1}{2}\left(p^k - p^{k-1} + \sqrt{(p^k-p^{k-1})^2 + 4(p^k-p^{k-1}+1)(p^{k-1}-1)}\right).$$

Proof. The polynomial of $P_{A(\Theta_{\mathbb{Z}_{p^k}})}(\gamma)$ in Theorem 3.3 yield the eigenvalues for $\Theta_{\mathbb{Z}_{p^k}}$. With multiplicity $p^{k-1}-2$, we obtain $\gamma_1 = 0$. Then we get $\gamma_2 = -1$ of multiplicity $p^k - p^{k-1}$, and

$$\gamma_{3,4} = \frac{1}{2}\left(p^k - p^{k-1} \pm \sqrt{(p^k-p^{k-1})^2 + 4(p^k-p^{k-1}+1)(p^{k-1}-1)}\right),$$

as the roots of the quadratic formula. Hence, the spectrum for $\Theta_{\mathbb{Z}_{p^k}}$ is as follows

$$\begin{aligned} Spec(\Theta_{\mathbb{Z}_{p^k}}) = \Bigg\{ & \left(\frac{1}{2}\left(p^k - p^{k-1} + \sqrt{(p^k-p^{k-1})^2 + 4(p^k-p^{k-1}+1)(p^{k-1}-1)}\right)\right)^1, \\ & (0)^{p^{k-1}-2}, (-1)^{p^k-p^{k-1}}, \\ & \left(\frac{1}{2}\left(p^k - p^{k-1} - \sqrt{(p^k-p^{k-1})^2 + 4(p^k-p^{k-1}+1)(p^{k-1}-1)}\right)\right)^1 \Bigg\}. \end{aligned}$$

The maximum of $|\gamma_i|$ is

$$\rho(\Theta_{\mathbb{Z}_{p^k}}) = \frac{1}{2}\left(p^k - p^{k-1} + \sqrt{(p^k-p^{k-1})^2 + 4(p^k-p^{k-1}+1)(p^{k-1}-1)}\right).$$

and this is spectral radius of $\Theta_{\mathbb{Z}_{p^k}}$. □

Theorem 3.5. *For the order GCD graph $\Theta_{\mathbb{Z}_{p^k}}$ on $\mathbb{Z}_{p^k}$ (p prime, $k \in \mathbb{N}$, then the energy for $\Theta_{\mathbb{Z}_{p^k}}$ is*

$$E(\Theta_{\mathbb{Z}_{p^k}}) = p^k - p^{k-1} + \sqrt{(p^k-p^{k-1})^2 + 4(p^k-p^{k-1}+1)(p^{k-1}-1)}.$$

Proof. Referring to the spectrum given in Theorem 3.4, we can compute the energy of $\Theta_{\mathbb{Z}_{p^k}}$ as shown below.

$$\begin{aligned} E(\Theta_{\mathbb{Z}_{p^k}}) = & (p^{k-1}-2)|0| + (p^k - p^{k-1})\,|-1| + \\ & \left|\frac{1}{2}\left(p^k - p^{k-1} \pm \sqrt{(p^k-p^{k-1})^2 + 4(p^k-p^{k-1}+1)(p^{k-1}-1)}\right)\right| \\ = & p^k - p^{k-1} + \sqrt{(p^k-p^{k-1})^2 + 4(p^k-p^{k-1}+1)(p^{k-1}-1)}. \end{aligned}$$

□

3.2. **Spectral properties of $\Theta_{\mathbb{Z}_{2p^k}}$.** This part focuses on the case $n = 2 \cdot p^k$, where p is a prime number and k is a positive integer, to investigate the spectral properties of the order GCD graph defined on the ring of integers modulo n. As stated in Example 3.2, the integer $n = 2^3$ can alternatively be expressed as $n = 2 \cdot 2^2$.

Theorem 3.6. *For the order GCD graph $\Theta_{\mathbb{Z}_{2p^k}}$ on $\mathbb{Z}_{2p^k}$ (p odd prime, $k \in \mathbb{N}$, the characteristic polynomial of $A(\Theta_{\mathbb{Z}_{2p^k}})$ is*

$$\begin{aligned} P_{A(\Theta_{\mathbb{Z}_{2p^k}})}(\gamma) =& \gamma^{2p^{k-1}-3}(\gamma+1)^{2p^k-2p^{k-1}} \\ & \big(\gamma^3 - (2p^k - 2p^{k-1})\gamma^2 - (2p^{k-1} - 2 + (2p^{k-1}-1)(2p^k - 2p^{k-1}))\gamma - \\ & (2p^k - 2p^{k-1} + 2)(2p^{k-1} - 2)\big). \end{aligned}$$

Proof. Let $W_1 = \{p, 2p, 3p, \dots, (2p^{k-1}-1)\,p\}$ and $W_2 = \mathbb{Z}_{2p^k} \backslash W_1$, where $W_1 \cup W_2 = \mathbb{Z}_{2\cdot p^k}$. According to the definition of the GCD graph, each vertex in W_1 is non-adjacent to every other vertex within W_1. However, it is adjacent to all vertices in W_2, along with two additional vertices from W_2. Hence, by Equation 2.1, the $2p^k \times 2p^k$ adjacency matrix of $\Theta_{\mathbb{Z}_{2p^k}}$ must take the form

$$A(\Theta_{\mathbb{Z}_{2p^k}}) = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

Equivalently, the adjacency matrix can be represented in the form of block matrices as follows:

$$A(\Theta_{\mathbb{Z}_{2p^k}}) = \begin{pmatrix} (J-I)_2 & J_{2\times(2p^{k-1}-2)} & J_{1\times(2p^k-2p^{k-1})} \\ J_{(2p^{k-1}-2)\times 2} & 0_{2p^{k-1}-2} & J_{(2p^{k-1}-2)\times(2p^k-2p^{k-1})} \\ J_{(2p^k-2p^{k-1})\times 2} & J_{(2p^k-2p^{k-1})\times(2p^{k-1}-2)} & (J-I)_{2p^k-2p^{k-1}} \end{pmatrix}.$$

Based on Theorem 3.1, with $n_1 = 2p^{k-1}$, $n_2 = 2p^k - 2p^{k-1}$, then we get

$$\begin{aligned} P_{A(\Theta_{\mathbb{Z}_{2p^k}})}(\gamma) =& \gamma^{2p^{k-1}-3}(\gamma+1)^{2p^k-2p^{k-1}} \\ & \big(\gamma^3 - (2p^k - 2p^{k-1})\gamma^2 - (2p^{k-1} - 2 + (2p^{k-1}-1)(2p^k - 2p^{k-1}))\gamma - \\ & (2p^k - 2p^{k-1} + 2)(2p^{k-1} - 2)\big). \end{aligned}$$

□

4. CONCLUSIONS

We presented the order GCD graph $\Theta_{\mathbb{Z}_n}$ on the ring $\mathbb{Z}_n$, with adjacency defined via the multiplicative orders of vertices, and examined its fundamental structural and spectral properties. Our results show that the arithmetic structure and prime factorization of n strongly influence the graph's topology and adjacency spectrum. Future work may address the Laplacian and signless Laplacian spectra, graph energy, and spectral integrality, as well as extensions of the order GCD graph to other algebraic settings such as finite fields and finite commutative rings.

Acknowledgement. Part of this work is partially supported by Universitas Mataram.

Competing interests. The authors declare no competing interests.

REFERENCES

[1] A. Abdurahim, M.U. Romdhini, F. Al-Sharqi, N.A. Robbaniyyah, Relative Prime Coprime Graph of Integers Modulo Group and Its Reverse Topological Indices, Pan-Amer. J. Math. 4 (2025), 10. https://doi.org/10.28919/cpr-pajm/4-10.

[2] Abdurahim, M.U. Romdhini, J. Qudsi, F. Al-Sharqi, Z.M. Rodzi, Delta Degree-Based Indices of Prime Coprime Graph for Integers Modulo Group, Sci. Technol. Indones. 11 (2026), 10–18. https://doi.org/10.26554/sti.2026.11.1.10-18.

[3] Abdurahim, M.U. Romdhini, J. Qudsi, S.K.S. Husain, Zagreb-Based Indices of Prime Coprime Graph for Integers Modulo Power of Primes, Malays. J. Math. Sci. 19 (2025), 961–974. https://doi.org/10.47836/mjms.19.3.10.

[4] P.J. Cameron, What Can Graphs and Algebraic Structures Say to Each Other?, AKCE Int. J. Graphs Comb. 21 (2023), 249–254. https://doi.org/10.1080/09728600.2023.2290036.

[5] W. Klotz, T. Sander, Some Properties of Unitary Cayley Graphs, Electron. J. Comb. 14 (2007), R45. https://doi.org/10.37236/963.

[6] K.R. McLean, The Characteristic Subgroups of an Abelian p-Group, J. Algebr. 663 (2025), 838–856. https://doi.org/10.1016/j.jalgebra.2024.10.001.

[7] P. Rana, A. Sehgal, P. Bhatia, P. Kumar, Topological Indices and Structural Properties of Cubic Power Graph of Dihedral Group, Contemp. Math. 5 (2024), 761–779. https://doi.org/10.37256/cm.5120243029.

[8] M.U. Romdhini, A. Nawawi, B.N. Syechah, Seidel Laplacian and Seidel Signless Laplacian Energies of Commuting Graph for Dihedral Groups, Malays. J. Fundam. Appl. Sci. 20 (2024), 701–713. https://doi.org/10.11113/mjfas.v20n3.3465.

[9] M.U. Romdhini, A. Abdurahim, F. Al-Sharqi, S. Bahri, A.E.S.H. Maharani, On Energy of Prime Coprime Graph for Dihedral Groups, Math. Stat. 13 (2025), 175–179. https://doi.org/10.13189/ms.2025.130306.

[10] M.U. Romdhini, P. Rana, A. Sehgal, P. Bhatia, Structural Properties and Characteristic Polynomials of Cubic Power Graph of Dihedral Group, Math. Stat. 13 (2025), 97–104. https://doi.org/10.13189/ms.2025.130204.

[11] P. Sarkar, K. Patra, The Order Gcd Graph, Contemp. Math. (2024), 2926–2932. https://doi.org/10.37256/cm.5320243600.

[12] A. Siwach, V. Bhatia, A. Sehgal, P. Rana, Characteristic Polynomial of Maximum and Minimum Matrix of Square Power Graph of Dihedral Group of Order 2n with Odd Natural Number n, Int. J. Stat. Appl. Math. 9 (2024), 57–64. https://doi.org/10.22271/maths.2024.v9.i3a.1731.