ISSN: 2832-4293

Pan-American Journal of Mathematics 5 (2026), 8
https://doi.org/10.28919/cpr-pajm/5-8
© 2026 by the authors

A NEW METHOD FOR FUZZY MULTIOBJECTIVE LINEAR FRACTIONAL OPTIMIZATION BASED ON MEDIAN FUNCTIONS OF α-CUTS

YOUSSOUF OUEDRAOGO AND ABDOULAYE COMPAORE*

ABSTRACT. This paper proposes a novel method for fuzzy multiobjective linear fractional optimization based on median functions of α-cuts. The aim is to provide a robust framework that faithfully captures the inherent fuzziness of the data, thereby improving solution quality in uncertain environments. The methodology consists of four stages: defuzzification, linearization of the fractional objectives, aggregation, and resolution. A key theoretical result establishes the preservation of Pareto optimality between the original fuzzy problem and the transformed deterministic model. Numerical experiments, including a practical case study, show that the proposed method yields more stable and higher-performing solutions than existing approaches. The main contribution lies in the use of median functions, which provide a natural compromise between the lower and upper bounds of a fuzzy interval and offer a unified treatment for both triangular and trapezoidal fuzzy numbers.

1. INTRODUCTION

In fields such as portfolio management, supply chain operations, and inventory control, optimization problems frequently involve data that are imprecise, poorly known, or subject to market fluctuations, resource variability, or changing operational conditions. Such uncertainty renders classical deterministic optimization approaches inadequate, thereby necessitating the incorporation of models capable of representing imprecision in a realistic manner.

Fuzzy set theory, introduced by Zadeh [1], has emerged as a fundamental framework for modeling these situations by replacing exact numerical values with fuzzy numbers—typically triangular, trapezoidal, or more general forms—that capture both central tendencies and the associated margins of uncertainty.

Among optimization problems under uncertainty, fuzzy multiobjective linear fractional programs play a strategic role. These models enable the simultaneous optimization of several ratios—such as profit-to-cost, return-to-resource, or efficiency-to-time, whose coefficients are represented by fuzzy numbers. This formulation is particularly relevant in management, logistics, and engineering, where decisions must be based on relative performance rather than absolute values. However, the fractional nature of the objectives, combined with the fuzzy structure of the data, significantly complicates their solution.

Various methods have been proposed to address these problems. Among them, the approach introduced by Nayak and Maharana [2] solves multiobjective fuzzy linear fractional programming problems with triangular fuzzy coefficients by employing a parametric transformation based on the Charnes–Cooper

LABORATOIRE DE MATHEMATIQUES INFORMATIQUE ET APPLICATION, UNIVERSITÉ NORBERT ZONGO, KOUDOUGOU BP376, BURKINA FASO

E-mail addresses: ouedraogoyoussouf185@gmail.com, ab.compaore1@gmail.com.
Submitted on Dec. 11, 2025.

2020 *Mathematics Subject Classification.* 90C70, 03E72, 90C29.

Key words and phrases. fuzzy multiobjective fractional optimization; fuzzy numbers; median function.
*Corresponding author.

method. First, each fuzzy fractional objective is converted into a fuzzy linear objective through a change of variables. The resulting multiobjective fuzzy problem is then transformed into a deterministic model using fuzzy goal programming, in which the objectives are expressed as aspiration levels to be achieved with minimal deviations. The method minimizes a weighted sum of negative deviations to obtain an efficient solution, thereby preserving the fuzzy structure throughout the solution process without resorting to premature defuzzification.

In [3], the same authors propose an alternative method for multiobjective fuzzy fractional programming problems involving trapezoidal fuzzy numbers. This approach utilizes α-cuts and δ-cuts to convert fuzzy objectives and constraints into interval-valued expressions. The fractional functions are then linearized via a first-order Taylor expansion around individual optimal solutions. Subsequently, the objectives are aggregated using the weighted-sum technique, and the sum of the lower and upper bounds of the aggregated objective is maximized to generate a set of nondominated solutions.

Finally, the method developed by Sama et al. [4] tackles multiobjective fuzzy linear fractional programming problems through an iterative scheme combining parametric transformation and core-based defuzzification. The procedure begins by linearizing each fuzzy fractional objective into a parametric linear optimization problem, leveraging an optimality condition that relates the fuzzy ratio to a linear fuzzy expression. An iterative process is then applied: at each iteration, fuzzy numbers are defuzzified using their core (i.e., modal) values, yielding a deterministic multiobjective linear program, which is solved via the ϵ-constraint method. The parameters are updated using the solution from the previous iteration, and the algorithm proceeds until convergence to a Pareto-optimal solution is achieved.

Despite their contributions, these methods suffer from several limitations: they may introduce biases by focusing solely on the lower and upper bounds of fuzzy intervals, lose valuable information by neglecting the internal distribution of the α-cuts, or lack robustness with respect to variations in the membership level α.

To address these shortcomings, in this work we propose a new optimization method for fuzzy multiobjective linear fractional problems that relies on a more balanced and faithful representation of fuzzy information: the median functions of the α-cuts. Rather than operating solely on the endpoints of fuzzy intervals, our approach uses the median of each α-cut, defined as the arithmetic mean of its lower and upper bounds, as a central indicator of the fuzzy value at each level $\alpha \in [0, 1]$. This transformation preserves the fuzzy structure while producing a continuous function that is easier to handle numerically.

Our method follows a four-step process:

(1) **Median-based defuzzification**: each fuzzy number is replaced by its median function with respect to α;
(2) **Linearization**: the fractional problem is transformed into an equivalent linear problem using Dinkelbach's theorem;
(3) **Aggregation**: the objectives are combined using a weighted-sum approach to obtain a single-objective problem;
(4) **Solution**: the transformed problem is solved using the simplex method for different values of α, allowing for a sensitivity analysis of the solutions with respect to uncertainty.

A central theoretical result of this work is the preservation of Pareto optimality between the original fuzzy problem and the deterministic model obtained via median functions. It is rigorously established that every Pareto-optimal solution of the transformed problem corresponds to a Pareto-optimal solution of the initial fuzzy problem, thereby ensuring the validity and robustness of the proposed approach.

The originality of the method lies in the use of median functions as a defuzzification operator, which provides a natural compromise between the lower and upper bounds of a fuzzy interval without systematically favoring extreme values. In contrast to classical approaches, the framework applies uniformly to both triangular and trapezoidal fuzzy numbers and enables a detailed sensitivity analysis with respect to the membership level α. The method is illustrated through several numerical examples, including a practical case study inspired by a production chain, allowing for direct comparison with existing techniques. Computational results demonstrate that the proposed approach yields higher-quality solutions, both in terms of objective function values and robustness to variations in α. A comparative analysis further underscores its advantages, particularly regarding solution stability and fidelity to the original fuzzy representation.

The remainder of the paper is organized as follows. Section 2 reviews the necessary preliminaries on fuzzy sets. Section 3 presents the proposed methodology and the main theoretical results. Section 4 reports the numerical experiments and discusses the practical case study. Finally, Section 5 concludes the paper and outlines directions for future research.

.

2. Preliminaries

2.1. **Fuzzy Numbers.** This section recalls the notion of fuzzy numbers and some basic arithmetic operations.

Definition 2.1. [4,5,7,8] Let $\mathcal{X}$ be a set. A fuzzy subset $\tilde{A}$ of $\mathcal{X}$ is characterized by a membership function $\mu_{\tilde{A}} : \mathcal{X} \to [0,1]$ and is represented as the set of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in \mathcal{X}\}. \tag{2.1}$$

The value $\mu_{\tilde{A}}(x) \in [0,1]$ represents the degree to which x belongs to the fuzzy set $\tilde{A}$.

Definition 2.2. [4,5,7,8] Let $\tilde{A}$ be a fuzzy set on $\mathcal{X}$ and $\alpha \in [0,1]$. The α-level set (or α-cut) of $\tilde{A}$, denoted $\tilde{A}_\alpha$, is defined by

$$\tilde{A}_\alpha = \{x \in \mathcal{X} \mid \mu_{\tilde{A}}(x) \geq \alpha\}. \tag{2.2}$$

In what follows, we identify $\mathcal{X}$ with $\mathbb{R}$.

Definition 2.3. [4,7,9–11] A fuzzy subset $\tilde{A}$ of $\mathbb{R}$ is called a *fuzzy number* if the following conditions hold:

(i) $\tilde{A}$ is *normal*, i.e., there exists $x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$;
(ii) $\tilde{A}$ is *convex*, i.e., $\mu_{\tilde{A}}$ is quasi-concave;
(iii) $\mu_{\tilde{A}}$ is upper semi-continuous, i.e., $\tilde{A}_\alpha$ is a closed subset of $\mathbb{R}$ for any $\alpha \in [0,1]$;
(iv) The 0-level set $\tilde{A}_0$ is a compact subset of $\mathbb{R}$.

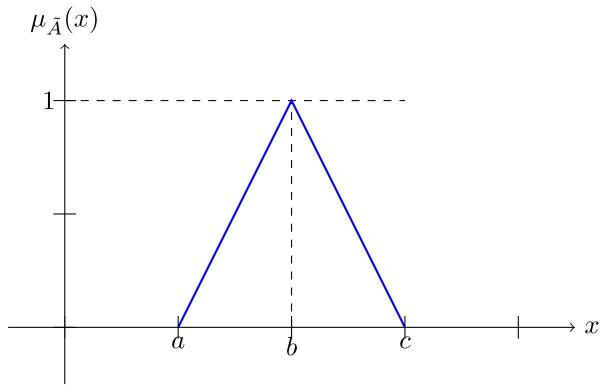
We denote by $\mathcal{N}(\mathbb{R})$ the set of all fuzzy numbers. For $\tilde{A} \in \mathcal{N}(\mathbb{R})$, its α-level set is a compact and convex interval in $\mathbb{R}$, denoted $\tilde{A}_\alpha = [\tilde{a}^l_\alpha, \tilde{a}^r_\alpha]$ for $\alpha \in [0,1]$.

Definition 2.4. [4, 12, 13] Let $\tilde{A} = (a, b, c)$ with $a, b, c \in \mathbb{R}$. $\tilde{A}$ is called a *triangular fuzzy number* if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a, \\ \dfrac{x-a}{b-a}, & a < x \leq b, \\ \dfrac{x-c}{b-c}, & b \leq x < c, \\ 0, & x \geq c. \end{cases} \tag{2.3}$$

The α-cut of a triangular fuzzy number is expressed as

(2.4) $$\tilde{A}^{\alpha} = [a^l_{\alpha}, a^r_{\alpha}] = [\alpha(b-a)+a, \alpha(b-c)+c], \quad \forall \alpha \in [0,1].$$



Membership function of a triangular fuzzy number

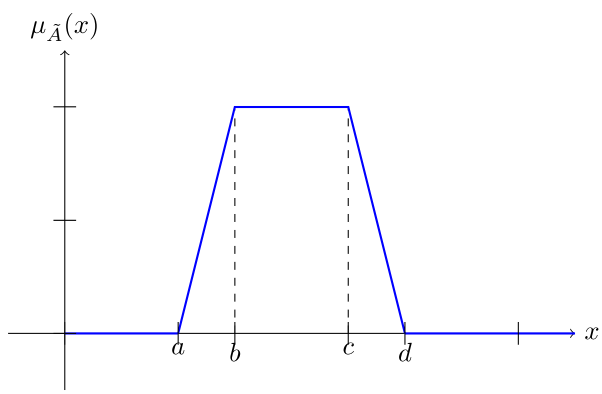
Remark 2.5. A triangular fuzzy number $\tilde{A} = (a, b, c)$ is symmetric if and only if $b - a = c - b$.

Definition 2.6. [4,6,8,12,13] Let $\tilde{A} = (a, b, c, d)$ with $a, b, c, d \in \mathbb{R}$. $\tilde{A}$ is called a *trapezoidal fuzzy number* if its membership function is

(2.5) $$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x < b, \\ 1, & b \leq x \leq c, \\ \dfrac{d-x}{d-c}, & c < x \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

Its α-cut is given by

(2.6) $$\tilde{A}^{\alpha} = [\alpha(b-a)+a, \alpha(c-d)+d], \quad \forall \alpha \in [0,1].$$



Membership function of a trapezoidal fuzzy number

Remark 2.7. A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is symmetric if and only if $b - a = d - c$.

2.2. **Arithmetic Operations and Median Function of Fuzzy Numbers.**

Theorem 2.8. *[8,14] Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers. The α-cuts of the fuzzy arithmetic operation $\tilde{A} \circledast \tilde{B}$ are given by*

(2.7) $$[\tilde{A} \circledast \tilde{B}]^{\alpha} = [\tilde{A}]^{\alpha} \circledast [\tilde{B}]^{\alpha}, \quad \forall \alpha \in [0,1],$$

where $\circledast$ denotes one of the arithmetic operations $\oplus, \ominus, \otimes, \oslash$.

Proof. See the proof of Theorem 2.4 in [14]. □

Proposition 2.9. [14] Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers with α-cuts $\tilde{A}^\alpha = [a^l_\alpha, a^r_\alpha]$ and $\tilde{B}^\alpha = [b^l_\alpha, b^r_\alpha]$, $\alpha \in [0,1]$. Then:

(i) The sum $\tilde{A} \oplus \tilde{B}$ has α-cuts

$$[\tilde{A} \oplus \tilde{B}]^\alpha = [a^l_\alpha + b^l_\alpha,\ a^r_\alpha + b^r_\alpha]. \tag{2.8}$$

(ii) The difference $\tilde{A} \ominus \tilde{B}$ has α-cuts

$$[\tilde{A} \ominus \tilde{B}]^\alpha = [a^l_\alpha - b^r_\alpha,\ a^r_\alpha - b^l_\alpha]. \tag{2.9}$$

(iii) Scalar multiplication $\lambda\tilde{A}$ has α-cuts

$$[\lambda\tilde{A}]^\alpha = \begin{cases} [\lambda a^l_\alpha, \lambda a^r_\alpha], & \lambda \geq 0, \\ [\lambda a^r_\alpha, \lambda a^l_\alpha], & \lambda < 0. \end{cases} \tag{2.10}$$

(iv) Multiplication $\tilde{A} \otimes \tilde{B}$ in $\mathcal{N}(\mathbb{R}^+)$ has α-cuts

$$[\tilde{A} \otimes \tilde{B}]^\alpha = [\min P_\alpha, \max P_\alpha], \tag{2.11}$$

with $P_\alpha = \{a^l_\alpha b^l_\alpha, a^l_\alpha b^r_\alpha, a^r_\alpha b^l_\alpha, a^r_\alpha b^r_\alpha\}$.

(v) Division $\tilde{A} \oslash \tilde{B}$ in $\mathcal{N}(\mathbb{R}^+)$ has α-cuts

$$[\tilde{A} \oslash \tilde{B}]^\alpha = \frac{[\tilde{A}]^\alpha}{[\tilde{B}]^\alpha} = [\min P_\alpha, \max P_\alpha], \tag{2.12}$$

with $P_\alpha = \left\{\frac{a^l_\alpha}{b^l_\alpha}, \frac{a^l_\alpha}{b^r_\alpha}, \frac{a^r_\alpha}{b^l_\alpha}, \frac{a^r_\alpha}{b^r_\alpha}\right\}$ and $\tilde{B} \neq \tilde{0}$.

Definition 2.10. [4,13] Let $\mathcal{R} : \mathcal{N}(\mathbb{R}) \to \mathbb{R}$ be a ranking function.

(i) For triangular fuzzy numbers: $\mathcal{R} = \dfrac{a + 2b + c}{4}$.

(ii) For trapezoidal fuzzy numbers: $\mathcal{R} = \dfrac{a + 2b + 2c + d}{6}$.

Definition 2.11. [15] A fuzzy number $\tilde{s} \in \mathcal{N}(\mathbb{R})$ is called *symmetric in the sense of Mareš* if

$$\mu_{\tilde{s}}(x) = \mu_{\tilde{s}}(-x), \quad \forall x \in \mathbb{R}, \tag{2.13}$$

i.e., $\tilde{s} = -\tilde{s}$. Let φ denote the set of all such symmetric fuzzy numbers.

Remark 2.12. [15] A fuzzy number is symmetric in the sense of Mareš if it is symmetric about zero. Symmetry about a nonzero value does not satisfy Mareš symmetry.

Definition 2.13. [15–18] For $\tilde{A} \in \mathcal{N}(\mathbb{R})$, the *median function* is

$$\begin{aligned} \mathscr{M}_{\tilde{A}} : [0,1] &\to \mathbb{R} \\ \alpha &\mapsto \frac{\tilde{a}^l_\alpha + \tilde{a}^r_\alpha}{2}. \end{aligned} \tag{2.14}$$

Remark 2.14. The median function of a fuzzy number symmetric in the sense of Mareš is zero.

Remark 2.15. For a symmetric triangular fuzzy number $\tilde{A} = (a, b, c)$, $\mathscr{M}_{\tilde{A}}(\alpha) = b$.

Remark 2.16. For a symmetric trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$, $\mathscr{M}_{\tilde{A}}(\alpha) = \frac{b+c}{2}$.

Remark 2.17. Let $\tilde{A}, \tilde{B}$ be fuzzy numbers. Then

$$\frac{\mathscr{M}_{\tilde{A}}(\alpha)}{\mathscr{M}_{\tilde{B}}(\alpha)} = \frac{\tilde{a}^l_\alpha + \tilde{a}^r_\alpha}{\tilde{b}^l_\alpha + \tilde{b}^r_\alpha}, \quad \text{if } \mathscr{M}_{\tilde{B}} \neq 0.$$

Hereafter, unless stated otherwise, we work in $\mathcal{T}(\mathbb{R})$, the set of fuzzy numbers excluding Mareš-symmetric cases.

Definition 2.18. Let $\tilde{A}, \tilde{B} \in \mathcal{T}(\mathbb{R})$. We define a partial order as follows:

(i) $\tilde{A} \preceq \tilde{B}$ if $\mathscr{M}_{\tilde{A}}(\alpha) \leq \mathscr{M}_{\tilde{B}}(\alpha), \quad \forall \alpha \in [0,1]$.
(ii) $\tilde{A} \prec \tilde{B}$ if $\tilde{A} \preceq \tilde{B}$ and there exists $\alpha_0 \in [0,1]$ such that $\mathscr{M}_{\tilde{A}}(\alpha_0) < \mathscr{M}_{\tilde{B}}(\alpha_0)$.
(iii) $\tilde{A} \approx \tilde{B}$ if $\tilde{A} \preceq \tilde{B}$ and $\tilde{B} \preceq \tilde{A}$.

Note that "$\preceq$" defines a partial order on $\mathcal{T}(\mathbb{R})$.

2.3. **Fuzzy Multiobjective Linear Fractional Programming Problem.** A fuzzy multiobjective linear fractional programming (FMOLFP) problem is defined as follows:

$$\text{(2.15)} \qquad \begin{cases} \max \tilde{F}_i(x) = \dfrac{\displaystyle\sum_{j=1}^{n} \tilde{A}_{ij} x_j \oplus \tilde{C}_i}{\displaystyle\sum_{j=1}^{n} \tilde{B}_{ij} x_j \oplus \tilde{D}_i} = \dfrac{\tilde{p}_i(x)}{\tilde{q}_i(x)}, \quad i = 1, 2, \dots, p, \\ \text{subject to } \displaystyle\sum_{j=1}^{n} \tilde{N}_{kj} x_j \ (\preceq, \approx, \succeq) \ \tilde{U}_k, \quad k = 1, \dots, m, \\ x_j \geq 0, \quad j = 1, \dots, n, \end{cases}$$

where $\tilde{A}_{ij}, \tilde{B}_{ij}, \tilde{C}_i, \tilde{D}_i, \tilde{N}_{kj}, \tilde{U}_k$ are fuzzy numbers in $\mathcal{T}(\mathbb{R})$, and $\tilde{q}_i(x) > \tilde{0}$ for all i.

Let

$$\tilde{\Omega} = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} \tilde{N}_{kj} x_j \ (\preceq, \approx, \succeq) \ \tilde{U}_k, \ k = 1, \dots, m, \ x_j \geq 0, \ j = 1, \dots, n \right\}$$

denote the nonempty feasible set of (2.15).

Definition 2.19. [4] Let $\bar{x} \in \tilde{\Omega}$ be a feasible solution of (2.15). $\bar{x}$ is said to be a *Pareto-optimal solution* of (2.15) if there does not exist $x \in \tilde{\Omega}$ such that

$$\tilde{F}_i(x) \succeq \tilde{F}_i(\bar{x}) \quad \text{for all } i = 1, 2, \dots, p$$

and

$$\tilde{F}_k(x) \succ \tilde{F}_k(\bar{x}) \quad \text{for at least one } k \in \{1, 2, \dots, p\}.$$

Definition 2.20. [4] A feasible solution $\bar{x} \in \tilde{\Omega}$ is called a *weakly Pareto-optimal solution* of (2.15) if there does not exist $x \in \tilde{\Omega}$ such that

$$\tilde{F}_i(\bar{x}) \succ \tilde{F}_i(x) \quad \text{for all } i = 1, 2, \dots, p.$$

3. Results and Discussions

To better present our results, we will first propose the theoretical approach and then the algorithm. Finally, didactic examples will be provided.

3.1. **New Solution Approach.** The proposed solution approach is based on a structured process, whose methodological development can be summarized in the following key steps:

Principle

The proposed method solves a fuzzy multiobjective fractional optimization problem through defuzzification, linearization, aggregation, and resolution.

Consider problem (2.15).

(i) **Defuzzification:** This consists of transforming the fuzzy problem into a deterministic problem. We start by computing the α-cuts of each fuzzy number involved in the model:

$$[\tilde{A}_{ij}]^\alpha = [A^l_{ij\alpha}, A^r_{ij\alpha}], \qquad [\tilde{B}_{ij}]^\alpha = [B^l_{i\alpha}, B^r_{i\alpha}]\,, \qquad [\tilde{C}_i]^\alpha = [C^l_{i\alpha}, C^r_{i\alpha}]\,, \qquad [\tilde{D}_i]^\alpha = [D^l_{i\alpha}, D^r_{i\alpha}]\,,$$
$$[\tilde{N}_{kj}]^\alpha = [N^l_{kj\alpha}, N^r_{kj\alpha}]\,, \qquad [\tilde{U}_k]^\alpha = [U^l_{k\alpha}, U^r_{k\alpha}]\,.$$

Next, we calculate the median functions associated with each objective function and constraints:

$$\mathscr{M}_{\tilde{A}_{ij}}(\alpha) = \frac{A^l_{ij\alpha} + A^r_{ij\alpha}}{2}\,, \qquad \mathscr{M}_{\tilde{B}_{ij}}(\alpha) = \frac{B^l_{ij\alpha} + B^r_{ij\alpha}}{2}\,, \qquad \mathscr{M}_{\tilde{C}_i}(\alpha) = \frac{C^l_{i\alpha} + C^r_{i\alpha}}{2},$$
$$\mathscr{M}_{\tilde{D}_i}(\alpha) = \frac{D^l_{i\alpha} + D^r_{i\alpha}}{2}, \qquad \mathscr{M}_{\tilde{N}_{kj}}(\alpha) = \frac{N^l_{kj\alpha} + N^r_{kj\alpha}}{2}\,, \qquad \mathscr{M}_{\tilde{U}_k}(\alpha) = \frac{U^l_{k\alpha} + U^r_{k\alpha}}{2}.$$

We can then rewrite problem (2.15) as follows:

$$\text{(3.1)} \qquad \begin{cases} \max \mathscr{M}_{\tilde{F}_i(x)}(\alpha) = \dfrac{\displaystyle\sum_{j=1}^{n} \mathscr{M}_{\tilde{A}_{ij}}(\alpha) x_j + \mathscr{M}_{\tilde{C}_i}(\alpha)}{\displaystyle\sum_{j=1}^{n} \mathscr{M}_{\tilde{B}_{ij}}(\alpha) x_j + \mathscr{M}_{\tilde{D}_i}(\alpha)} = \dfrac{\mathscr{M}_{\tilde{p}_i(x)}(\alpha)}{\mathscr{M}_{\tilde{q}_i(x)}(\alpha)}, \forall i = 1, 2, \ldots, p, \\ \text{subject to} \displaystyle\sum_{j=1}^{n} \mathscr{M}_{\tilde{N}_{kj}}(\alpha) x_j (\geq, =, \leq) \mathscr{M}_{\tilde{U}_k}(\alpha), \qquad k = 1, \ldots, m, \\ x_j \geq 0, \quad j = 1, \ldots, n. \end{cases}$$

Problem (3.1) is a deterministic multiobjective linear fractional optimization problem.

Let

$$\Omega = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} \mathscr{M}_{\tilde{N}_{kj}}(\alpha) x_j (\geq, =, \leq) \mathscr{M}_{\tilde{U}_k}(\alpha), \quad x_j \geq 0, \quad k = 1, \ldots, m \right\}.$$

Theorem 3.1. *$\bar{x} \in \Omega$ is a Pareto optimal solution of problem (3.1) for $\alpha \in [0,1]$ if $\bar{x}$ is a Pareto optimal solution of problem (2.15).*

Proof. Let $\bar{x}$ be an optimal solution of problem (3.1). Then, for all $x \in \Omega$, we have:

$$\mathscr{M}_{\tilde{F}_i(\bar{x})}(\alpha) \geq \mathscr{M}_{\tilde{F}_i(x)}(\alpha) \quad \text{for all } i, \quad \text{and} \quad \mathscr{M}_{\tilde{F}_k(\bar{x})}(\alpha) > \mathscr{M}_{\tilde{F}_k(x)}(\alpha) \quad \text{for at least one } k \in \{1, \ldots, m\},\ \forall \alpha \in [0,1].$$

This gives:

$$\frac{\displaystyle\sum_{j=1}^{n} \mathscr{M}_{\tilde{A}_{ij}}(\alpha) \bar{x}_j + \mathscr{M}_{\tilde{C}_i}(\alpha)}{\displaystyle\sum_{j=1}^{n} \mathscr{M}_{\tilde{B}_{ij}}(\alpha) \bar{x}_j + \mathscr{M}_{\tilde{D}_i}(\alpha)} \geq \frac{\displaystyle\sum_{j=1}^{n} \mathscr{M}_{\tilde{A}_{ij}}(\alpha) x_j + \mathscr{M}_{\tilde{C}_i}(\alpha)}{\displaystyle\sum_{j=1}^{n} \mathscr{M}_{\tilde{B}_{ij}}(\alpha) x_j + \mathscr{M}_{\tilde{D}_i}(\alpha)}, \quad \forall i,$$

$$\frac{\sum_{j=1}^{n}\mathscr{M}_{\tilde{A}_{kj}}(\alpha)\bar{x}_j+\mathscr{M}_{\tilde{C}_k}(\alpha)}{\sum_{j=1}^{n}\mathscr{M}_{\tilde{B}_{kj}}(\alpha)\bar{x}_j+\mathscr{M}_{\tilde{D}_k}(\alpha)}>\frac{\sum_{j=1}^{n}\mathscr{M}_{\tilde{A}_{kj}}(\alpha)x_j+\mathscr{M}_{\tilde{C}_k}(\alpha)}{\sum_{j=1}^{n}\mathscr{M}_{\tilde{B}_{kj}}(\alpha)x_j+\mathscr{M}_{\tilde{D}_k}(\alpha)},\quad \text{for at least one } k\in\{1,\ldots,m\},\forall\alpha\in[0,1].$$

Therefore, using α-cuts and median functions, we obtain:

$$\frac{\sum_{j=1}^{n}[\tilde{A}_{ij}]^{\alpha}\bar{x}_j+[\tilde{C}_i]^{\alpha}}{\sum_{j=1}^{n}[\tilde{B}_{ij}]^{\alpha}\bar{x}_j+[\tilde{D}_i]^{\alpha}}\geq\frac{\sum_{j=1}^{n}[\tilde{A}_{ij}]^{\alpha}x_j+[\tilde{C}_i]^{\alpha}}{\sum_{j=1}^{n}[\tilde{B}_{ij}]^{\alpha}x_j+[\tilde{D}_i]^{\alpha}},\quad \forall i,$$

$$\frac{\sum_{j=1}^{n}[\tilde{A}_{kj}]^{\alpha}\bar{x}_j+[\tilde{C}_k]^{\alpha}}{\sum_{j=1}^{n}[\tilde{B}_{kj}]^{\alpha}\bar{x}_j+[\tilde{D}_k]^{\alpha}}>\frac{\sum_{j=1}^{n}[\tilde{A}_{kj}]^{\alpha}x_j+[\tilde{C}_k]^{\alpha}}{\sum_{j=1}^{n}[\tilde{B}_{kj}]^{\alpha}x_j+[\tilde{D}_k]^{\alpha}},\quad \text{for at least one } k\in\{1,\ldots,m\},\forall\alpha\in[0,1].$$

Which implies:

$$\frac{\sum_{j=1}^{n}\tilde{A}_{ij}\bar{x}_j\oplus\tilde{C}_i}{\sum_{j=1}^{n}\tilde{B}_{ij}\bar{x}_j\oplus\tilde{D}_i}\succeq\frac{\sum_{j=1}^{n}\tilde{A}_{ij}x_j\oplus\tilde{C}_i}{\sum_{j=1}^{n}\tilde{B}_{ij}x_j\oplus\tilde{D}_i},\quad \forall i,$$

$$\frac{\sum_{j=1}^{n}\tilde{A}_{kj}\bar{x}_j\oplus\tilde{C}_k}{\sum_{j=1}^{n}\tilde{B}_{kj}\bar{x}_j\oplus\tilde{D}_k}\succ\frac{\sum_{j=1}^{n}\tilde{A}_{kj}x_j\oplus\tilde{C}_k}{\sum_{j=1}^{n}\tilde{B}_{k}x_j\oplus\tilde{D}_k},\quad \text{for at least one } k\in\{1,\ldots,m\}.$$

Thus, we obtain $\tilde{F}_i(\bar{x})\succeq\tilde{F}_i(x)$ for all $i=1,2,\ldots,p$ and $\tilde{F}_k(\bar{x})\succ\tilde{F}_k(x)$ for at least one $k\in\{1,\ldots,m\}$. Therefore, $\bar{x}$ is a Pareto optimal solution of problem (2.15). □

(ii) **Linearization:** This consists of transforming the fractional problem (3.1) into an equivalent linear optimization problem, using Dinkelbach's theorem [6]. Thus, problem (3.1) reduces to a linear problem formulated as follows:

$$\begin{cases}\max\mathscr{M}_{\tilde{F}_i(x)}(\alpha)=\mathscr{M}_{\tilde{p}_i(x)}(\alpha)-\gamma_i^{*}\mathscr{M}_{\tilde{q}_i(x)}(\alpha), & i=1,\ldots,p,\\ \text{subject to}\sum_{j=1}^{n}\mathscr{M}_{\tilde{N}_{kj}}(\alpha)x_j(\geq,=,\leq)\mathscr{M}_{\tilde{U}_k}(\alpha), & k=1,\ldots,m,\\ \alpha\in[0,1],\ x_j\geq0,\quad j=1,\ldots,n.\end{cases}\tag{3.2}$$

Define the sequence $(\gamma_i^{(n+1)}, i=1,2,\cdots,p)_{n\in\mathbb{N}}$ as

$$\gamma_i^{(n+1)}=\frac{\mathscr{M}_{\tilde{p}_i(x^{(n)})}(\alpha)}{\mathscr{M}_{\tilde{q}_i(x^{(n)})}(\alpha)}.$$

To determine γ_i^{*}, we apply Dinkelbach's algorithm [22], which proceeds iteratively as follows:
0) Choose γ^0 and fix ϵ.

1) For $n\in\mathbb{N}$, solve:

$$x^{(n+1)} = \arg\max_{x\in\Omega} \left(\mathscr{M}_{\tilde{p}_i(x)}(\alpha) - \gamma_i^{(n+1)}\mathscr{M}_{\tilde{q}_i(x)}(\alpha)\right),\ i = 1, 2, \cdots, p,$$

subject to

$$\sum_{j=1}^{n} \mathscr{M}_{\tilde{N}_{kj}}(\alpha) x_j (\geq, =, \leq) \mathscr{M}_{\tilde{U}_k}(\alpha), \qquad k = 1, \dots, m.$$

2) Update the parameter:

$$\gamma_i^{(n+1)} = \frac{\mathscr{M}_{\tilde{p}_i(x^{(n)})}(\alpha)}{\mathscr{M}_{\tilde{q}_i(x^{(n)})}(\alpha)}.$$

3) Repeat until:

$$|\gamma_i^{n+1} - \gamma_i^{n}| < \epsilon, \quad \text{stop, set } \gamma_i^* = \gamma_i^{n+1},\ x^* = x^{(n)}.$$

4) Otherwise, set $n = n + 1$ and return to step 1).

Theorem 3.2. *Let $\bar{x} \in \Omega$. If $\bar{x}$ is a Pareto optimal solution of (3.2) for $\alpha \in [0, 1]$, then $\bar{x}$ is a Pareto optimal solution of problem (3.1) for $\alpha \in [0, 1]$.*

Proof. Let $\bar{x} \in \Omega$ be a Pareto optimal solution of problem (3.2). Suppose $\bar{x}$ is not a solution of problem (3.1). Then there exists $\hat{x} \in \Omega$ such that

$$\mathscr{M}_{\tilde{F}_i(\hat{x})}(\alpha) \geq \mathscr{M}_{\tilde{F}_i(\bar{x})}(\alpha), \quad \forall i,$$

and

$$\mathscr{M}_{\tilde{F}_k(\hat{x})}(\alpha) > \mathscr{M}_{\tilde{F}_k(\bar{x})}(\alpha) \quad \text{for at least one } k \in \{1, \dots, m\},\ \alpha \in [0, 1].$$

Since $\gamma_i^* = \dfrac{\mathscr{M}_{\tilde{p}_i(\bar{x})}(\alpha)}{\mathscr{M}_{\tilde{q}_i(\bar{x})}(\alpha)}$ for all i, we have:

$$\mathscr{M}_{\tilde{p}_i(\bar{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\bar{x})}(\alpha) = 0.$$

Also,

$$\frac{\mathscr{M}_{\tilde{p}_i(\hat{x})}(\alpha)}{\mathscr{M}_{\tilde{q}_i(\hat{x})}(\alpha)} \geq \frac{\mathscr{M}_{\tilde{p}_i(\bar{x})}(\alpha)}{\mathscr{M}_{\tilde{q}_i(\bar{x})}(\alpha)} \quad \forall i,$$

and

$$\frac{\mathscr{M}_{\tilde{p}_k(\hat{x})}(\alpha)}{\mathscr{M}_{\tilde{q}_k(\hat{x})}(\alpha)} > \frac{\mathscr{M}_{\tilde{p}_k(\bar{x})}(\alpha)}{\mathscr{M}_{\tilde{q}_k(\bar{x})}(\alpha)} \quad \text{for at least one } k \in \{1, \dots, m\}.$$

Thus,

$$\mathscr{M}_{\tilde{p}_i(\hat{x})}(\alpha) \geq \mathscr{M}_{\tilde{q}_i(\hat{x})}(\alpha) \frac{\mathscr{M}_{\tilde{p}_i(\bar{x})}(\alpha)}{\mathscr{M}_{\tilde{q}_i(\bar{x})}(\alpha)}, \quad \forall i,$$

since $\mathscr{M}_{\tilde{q}_i(\hat{x})}(\alpha) > 0$, and

$$\mathscr{M}_{\tilde{p}_k(\hat{x})}(\alpha) > \mathscr{M}_{\tilde{q}_k(\hat{x})}(\alpha) \frac{\mathscr{M}_{\tilde{p}_k(\bar{x})}(\alpha)}{\mathscr{M}_{\tilde{q}_k(\bar{x})}(\alpha)} \quad \text{for at least one } k \in \{1, \dots, m\}.$$

This implies

$$\mathscr{M}_{\tilde{p}_i(\hat{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\hat{x})}(\alpha) \geq 0 = \mathscr{M}_{\tilde{p}_i(\bar{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\bar{x})}(\alpha), \quad \forall i,$$

and

$$\mathscr{M}_{\tilde{p}_k(\hat{x})}(\alpha) - \gamma_k^* \mathscr{M}_{\tilde{q}_k(\hat{x})}(\alpha) > 0 = \mathscr{M}_{\tilde{p}_k(\bar{x})}(\alpha) - \gamma_k^* \mathscr{M}_{\tilde{q}_k(\bar{x})}(\alpha), \quad \text{for at least one } k \in \{1, \dots, m\}.$$

$\Longrightarrow \mathscr{M}_{\tilde{p}_i(\hat{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\hat{x})}(\alpha) \geq \mathscr{M}_{\tilde{p}_i(\bar{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\bar{x})}(\alpha), \quad \forall i,$ and $\mathscr{M}_{\tilde{p}_k(\hat{x})}(\alpha) - \gamma_k^* \mathscr{M}_{\tilde{q}_k(\hat{x})}(\alpha) > \mathscr{M}_{\tilde{p}_k(\bar{x})}(\alpha) - \gamma_k^* \mathscr{M}_{\tilde{q}_k(\bar{x})}(\alpha)$.

Then $\bar{x}$ would not be a Pareto optimal solution of problem (3.2), which is a contradiction. Hence, $\bar{x}$ is also a Pareto optimal solution of problem (3.1). □

(iii) **Aggregation:** This consists of transforming the multiple objectives of the problem into a single objective using an aggregation function. For this purpose, we use the weighted sum. This choice is justified by the fact that the objectives obtained after the transformation are linear. Thus, using the weighted sum, problem (3.2) becomes:

$$\begin{cases} \max \mathscr{M}^{w}_{\tilde{F}_i(x)}(\alpha) = \sum_{i=1}^{p} w_i \left[\mathscr{M}_{\tilde{p}_i(x)}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(x)}(\alpha)\right], \\ \text{subject to} \sum_{j=1}^{n} \mathscr{M}_{\tilde{N}_{kj}}(\alpha) x_j (\geq, =, \leq) \mathscr{M}_{\tilde{U}_k}(\alpha), \qquad k = 1, \dots, m, \\ \alpha \in [0,1],\ x_j \geq 0, \quad j = 1, \dots, n, \end{cases} \tag{3.3}$$

where $w_i > 0$ denotes the weight associated with the i-th objective function for $i = 1, 2, \dots, p$ and $\sum_{i=1}^{p} w_i = 1$.

Theorem 3.3. *Let $\bar{x} \in \Omega$. If $\bar{x}$ is a Pareto optimal solution of (3.3) for all $\alpha \in [0,1]$, then $\bar{x}$ is a Pareto optimal solution of problem (3.2) for all $\alpha \in [0,1]$.*

Proof. Let $\bar{x} \in \Omega$ be a Pareto optimal solution of problem (3.3). Suppose $\bar{x}$ is not a solution of problem (3.2). According to the proof of Theorem 3.2, there exists $\hat{x} \in \Omega$ such that:

$$\mathscr{M}_{\tilde{p}_i(\hat{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\hat{x})}(\alpha) \geq \mathscr{M}_{\tilde{p}_i(\bar{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\bar{x})}(\alpha) \quad \forall i,$$

and

$$\mathscr{M}_{\tilde{p}_k(\hat{x})}(\alpha) - \gamma_k^* \mathscr{M}_{\tilde{q}_k(\hat{x})}(\alpha) > \mathscr{M}_{\tilde{p}_k(\bar{x})}(\alpha) - \gamma_k^* \mathscr{M}_{\tilde{q}_k(\bar{x})}(\alpha) \quad \text{for at least one } k \in \{1, \dots, m\}.$$

Let $w_i \geq 0$. Then:

$$w_i\left[\mathscr{M}_{\tilde{p}_i(\hat{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\hat{x})}(\alpha)\right] \geq w_i\left[\mathscr{M}_{\tilde{p}_i(\bar{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\bar{x})}(\alpha)\right] \quad \forall i,$$

and

$$w_k\left[\mathscr{M}_{\tilde{p}_k(\hat{x})}(\alpha) - \gamma_k^* \mathscr{M}_{\tilde{q}_k(\hat{x})}(\alpha)\right] > w_k\left[\mathscr{M}_{\tilde{p}_k(\bar{x})}(\alpha) - \gamma_k^* \mathscr{M}_{\tilde{q}_k(\bar{x})}(\alpha)\right] \quad \text{for at least one } k.$$

Summing over i, we get:

$$\sum_{i=1}^{p} w_i\left[\mathscr{M}_{\tilde{p}_i(\hat{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\hat{x})}(\alpha)\right] > \sum_{i=1}^{p} w_i\left[\mathscr{M}_{\tilde{p}_i(\bar{x})}(\alpha) - \gamma_i^* \mathscr{M}_{\tilde{q}_i(\bar{x})}(\alpha)\right],$$

which implies that $\bar{x}$ is not an optimal solution of problem (3.3), leading to a contradiction. Hence, $\bar{x}$ is also a Pareto optimal solution of problem (3.2). □

(iv) **Solution procedure:** This step consists of solving problem (3.3) obtained from the previous transformation. In our case, the solution is obtained using Dantzig's Simplex algorithm, which provides the corresponding deterministic solution.

(v) **Solution interpretation:** This step aims to determine the solution of the original problem (2.15) by transforming the deterministic solution obtained for problem (3.3) into a fuzzy solution using fuzzy arithmetic operations.

3.2. **Algorithm of the Proposed Method.**

Consider the problem

$$\max_{x\in\tilde{\nabla}}\left\{\frac{\tilde{p}_i(x)}{\tilde{q}_i(x)}\right\},\, i=1,2,\cdots,p;$$

Algorithm of the New Method

Step 1. Defuzzify the program from Step 1 using α-cuts and move to the median functions to obtain

$$\max_{x\in\nabla}\left\{\frac{\mathscr{M}_{\tilde{p}_i(x)}(\alpha)}{\mathscr{M}_{\tilde{q}_i(x)}(\alpha)}\right\},\, i=1,2,\cdots,p \qquad \text{and} \qquad \alpha\in[0,1];$$

Step 2. Linearize the problem from Step 2 in the form:

$$\max_{x\in\nabla}\left\{\mathscr{M}_{\tilde{p}_i(x)}(\alpha)-\gamma_i^*\mathscr{M}_{\tilde{q}_i(x)}(\alpha)\right\},\, i=1,2,\cdots,p \qquad \text{and} \qquad \alpha\in[0,1]$$

using Dinkelbach's theorem [6];

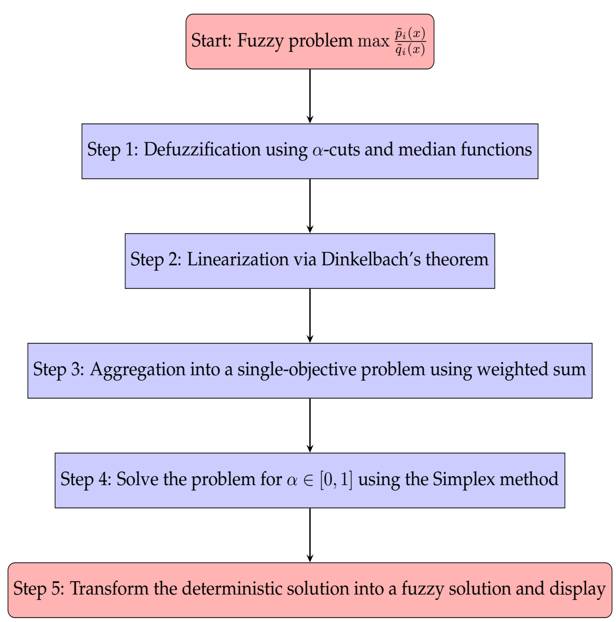
Step 3. Transform the problem from **Step 2** into a single-objective problem using the weighted sum method:

$$\max_{x\in\nabla}\mathscr{M}^{w}_{\tilde{F}_i(x)}(\alpha)=\sum_{i=1}^{p}w_i\left[\mathscr{M}_{\tilde{p}_i(x)}(\alpha)-\gamma_i^*\mathscr{M}_{\tilde{q}_i(x)}(\alpha)\right]$$

Step 4. Fix the weights w_i and determine the parameters γ_i^*, then solve the problem for different values of $\alpha\in[0,1]$ with a step of 0.1 using the Simplex method:

$$\max_{x\in\nabla}\mathscr{M}^{w}_{\tilde{F}_i(x)}(\alpha)=\sum_{i=1}^{p}w_i\left[\mathscr{M}_{\tilde{p}_i(x)}(\alpha)-\gamma_i^*\mathscr{M}_{\tilde{q}_i(x)}(\alpha)\right]$$

Step 5. Set up the solution and display it.



3.3. **Algorithm Complexity.**

① **Step 1: Defuzzification**
The defuzzification process depends on the number of decision variables involved in the problem. For a problem with n decision variables, each objective function and each constraint requires, in the worst case, $3n+3$ operations. Hence, for a problem with p objective functions and m constraints, the overall complexity can be expressed as:

$$O\big(3p(n+1)+3m(n+1)\big) = \max\{O(3pn),\, O(3mn)\} = \max\{O(pn),\, O(mn)\}, \quad n \in \mathbb{N}.$$

② **Step 2: Linearization**
The computational effort in this step depends on the number of objective functions and on the linearization parameter γ. The complexity of Dinkelbach's algorithm can be estimated as $O(n^3)$. Therefore, for a problem with $p \in \mathbb{N}$ objective functions, the worst-case complexity is $O(pn^3)$.

③ **Step 3: Aggregation**
In this step, the aggregation of the p objective functions is performed. Thus, the worst-case complexity is $O(p)$.

④ **Step 4: Solution via the Simplex Method**
The Simplex algorithm explores feasible decision points in a pairwise combinatorial manner. Consequently, for each weighting coefficient w_i, the computational complexity is of order $O(2^n)$.

Finally, the overall computational complexity of the algorithm can be written as:

$$C = p\max\big\{O(pn),\, O(mn),\, O(pn^3),\, O(p),\, O(2^n)\big\} = O(p2^n).$$

3.4. **Example.**

Example 1.: Consider the linear fuzzy multi-objective fractional problem taken from [4].

$$\begin{cases} \max \tilde{Z}_1(x) = \dfrac{(5.6,5.7,6.2)x_1 \oplus (4.7,4.9,5.5)x_2}{(1.7,1.9,2.5)x_1 \oplus (6.6,6.8,7.2)} \\ \max \tilde{Z}_2(x) = \dfrac{(1.6,1.7,2.2)x_1 \oplus (2.7,2.8,3.1)x_2}{(0.7,0.9,1.5)x_1 \oplus (0.8,0.9,1.4)x_2 \oplus (6.7,6.9,7.5)} \\ \text{s.t.} \\ (0.8,0.9,1.4)x_1 \oplus (1.8,1.9,2.4)x_2 \preceq (2.7,2.8,3.1), \\ (2.7,2.9,3.5)x_1 \oplus (1.7,1.8,2.1)x_2 \preceq (5.7,5.8,6.1), \\ x_1, x_2 \geq 0 \end{cases}$$

Step 1.: Defuzzification

• Objective functions:

$(5.6,5.7,6.2)^\alpha = [0.1\alpha + 5.6, -0.5\alpha + 6.2]$; $(4.7,4.9,5.5)^\alpha = [0.2\alpha + 4.7, -0.6\alpha + 5.5]$;

$(1.7, 1.9, 2.5)^\alpha = [0.2\alpha + 1.7, -0.6\alpha + 2.5]$; $(6.6, 6.8, 7.2)^\alpha = [0.2\alpha + 6.6, -0.4\alpha + 7.2]$;

$(1.6, 1.7, 2.2)^\alpha = [0.1\alpha + 1.6, -0.5\alpha + 2.2]$; $(2.7, 2.8, 3.1)^\alpha = [0.1\alpha + 2.7, -0.3\alpha + 3.1]$;

$(0.7, 0.9, 1.5)^\alpha = [0.2\alpha + 0.7, -0.6\alpha + 1.5]$; $(0.8, 0.9, 1.4)^\alpha = [0.1\alpha + 0.7, -0.3\alpha + 1.4]$;

$(6.7, 6.9, 7.5)^\alpha = [0.2\alpha + 6.7, -0.6\alpha + 7.5]$.

$\Longrightarrow \mathscr{M}_{\tilde{Z}_1(x)}(\alpha) = \frac{(-0.4\alpha+11.8)x_1+(-0.4\alpha+10.2)x_2}{(-0.4\alpha+4.2)x_1+(-0.2\alpha+13.8)}$ and

$\mathscr{M}_{\tilde{Z}_2(x)}(\alpha) = \frac{(-0.4\alpha+3.8)x_1+(-0.2\alpha+5.8)x_2}{(-0.4\alpha+2.2)x_1+(-0.2\alpha+1.8)x_2+(-0.4\alpha+14.2)}$.

• Constraints:

$(0.8, 0.9, 1.4)^\alpha = [0.1\alpha + 0.8, -0.5\alpha + 1.4]$; $(1.8, 1.9, 2.4)^\alpha = [0.1\alpha + 1.8, -0.5\alpha + 2.4]$;

$(2.7, 2.9, 3.1)^\alpha = [0.1\alpha + 2.7, -0.3\alpha + 3.1]$; $(2.7, 2.9, 3.5)^\alpha = [0.2\alpha + 2.7, -0.6\alpha + 3.5]$;

$(1.7, 1.8, 2.1)^\alpha = [0.1\alpha + 1.7, -0.3\alpha + 2.1]$; $(5.7, 5.8, 6.1)^\alpha = [0.1\alpha + 5.7, -0.3\alpha + 6.1]$.

Hence, the problem constraints can be written as:

$$\begin{cases} (-0.4\alpha + 2.2)x_1 + (-0.4\alpha + 4.2)x_2 \leq -0.2\alpha + 5.8 \\ (-0.4\alpha + 6.2)x_1 + (-0.2\alpha + 3.8)x_2 \leq -0.2\alpha + 11.8 \end{cases}$$

We obtain the following fuzzy multi-objective fractional problem:

$$(3.4)\quad \begin{cases} \max \mathscr{M}_{\tilde{Z}_1(x)}(\alpha) = \frac{(-0.4\alpha+11.8)x_1+(-0.4\alpha+10.2)x_2}{(-0.4\alpha+4.2)x_1+(-0.2\alpha+13.8)} \\ \max \mathscr{M}_{\tilde{Z}_2(x)}(\alpha) = \frac{(-0.4\alpha+3.8)x_1+(-0.2\alpha+5.8)x_2}{(-0.4\alpha+2.2)x_1+(-0.2\alpha+1.8)x_2+(-0.4\alpha+14.2)} \\ \text{s.t.} \\ (-0.4\alpha + 2.2)x_1 + (-0.4\alpha + 4.2)x_2 \leq -0.2\alpha + 5.8 \\ (-0.4\alpha + 6.2)x_1 + (-0.2\alpha + 3.8)x_2 \leq -0.2\alpha + 11.8 \\ \alpha \in [0, 1],\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Step 2.: Linearization

Problem (3.4) becomes:

$$(3.5)\quad \begin{cases} \max \mathscr{M}_{\tilde{Z}_1(x)}(\alpha) = \big((-0.4\alpha + 11.8)x_1 + (-0.4\alpha + 10.2)x_2\big) - \lambda_1\big((-0.4\alpha + 4.2)x_1 + (-0.2\alpha + 13.8)\big) \\ \max \mathscr{M}_{\tilde{Z}_2(x)}(\alpha) = \big((-0.4\alpha + 3.8)x_1 + (-0.2\alpha + 5.8)x_2\big) - \lambda_2\big((-0.4\alpha + 2.2)x_1 + (-0.2\alpha + 1.8)x_2 + (-0.4\alpha + 14.2)\big) \\ \text{s.t.} \\ (-0.4\alpha + 2.2)x_1 + (-0.4\alpha + 4.2)x_2 \leq -0.2\alpha + 5.8 \\ (-0.4\alpha + 6.2)x_1 + (-0.2\alpha + 3.8)x_2 \leq -0.2\alpha + 11.8 \\ \alpha \in [0, 1],\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Step 3.: Aggregation

From problem (3.5), we obtain the single-objective problem:

$$(3.6)\quad \begin{cases} \max \mathscr{M}_{\tilde{Z}(x)}(\alpha) = w_1\big[((-0.4\alpha+11.8)x_1+(-0.4\alpha+10.2)x_2)-\lambda_1((-0.4\alpha+4.2)x_1+(-0.2\alpha+13.8))\big] \\ \quad +w_2\big[((-0.4\alpha+3.8)x_1+(-0.2\alpha+5.8)x_2)-\lambda_2((-0.4\alpha+2.2)x_1+(-0.2\alpha+1.8)x_2+(-0.4\alpha+14.2))\big] \\ \text{s.t.} \\ (-0.4\alpha+2.2)x_1+(-0.4\alpha+4.2)x_2 \le -0.2\alpha+5.8 \\ (-0.4\alpha+6.2)x_1+(-0.2\alpha+3.8)x_2 \le -0.2\alpha+11.8 \\ \alpha\in[0,1],\ x_1\ge 0, x_2\ge 0 \end{cases}$$

Step 4.: Resolution

Problem (3.6) reduces to a single-objective linear program, which we solve using the Simplex method. Solving each objective separately with Dinkelbach's algorithm [22], we obtain $\lambda_1 = 1.27$ and $\lambda_2 = 0.52$. Choosing $w_1 = w_2 = 0.5$, we get:

$$(3.7)\quad \begin{cases} \max \mathscr{M}_{\tilde{Z}(x)}(\alpha) = 0.5\big[((-0.4\alpha+11.8)x_1+(-0.4\alpha+10.2)x_2)-1.27((-0.4\alpha+4.2)x_1+(-0.2\alpha+13.8))\big] \\ \quad +0.5\big[((-0.4\alpha+3.8)x_1+(-0.2\alpha+5.8)x_2)-0.52((-0.4\alpha+2.2)x_1+(-0.2\alpha+1.8)x_2+(-0.4\alpha+14.2))\big] \\ \text{s.t.} \\ (-0.4\alpha+2.2)x_1+(-0.4\alpha+4.2)x_2 \le -0.2\alpha+5.8 \\ (-0.4\alpha+6.2)x_1+(-0.2\alpha+3.8)x_2 \le -0.2\alpha+11.8 \\ \alpha\in[0,1],\ x_1\ge 0, x_2\ge 0 \end{cases}$$

The solutions obtained are listed in the table below.

TABLE 1. Optimal solutions for $\alpha \in [0,1]$ with step 0.1 for $w_1 = w_2 = 0.5$

α	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
x_1	1.5372	1.5399	1.5423	1.5446	1.5468	1.5487	1.5506	1.5523	1.5538	1.5552	1.5565
x_2	0.7455	0.7256	0.7062	0.6873	0.6687	0.6506	0.6328	0.6154	0.5984	0.5818	0.5656
$\max \mathscr{M}_{\tilde{Z}_1}(\alpha)$	1.2772	1.2673	1.2577	1.2484	1.2391	1.2299	1.2212	1.2122	1.2036	1.1949	1.1867
$\max \mathscr{M}_{\tilde{Z}_2}(\alpha)$	0.5294	0.5252	0.5213	0.5170	0.5131	0.5099	0.5065	0.5033	0.4995	0.4965	0.4933

To validate the efficiency of our approach, we compare the results of the proposed method with those reported in the literature. For each α, an efficient solution pair is obtained. For $\alpha = 1$, we have: $x_1 = 1.5372$, $x_2 = 0.7455$, with $\max \tilde{Z}_1 = (1.0968, 1.2771, 1.4794)$ and $\max \tilde{Z}_2 = (0.4122, 0.5249, 0.6799)$.

Applying Definition 2.10, we obtain:

TABLE 2. Results obtained by the proposed method and other methods

Methods	$\tilde{Z}_{1\max}$	$\tilde{Z}_{2\max}$	$\mathcal{R}(\tilde{Z}_1)$	$\mathcal{R}(\tilde{Z}_2)$
Proposed Method	**(1.0968,1.2771,1.4794)**	**(0.4122,0.5249,0.6799)**	**1.2764**	**0.5354**
Sama et al. [6]	(1.0914,1.2711,1.4723)	(0.4097,0.5213,0.6763)	1.2764	0.5321
Durga et al. [23]	(1.0864,1.2515,1.4488)	(0.4198,0.5278,0.6646)	1.2595	0.535

Example 2.: Consider the example taken from [3].

$$\begin{cases} \max \widetilde{Z}_1(x) = \frac{(1,2,4,5)x_1 \oplus (0,1,3,4)x_2 \oplus (4,6,8,10)x_3 \oplus (1,3,7,9)}{(0,2,7,9)x_1 \oplus (2,4,6,8)x_2 \oplus (2,3,5,6)x_3 \oplus (4,5,7,8)} \\ \max \widetilde{Z}_2(x) = \frac{(4,6,8,10)x_1 \oplus (1.5,2.5,6,7)x_2 \oplus (5,7,9,11)x_3 \oplus (3,5,8,10)}{(4,5,9,10)x_1 \oplus (2.5,3.5,5.5,6.5)x_2 \oplus (1,3,5,7)x_3 \oplus (3,4,9,10)} \\ \max \widetilde{Z}_3(x) = \frac{(8,9,11,12)x_1 \oplus (4,7,9,12)x_2 \oplus (5,7,9.5,12)x_3 \oplus (1,3,10,12)}{(8,9,14,15)x_1 \oplus (8,10,12,14)x_2 \oplus (1,5,9,13)x_3 \oplus (1,4,5,8)} \\ \text{s.t.} \\ (0.5, 2.5, 8, 10)x_1 \oplus (2.5, 3.5, 7, 8)x_2 \oplus (4, 6, 8, 10)x_3 \preceq (7, 8.5, 9, 11.5) \\ (1.5, 3, 4, 6)x_1 \oplus (0, 1, 7, 9)x_2 \oplus (1, 3, 7, 10)x_3 \preceq (4, 7, 11, 14) \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{cases}$$

Solving each objective function separately using Dinkelbach's algorithm, we obtain $\lambda_1 = 1.22$, $\lambda_2 = 1.43$, $\lambda_3 = 1.26$, and we chose weights $w_1 = 0.4$, $w_2 = w_3 = 0.3$. After solving, we obtain the solutions listed in the table below.

TABLE 3. Optimal solutions and objective values for $\alpha \in [0, 1]$ with step 0.1

α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
x_1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
x_2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
x_3	1.321	1.314	1.307	1.300	1.293	1.286	1.279	1.271	1.264	1.257	1.250
$\mathscr{M}_{\tilde{Z}_1}(\alpha)$	1.2625	1.2614	1.2602	1.2589	1.2577	1.2556	1.2552	1.2538	1.2525	1.2513	1.2500
$\mathscr{M}_{\tilde{Z}_2}(\alpha)$	1.4484	1.4472	1.4457	1.4444	1.4431	1.4418	1.4405	1.4389	1.4375	1.4360	1.4348
$\mathscr{M}_{\tilde{Z}_3}(\alpha)$	1.2896	1.2876	1.2854	1.2833	1.2812	1.2799	1.2769	1.2749	1.2729	1.2709	1.2689

To validate the efficiency of our approach, we compare the results obtained using the proposed method with those reported in the literature. For each value of α, an efficient solution set is obtained. For $\alpha = 1$, we have: $x_1 = x_2 = 0$ and $x_3 = 1.25$. The comparison table is as follows:

TABLE 4. Results obtained by the proposed method and other methods

Methods	$\tilde{Z}_{1\max}$	$\tilde{Z}_{2\max}$	$\tilde{Z}_{3\max}$	$\mathcal{R}(\tilde{Z}_1)$	$\mathcal{R}(\tilde{Z}_2)$	$\mathcal{R}(\tilde{Z}_3)$
New Method	**(0.370,0.7924,1.9428,3.3076)**	**(0.4933,0.9016,2.4838,5.5882)**	(0.2989,0.7230,2.1341,12.0000)	**1.5275**	**2.1420**	3.0021
Sujit et al. [3]	(0.3411,0.7274,1.8394,3.0938)	(0.4585,0.8340,2.3992,5.0803)	**(0.2781,0.7093,2.1847,12.0000)**	1.4280	2.0008	**3.0110**

Example 3: (Practical Problem) [3]

A company produces three types of products A, B, and C with approximate production rates of 5, 8, and 9 units per hour, requiring about 4, 7, and 8 minutes per unit, respectively. Additionally, a production of about 9 units with a processing time of around 6 minutes is also required during the manufacturing process.

The production cost per unit of A, B, and C is approximately \$4, \$7, and \$8, respectively, while the profit per unit is approximately \$6, \$8, and \$6.5. For business development, a profit of about \$3 and a production cost of around \$6 is required.

The company's objective is to simultaneously optimize the production/time and profit/cost ratios to promote growth. The required raw material per unit is approximately 5.5, 3.5, and 5.75 tons for A, B, and C, respectively, with a total raw material availability of about 6.75 tons.

Since parameters such as cost, profit, raw materials, and production rate are uncertain, all are considered as trapezoidal fuzzy numbers to handle uncertainty. Thus, this problem can be formulated as a multi-objective fuzzy fractional programming problem.

$$\begin{cases} \max\ \tilde{Z}_1(x) = \dfrac{\tilde{5}x_1 + \tilde{8}x_2 + \tilde{9}x_3 + \tilde{9}}{\tilde{4}x_1 + \tilde{7}x_2 + \tilde{8}x_3 + \tilde{6}} \\ \max\ \tilde{Z}_2(x) = \dfrac{\tilde{6}x_1 + \tilde{8}x_2 + \tilde{6.5}x_3 + \tilde{3}}{\tilde{4}x_1 + \tilde{7}x_2 + \tilde{8}x_3 + \tilde{6}} \\ \text{s.t.}\quad \tilde{5.5}x_1 + \tilde{3.5}x_2 + \tilde{5.75}x_3 \leq \tilde{6.75}, \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{cases} \tag{3.8}$$

where the trapezoidal fuzzy numbers are defined as follows:

$$\tilde{3} = (1,3,5,7), \quad \tilde{3.5} = (3,4,6,7), \quad \tilde{4} = (1,2,5,6), \quad \tilde{5} = (2,4,5,7),$$
$$\tilde{5.5} = (5,7,9,11), \quad \tilde{5.75} = (5,6,7,8), \quad \tilde{6} = (4,6,8,10),$$
$$\tilde{6.5} = (5,6,7,8), \quad \tilde{6.75} = (3,4,6,7), \quad \tilde{7} = (3,6,9,12), \quad \tilde{8} = (4,5,8,9), \quad \tilde{9} = (3,5,9,11).$$

The fuzzy optimization model considered is:

$$\begin{cases} \max\ \tilde{Z}_1(x) = \dfrac{(2,4,5,7)x_1 + (4,5,8,9)x_2 + (3,5,9,11)x_3 + (3,5,9,11)}{(1,2,5,6)x_1 + (3,6,9,12)x_2 + (4,5,8,9)x_3 + (4,6,8,10)} \\ \max\ \tilde{Z}_2(x) = \dfrac{(4,6,8,10)x_1 + (5,7,8,10)x_2 + (5,6,7,8)x_3 + (1,3,5,7)}{(1,2,5,6)x_1 + (3,6,9,12)x_2 + (4,5,8,9)x_3 + (4,6,8,10)} \\ \text{s.t.}\quad (5,7,9,11)x_1 + (3,4,6,7)x_2 + (5,6,7,8)x_3 \leq (3,4,6,7) \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{cases} \tag{3.9}$$

Applying the proposed approach, we obtain:

$$x_1 = 0.625, \quad x_2 = 0, \quad x_3 = 0,$$

with the fuzzy optimal solution:

$$\tilde{Z}_1(x)_{\max} = (0.3090, 0.6741, 1.6724, 3.3243), \quad \tilde{Z}_2(x)_{\max} = (0.2545, 0.6067, 1.3793, 2.8648).$$

To validate the efficiency of our approach, we compare the results obtained by the proposed method with those reported in the literature:

TABLE 5. Results obtained by the proposed method and other methods

Methods	$\tilde{Z}_{1\max}$	$\tilde{Z}_{2\max}$	$\mathcal{R}(\tilde{Z}_1)$	$\mathcal{R}(\tilde{Z}_2)$
Proposed Method	**(0.3090, 0.6741,1.6724,3.3243)**	**(0.2545,0.6067,1.3793,2.8648)**	**1.3877**	**1.1818**
Sujit et al. [3]	(0.3087,0.6756,1.6649,3.30)	(0.2486,0.5986,1.3556,2.8145)	1.3816	1.1619

3.5. **Discussions.**

In Example 1, the considered numerical problem was initially solved by Sama et al. [6] and Durga et al. [23]. Since this numerical problem involves triangular fuzzy numbers, the proposed method allows obtaining optimal values directly in the form of triangular fuzzy numbers. The optimal values reported by Sama et al. [6] are:

$$\tilde{Z}_1 = (1.0914, 1.2711, 1.4723), \quad \tilde{Z}_2 = (0.4097, 0.5213, 0.6763),$$

while Durga et al. [23] report:

$$\tilde{Z}_1 = (1.0864, 1.2515, 1.4488), \quad \tilde{Z}_2 = (0.4198, 0.5278, 0.6646).$$

By applying our method, we obtain:

$$\tilde{Z}_1 = (1.0968, 1.2771, 1.4794), \quad \tilde{Z}_2 = (0.4122, 0.5249, 0.6799).$$

These results show that the optimal values obtained using our approach encompass or closely approximate the solutions reported in the literature, while providing a higher ranking function value $\mathcal{R}(\tilde{Z})$, as indicated in Table 2.

In Example 2, the problem was initially solved by Sujit et al. [3]. Since this numerical problem involves trapezoidal fuzzy numbers, the proposed method allows obtaining optimal values directly in the form of trapezoidal fuzzy numbers. The values obtained by Sujit et al. are:

$$\tilde{Z}_1 = (0.3411, 0.7274, 1.8394, 3.0938), \quad \tilde{Z}_2 = (0.4585, 0.8340, 2.3992, 5.0803), \quad \tilde{Z}_3 = (0.2781, 0.7093, 2.1847, 12.0000),$$

while our methodology provides:

$$\tilde{Z}_1 = (0.3700, 0.7924, 1.9428, 3.3076), \quad \tilde{Z}_2 = (0.4933, 0.9016, 2.4838, 5.5882), \quad \tilde{Z}_3 = (0.2989, 0.7230, 2.1341, 12.0000).$$

Comparative analysis reveals that the optimal values determined by Sujit et al. [3] are included within or very close to the values generated by our approach, confirming the consistency of the results. Moreover, the values of the ranking functions $\mathcal{R}(\tilde{Z}_1)$ and $\mathcal{R}(\tilde{Z}_2)$ are higher with the proposed method, as shown in Table 4, indicating better performance in terms of solution quality.

In Example 3, which constitutes a practical production case, the trapezoidal fuzzy numbers used in this example are symmetric. Therefore, the resulting defuzzified problem no longer depends on the parameter α, in accordance with Remark 2.16. This problem was initially addressed by Sujit et al. [3]. Our method provides significantly better solutions, as indicated in Table 5.

Overall, these comparisons show that the fuzzy optimal values generated by our method, based on the median functions of α-cuts, are not only consistent with previous results but also provide more stable solutions that faithfully reflect the original fuzzy structure. These findings confirm the relevance and feasibility of our approach for solving multi-objective fractional fuzzy optimization problems.

4. Conclusion

In this work, we have proposed a new method for solving multi-objective fractional fuzzy optimization problems based on the median functions of α-cuts. The originality of this approach lies in using the median as a means of defuzzification. The resulting transformation leads to an equivalent deterministic model, linearized using Dinkelbach's theorem, then aggregated and efficiently solved using the simplex method.

The numerical results, applied both to examples from the literature and to a practical problem inspired by industrial production, showed that our method generates stable, regular Pareto-optimal solutions that are close to the classical values reported by other authors, while preserving the original fuzzy structure. Comparative analysis confirmed that the proposed method outperforms existing approaches, particularly in terms of robustness with respect to variations in α and the quality of the obtained solutions.

These findings highlight the relevance and feasibility of the approach based on the median functions of α-cuts. They also open several promising perspectives: extending the methodology to nonlinear or dynamic models, integrating other defuzzification methods for comparison, and developing hybrid algorithms that combine this approach with metaheuristic methods to tackle large-scale problems. Altogether, these perspectives offer promising advances in the modeling and solution of optimization problems under uncertainty.

Competing interests. The authors declare no competing interests.

References

[1] L. Zadeh, Fuzzy Sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[2] S. Nayak, S. Maharana, An Efficient Fuzzy Mathematical Approach to Solve Multi-Objective Fractional Programming Problem Under Fuzzy Environment, J. Appl. Math. Comput. 69 (2023), 2873–2899. https://doi.org/10.1007/s12190-023-01860-0.

[3] S. Maharana, S. Nayak, A Fuzzy Mathematical Model to Solve Multi-Objective Trapezoidal Fuzzy Fractional Programming Problems, Int. J. Syst. Assur. Eng. Manag. 15 (2024), 2757–2771. https://doi.org/10.1007/s13198-024-02298-8.

[4] J.D.L.C. Sama, D.P. Traore, K. Some, New Approach to Solving Fuzzy Multiobjective Linear Fractional Optimization Problems, Int. J. Anal. Appl. 22 (2024), 11. https://doi.org/10.28924/2291-8639-22-2024-11.

[5] R.E. Bellman, L.A. Zadeh, Decision-Making in a Fuzzy Environment, Manage. Sci. 17 (1970), B141–B164. https://www.jstor.org/stable/2629367.

[6] J.D.L.C. Sama, Abdoulaye Compaore, Kounhinir Some, Hybrid Approach for Solving Fuzzy Fractional Linear Optimization Problems, Ann. Fuzzy Math. Inform. 25 (2023), 111–123. https://doi.org/10.30948/afmi.2023.25.2.111.

[7] J.D.L.C. Sama, K. Some, Solving Fuzzy Nonlinear Optimization Problems Using Null Set Concept, Int. J. Fuzzy Syst. 26 (2023), 674–685. https://doi.org/10.1007/s40815-023-01626-7.

[8] A. Compaore, K. Some, B. Some, New Approach to the Resolution of Triangular Fuzzy Linear Programs: MOMA-Plus Method, Int. J. Appl. Math. Res. 6 (2017), 115–120. https://doi.org/10.14419/ijamr.v6i4.8069.

[9] H.C. Wu, Solutions of Fuzzy Multiobjective Programming Problems Based on the Concept of Scalarization, J. Optim. Theory Appl. 139 (2008), 361–378. https://doi.org/10.1007/s10957-008-9419-x.

[10] M. Shafiq, R. Viertl, On the Estimation of Parameters, Survival Functions, and Hazard Rates Based on Fuzzy Life Time Data, Commun. Stat. - Theory Methods 46 (2016), 5035–5055. https://doi.org/10.1080/03610926.2015.1099667.

[11] R. Viertl, Measurement Results and Fuzzy Models, in: Series on Advances in Mathematics for Applied Sciences, World Scientific, 2022: pp. 126–141. https://doi.org/10.1142/9789811242380_0006.

[12] R. Arya, P. Singh, S. Kumari, M.S. Obaidat, An Approach for Solving Fully Fuzzy Multi-Objective Linear Fractional Optimization Problems, Soft Comput. 24 (2019), 9105–9119. https://doi.org/10.1007/s00500-019-04442-0.

[13] S. Nayak, S. Maharana, An Efficient Fuzzy Mathematical Approach to Solve Multi-Objective Fractional Programming Problem Under Fuzzy Environment, J. Appl. Math. Comput. 69 (2023), 2873–2899. https://doi.org/10.1007/s12190-023-01860-0.

[14] L.C. de Barros, R.C. Bassanezi, W.A. Lodwick, A First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics: Theory and Applications, Springer, 2017. https://doi.org/10.1007/978-3-662-53324-6.

[15] D. Qiu, C. Lu, W. Zhang, Y. Lan, Algebraic Properties and Topological Properties of the Quotient Space of Fuzzy Numbers Based on Mareš Equivalence Relation, Fuzzy Sets Syst. 245 (2014), 63–82. https://doi.org/10.1016/j.fss.2014.01.003.

[16] K.D. Jamison, A Normed Space of Fuzzy Number Equivalence Classes, UCD/CCM Report, (1997).

[17] D. Qiu, W. Zhang, C. Lu, On Fuzzy Differential Equations in the Quotient Space of Fuzzy Numbers, Fuzzy Sets Syst. 295 (2016), 72–98. https://doi.org/10.1016/j.fss.2015.03.010.

[18] N. Yu, D. Qiu, The Karush-Kuhn-Tucker Optimality Conditions for the Fuzzy Optimization Problems in the Quotient Space of Fuzzy Numbers, Complexity 2017 (2017), 1242841. https://doi.org/10.1155/2017/1242841.

[19] W. Zhang, D. Qiu, M. Dong, Optimizations of Convex and Generalized Convex Fuzzy Mappings in The Quotient Space of Fuzzy Numbers, 47 (2017), 1–6.

[20] D. Qiu, H. Li, On Convexity of Fuzzy Mappings and Fuzzy Optimizations, Italian J. Pure Appl. Math. 35 (2015), 293–304.

[21] D. Qiu, H. Li, Fuzzy Optimizations of Convex Fuzzy Mappings in the Quotient Space of Fuzzy Numbers, J. Intell. Fuzzy Syst. 31 (2016), 1235–1244. https://doi.org/10.3233/ifs-162188.

[22] W. Dinkelbach, On Nonlinear Fractional Programming, Manag. Sci. 13 (1967), 492–498. https://doi.org/10.1287/mnsc.13.7.492.

[23] P.D.P. Dash, R.B. Dash, Solving Multi Objective Fuzzy Fractional Programming Problem, Ultra Scientist, 24 (2012), 429–434.