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EXISTENCE AND APPROXIMATION OF SOLUTION FOR VECTOR MIXED VARIATIONAL-LIKE INEQUALITY UNDER DENSELY (η, f)-C-PSEUDOMONOTONE MAPPINGS

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ABSTRACT. In this paper we study vector mixed variational inequality problem under (η, C)-pseudo-monotonicity and densely (η, f)-C-pseudo-monotonocity in reflexive Banach space. The existence and uniqueness of solutions have been established with the help of KKM technique and further we have proposed iterative algorithm to find the approximate solution of vector mixed variational inequality by defining an auxiliary problem.

1. INTRODUCTION

The theory of vector variational inequalities have wide scope and in their diverse form handle mathematical models to a number of interesting physical phenomena that covers areas like equilibrium problems, Optimization problems, etc. It was Giannessi [6] in 1980 who framed vector variational inequalities in the context of finite dimensional Euclidean space by extending the classical variational inequality. People have done many works on its different aspects; see e.g. [1, 4, 12, 23] and the references therein. In the literature, many papers appeared that establishes vector variational inequality as a powerful tool to study vector optimization problems; have a look for instance [1,21,23] and the references therein.

Let X be a reflexive Banach space and Y be a real Banach space. Let $K \subset X$ be nonempty, closed and convex set, and $C \subset Y$ be a closed, convex and pointed cone with apex at the origin. The dual of Y is denoted by Y^* and the positive polar cone of C is denoted by C^* ,

$$\text{i.e } C^* = \{u \in B^* : \langle u, v\rangle \geq 0,\ \forall v \in C\},$$

and

$$intC^* \subset \{u \in B^* : \langle u, v\rangle > 0,\ \forall v \in C\}.$$

Then C induces a vector ordering in B as follows:

$$\begin{aligned} u \leq v &\text{ if and only if } v - u \in C,\\ u \not\leq v &\text{ if and only if } v - u \notin C,\\ u < v &\text{ if and only if } v - u \in intC,\\ u \not< v &\text{ if and only if } v - u \notin intC. \end{aligned}$$

Some elementary properties regarding the ordering are as follows:

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$$u \notin -intC \implies -u \notin intC,$$
$$u, v \notin -intC, \implies (u+v) \notin -intC,$$
$$C + intC \subseteq intC.$$

It is clear that the above order relation ($\leq$) is a partial order relation and therefore $(Y, \leq)$ is an ordered Banach space. Let $L(X,Y)$ denotes the space of all continuous linear mappings from X into Y and let $\langle T, x\rangle$ denotes the value of $T \in L(X,Y)$ at $x \in X$. For $l \in intC^*$ and $T \in L(X,Y)$, we define the operator $T_l : X \to X^*$ by $\langle T_l(u), v\rangle = \langle l, \langle Tu, v\rangle\rangle$ for all $u, v \in X$ and there exists $l \in intC^*$ such that $\langle l, u\rangle \geq 0$ for all $u \notin -intC$, see [22]. Let $f : K \to Y$, $\eta : K \times K \to X$ and $G : K \to L(X,Y)$ be the mappings. The vector mixed variational inequality (in short VMVI) is defined as follows:

$$\text{find } u^* \in K : \ \langle G(u^*), \eta(u, u^*)\rangle + f(u) - f(u^*) \notin -intC, \ \forall u \in K. \tag{1.1}$$

We are denoting this problem as VMVI(1.1) and U^* denotes the set of solutions for the VMVI(1.1).
We consider another vector mixed variational inequality problem defined as follows

$$\text{find } u^* \in K : \ \langle G(u), \eta(u, u^*)\rangle + f(u) - f(u^*) \notin -intC, \ \forall u \in K. \tag{1.2}$$

We are denoting this problem as VMVI(1.2) and U_* denotes the set of solutions for the VMVI(1.2).

1.1. **Some Special Cases:**

- When $X = \mathbb{R}^n, Y = \mathbb{R}$ and $C = [0, \infty)$ then VMVI(1.1) gets reduced into the form which has been considered by Ansari [18].
- When $X = \mathbb{R}^n, Y = \mathbb{R}, \eta(u, u^*) = u - u^*$ and $C = [0, \infty)$ then VMVI (1.1) reduces to the classical Mixed variational inequality problem:

  $$\text{find } u^* \in K : \ (G(u^*), u - u^*) + f(u) - f(u^*) \geq 0 \ \forall u \in K.$$

  which was first studied by Lescarret and Browder [3,5] in 1966 and afterwards carried out by many authers (e.g., [2,13,17]).
  This theory has seen explosive growth in recent years and has been extended and enlarged by many researchers. The importance and application of mixed variational inequality have been documented in the literature. One of the practiced approach to solve the mixed variational inequalities is to convert them into the minimization of gap function and then use different possible numerical techniques. I. V. Konnov, Pinyagina [9–11,17] has used descent method and regularization method to establish the solution of classical mixed variational inequality and for the more see the references therein.

2. PRELIMINARIES

Definition 2.1. Let K be a non-empty subset of a Hausdorff topological vector space X then the set-valued mapping $F : K \to 2^X$ is said to be a KKM mapping if for any finite subset $\{u_1, u_2, \ldots, u_n\} \subset K$, we have

$$co\{u_1, u_2, \ldots, u_n\} \subset \bigcup_{i=1}^{n} F(u_i).$$

Lemma 2.2. *[7] Let K be a nonempty subset of a Hausdorff topological vector space X and $F : K \to 2^X$ be a KKM mapping. If $F(y)$ is closed in X for all $y \in K$ and compact for some $y \in K$, then*

$$\bigcap_{y \in K} F(y) \neq \emptyset.$$

Let K be a convex subset in X and K_0 be a subset of K. By Luc [14], the set K_0 is *segment-dense* in K if for each $x \in K$, there exists an $x_0 \in K_0$ such that x is a cluster point of the set $[x, x_0] \cap K_0$. Very recently Sahu et al. [20] generalized the densely pseudomonotonicity of Luc [14] to generalized densely relaxed η-α pseudomonotonicity. Motivated by Sahu et al. [20] and Luc [14], we have defined the notion of densely (η, f)-C-pseudomonotonicity.

Definition 2.3. A map $\eta : K \times K \to E$ is said to be Lipschitz continuous if there exists $\tau > 0$ such that $||\eta(u, v)|| \leq \tau||u - v||$ for all $u, v \in K$.

Definition 2.4. A mapping $G : K \to L(X, Y)$ is said to be (η, f)-C- pseudomonotone if for all $u, v \in K$, we have

$$\begin{aligned} &\langle G(u), \eta(v, u)\rangle + f(v) - f(u) \notin -intC \\ &\implies \langle G(v), \eta(v, u)\rangle + f(v) - f(u) \notin -intC. \end{aligned}$$

Definition 2.5. Let $K \subset X$ be a convex set. A mapping $f : K \to Y$ is said to be C- convex if for all $u, v \in K$ and for each $t \in [0, 1]$, we have

$$f(tu + (1 - t)v) - tf(u) - (1 - t)f(v) \in -C$$

Example 2.6. Let $X = Y = \mathbb{R}^2$ and $K = [0, 1] \times [0, 1]$. Let $C \subset \mathbb{R}$ be a cone defined as $C = \{(x, y) : y \leq 2x, \text{ and } x, y \geq 0\}$. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}^2$ be defined as $f(x) = (x_1^2, x_2^2)$ where $x = (x_1, x_2)$.
Now,
$\lambda f(x) + (1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y) = \lambda(x_1^2, x_2^2) + (1 - \lambda)(y_1^2, y_2^2)$
$-f(\lambda x_1 + (1 - \lambda)y_1, \lambda x_2 + (1 - \lambda)y_2)$

$$= \left(\lambda x_1^2 + (1 - \lambda)y_1^2, \lambda x_2^2 + (1 - \lambda)y_2^2\right) - \left((\lambda x_1 + (1 - \lambda)y_1)^2, (\lambda x_2 + (1 - \lambda)y_2)^2\right)$$

$$= \left(\lambda x_1^2 + (1 - \lambda)y_1^2 - (\lambda x_1 + (1 - \lambda)y_1)^2, \lambda x_2^2 + (1 - \lambda)y_2^2 - \lambda x_2 + (1 - \lambda)y_2)^2\right).$$

Since, we know that $0 \leq x - x^2 \leq x$ for $x \in [0, 1]$, hence we can deduce that

$$\lambda f(x) + (1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y) \in C.$$

Hence f is C-convex.

Definition 2.7. A mapping $G : K \to L(X, Y)$ is said to be densely (η, f)-C-pseudomonotone on K if there exists a segment-dense subset $K_0 \subset K$ such that G is (η, f)-C-pseudomonotone at every point of K_0.

Definition 2.8. Let $K \subset X$ be a convex set and $f : K \to Y$ be a mapping. Then f is said to be

(i) C- upper semi-continuous at u if for any sequence $\{u_n\} \subset X$ converging to u, we have $\limsup_{n\to\infty} f(u_n) - f(u) \in -C$.
(ii) C- lower semi-continuous at u if for any sequence $\{u_n\} \subset X$ converging to u, we have $\liminf_{n\to\infty} f(u_n) - f(u) \in C$.
(iii) C-hemicontinuous if for any $u, v \in K$ and $t \in [0, 1]$, the mapping $t \mapsto f(u + t(v - u))$ is continuous at 0^+.

Definition 2.9. Let $G : K \to L(X, Y)$ and $\eta : K \times K \to X$ be two mappings. G is said to be η-C-hemicontinuous if for any $u, v \in K$ and $t \in [0, 1]$, the mapping $t \mapsto \langle G(u + t(v - u)), \eta(v, u)\rangle$ is continuous at 0^+.

Definition 2.10. Let $T : K \to L(X, Y)$ be an operator and $\mathbf{a} \in intC$, then T is said to be η-C strongly monotone with modulus $\mathbf{a}$ if

$$\langle Tu - Tv, \eta(u, v)\rangle - ||u - v||^2\mathbf{a} \in C, \ \forall u, v \in K.$$

3. Existence Results for Vector Mixed Variational-like Inequalities

We begin with some lemmas with proof which will be useful for establishing the main results.

Lemma 3.1. *Let K be a nonempty closed and convex subset of a real reflexive Banach space E and B be a real Banach space ordered by a closed, convex and pointed cone C. Let $G : K \to L(E, B)$ is (η, f)-C-pseudomonotone, then $U^* \subset U_*$.*

Proof. Let $u^* \in U^*$, then we have

$$\langle G(u^*), \eta(u, u^*)\rangle + f(u) - f(u^*) \notin -intC, \ \forall u \in K.$$

Since G is (η, f)-C-pseudomonotone, we have

$$\langle G(u), \eta(u, u^*)\rangle + f(u) - f(u^*) \notin -intC, \ \forall u \in K.$$

This shows that $u^* \in U_*$. □

Lemma 3.2. *Let K be a nonempty closed and convex subset of a real Banach space E and B be a real Banach spaces ordered by a closed, convex and pointed cone C. Let the mapping $G : K \to L(E, B)$ be η-C-hemicontinuous and the map $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex. Suppose the mapping $f : K \to B$ is C-convex and $\eta(x, x) = 0$, then $U_* \subset U^*$.*

Proof. Let $u^* \in U_*$ i.e there is an $u^* \in K$ such that

$$\langle G(u), \eta(u, u^*)\rangle + f(u) - f(u^*) \notin -intC, \ \forall u \in K. \tag{3.1}$$

Let $u \in K$ be any point and $\bar{u} = (1 - t)u^* + tu$, $t \in (0, 1)$. Since K is convex, $\bar{u} \in K$ and substituting $u = \bar{u}$ in (3.1), we have

$$\langle G(\bar{u}), \eta(\bar{u}, u^*)\rangle + f(\bar{u}) - f(u^*) \notin -intC.$$

Since $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex and $\eta(x, x) = 0$, we obtain

$$t\langle G(\bar{u}), \eta(u, u^*)\rangle + f((1 - t)u^* + tu) - f(u^*) \notin -intC. \tag{3.2}$$

Now, by making use of the condition that f is C- convex, we have

$$t\langle G((1 - t)u^* + tu), \eta(u, u^*)\rangle + t(f(u) - f(u^*)) \notin -intC.$$

Thus,

$$\langle G((1 - t)u^* + tu), \eta(u, u^*)\rangle + (f(u) - f(u^*)) \notin -intC.$$

Since G is a η-C-hemicontinuous, we have

$$\langle G(u^*), \eta(u, u^*)\rangle + f(u) - f(u^*) \notin -intC.$$

This shows that $u^* \in U^*$ i.e $U_* \subset U^*$. □

Therefore, under the conditions of Lemma 3.1 and Lemma 3.2, the solution set of VMVI(1.1) and VMVI(1.2) coincide and is summarized as the following corollary.

Corollary 1. Let K be a nonempty closed and convex subset of a real reflexive Banach space E and B be a real Banach space ordered by a closed, convex and pointed cone C. Let $G : K \to L(E, B)$ is (η, f)-C-pseudomonotone and η-C-hemicontinuous both. Let the map $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex, the mapping $f : K \to B$ is C-convex and $\eta(x, x) = 0$, then VMVI(1.1) and VMVI(1.2) share the same solution set.

Theorem 3.3. *Let K be a nonempty closed, convex and bounded subset of a real reflexive Banach space E and B be a real Banach spaces ordered by a closed, convex and pointed cone C. Let $G : K \to L(E, B)$ be an η-C-hemicontinuous mapping and $f : K \to B$ be C-convex function. Suppose that the following conditions hold:*

(i) G is (η, f)-C-pseudomonotone;
(ii) the map $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex;
(iii) $u \mapsto \langle G(w), \eta(v, u)\rangle$ is C-upper semicontinuous;
(iv) f is C-lower semicontinuous;
(v) $\eta(u, u) = 0$.

Then the problem VMVI(1.1) *has a solution.*

Proof. Let us define two set valued mappings $\sigma, \rho : K \to 2^X$ by

$$\sigma(v) = \{u \in K |\ \langle G(u), \eta(v, u)\rangle + f(v) - f(u) \notin -intC\},$$

and

$$\rho(v) = \{u \in K |\ \langle G(v), \eta(v, u)\rangle + f(v) - f(u) \notin -intC\}.$$

We will show that σ, ρ are KKM mappings. Suppose σ is not a KKM mapping. Then there exist a set $\{u_1, ..., u_n\} \subset K$ such that $co\{u_1, ..., u_n\} \nsubseteq \bigcup_{i=1}^{i=n} \sigma(u_i)$ i.e. there exist $\bar{u} \in co\{u_1, ..., u_n\}, \bar{u} = \sum_{i=1}^{i=n} \lambda_i u_i$ with $\sum_{i=1}^{i=n} \lambda_i = 1$ where $\lambda_i \geq 0$, but $\bar{u} \notin \bigcup_{i=1}^{i=n} \sigma(u_i)$ i.e $\bar{u} \notin \sigma(u_i)$ for any $i \in \{1, ..., n\}$. Thus,

$$\langle G(\bar{u}), \eta(u_i, \bar{u})\rangle + f(u_i) - f(\bar{u}) \in -intC,\ \forall i \in \{1, ..., n\}.$$

Multiplying by λ_i and taking sum

$$\sum_{i=1}^{i=n} \lambda_i \left\langle G(\bar{u}), \eta(u_i, \bar{u}) \right\rangle + \sum_{i=1}^{i=n} \lambda_i f(u_i) - f(\bar{u}) \in -intC. \tag{3.3}$$

Since map $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex, we have

$$\left\langle G(\bar{u}), \eta\left(\sum_{i=1}^{i=n} \lambda_i u_i, \bar{u}\right) \right\rangle - \sum_{i=1}^{i=n} \lambda_i \left\langle G(\bar{u}), \eta(u_i, \bar{u}) \right\rangle \in -C. \tag{3.4}$$

Again since f is C- convex, we get

$$f\left(\sum_{i=1}^{i=n} \lambda_i u_i\right) - \sum_{i=1}^{i=n} \lambda_i f(u_i) \in -C. \tag{3.5}$$

From (3.3)-(3.5), we deduce that

$$0 = \langle G(\bar{u}), \eta(\bar{u}, \bar{u})\rangle \in -intC.$$

Which contradicts the assumption that C is a pointed cone. Hence σ is a KKM mapping. Clearly $\sigma(v) \subset \rho(v)$. Indeed, for given $v \in K$, let $u \in \sigma(v)$, then

$$\langle G(u), \eta(v, u)\rangle + f(v) - f(u) \notin -intC.$$

By (η, f)-C-pseudomonotonicity of G, we have

$$\langle G(v), \eta(v, u)\rangle + f(v) - f(u) \notin -intC.$$

Thus, $u \in \rho(v)$. Therefore we conclude that ρ is also a KKM mapping. Since K is closed, bounded and convex subset of the reflexive Banach space E, it is weakly compact. Now we will show that $\rho(v)$ is weakly closed for all $v \in K$. Let $\{u_n\}$ be a sequence in $\rho(v)$ such that $u_n \rightharpoonup u$. Then, we have

$$\langle G(v), \eta(v, u_n)\rangle + f(v) - f(u_n) \notin -intC.$$

Which implies that

$$\limsup[\langle G(v), \eta(v, u_n)\rangle + f(v) - f(u_n)] \notin -intC. \tag{3.6}$$

Since the map $u \mapsto \langle G(w), \eta(v, u)\rangle$ is C-upper semicontinuous and the mapping f is C-lower semicontinuous, we have

$$\langle G(v), \eta(v, u)\rangle - f(u) - \limsup[\langle G(v), \eta(v, u_n)\rangle - f(u_n)] \in -C. \tag{3.7}$$

Combining (3.6) and (3.7), we conclude that

$$\langle G(v), \eta(v, u)\rangle + f(v) - f(u) \notin -intC.$$

This shows that $u \in \rho(v)$. Thus, $\rho(v)$ is closed in K and therefore weakly compact. From Lemma 2.2, it follows that

$$\bigcap_{v\in K} \sigma(v) = \bigcap_{v\in K} \rho(v) \neq \emptyset.$$

Thus, there exists an $u \in K$ such that $\langle G(u), \eta(v, u)\rangle + f(v) - f(u) \notin intC$, for all $v \in K$. Which completes the proof of the theorem. □

Theorem 3.4. *Let K be a nonempty closed, convex and unbounded subset of a real reflexive Banach space E and B be a real Banach spaces ordered by a closed, convex and pointed cone C. Let $G : K \to L(E, B)$ be an η-C-hemicontinuous mapping and $f : K \to B$ be a C-convex function. Suppose that the following conditions hold:*

(i) G is (η, f)-C-pseudomonotone;
(ii) the map $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex;
(iii) $u \mapsto \langle G(w), \eta(v, u)\rangle$ is C-upper semicontinuous;
(iv) f is C-lower semicontinuous;
(v) $\eta(u, u) = 0$;
(vi) G is weakly coercive with respect to f that is there exists $v_0 \in K$ such that $\langle G(u), \eta(v_0, u)\rangle + f(v_0) - f(u) \in -intC$, whenever $||u|| \to +\infty$ and $u \in K$.

Then the problem VMVI(1.1) *has a solution.*

Proof. For $\delta > 0$, let $B_\delta = \{u \in K : ||u|| < \delta\}$. Consider the problem of finding $u_\delta \in K \cap B_\delta$ such that

$$\langle G(u_\delta), \eta(v, u_\delta)\rangle + f(v) - f(u_\delta) \notin -intC,\ \forall v \in K \cap B_\delta. \tag{3.8}$$

By Theorem 3.3, problem (3.8) has at least one solution $u_\delta \in K \cap B_\delta$. Choose $\delta > ||v_0||$ with v_0 as in condition (vi). Then we have

$$\langle G(u_\delta), \eta(v_0, u_\delta)\rangle + f(v_0) - f(u_\delta) \notin -intC.$$

If $||u_\delta|| = \delta$ for all δ, we may choose δ large enough and condition (vi) forces to have

$$\langle G(u_\delta), \eta(v_0, u_\delta)\rangle + f(v_0) - f(u_\delta) \in -intC.$$

Therefore, we are getting a contradiction. Hence, there exist a δ such that $||u_\delta|| < \delta$. Now, for any $v \in D$, there is a $t \in (0, 1)$ such that $\bar{u} = u_\delta + t(v - u_\delta) \in K \cap B_\delta$. From (3.8), we have

$$\langle G(u_\delta), \eta(\bar{u}, u_\delta)\rangle + f(\bar{u}) - f(u_\delta) \notin -intC.$$

Which implies that

$$\langle G(u_\delta), \eta(tv + (1-t)u_\delta, u_\delta)\rangle + f(tv + (1-t)u_\delta) - f(u_\delta) \notin -intC.$$

Since f is C-convex, by using condition (ii) and (v), we get

$$\langle G(u_\delta), \eta(v, u_\delta)\rangle + f(v) - f(u_\delta) \notin -intC,\ \forall v \in D.$$

This shows that u_δ is the solution of problem (1.1). □

Next result is about the existence of solution of VMVI(1.1) when the function G is densely (η, f)-C-monotone operator.

Theorem 3.5. *Let K be a nonempty convex and compact subset of a real reflexive Banach space E and B be a real Banach space ordered by a closed, convex and pointed cone C. Let $G : K \to L(E, B)$ be an η-C-hemicontinuous and densely (η, f)-C-pseudomonotone operator on K. Let $f : K \to B$ be a C-convex function. Suppose that the following conditions hold:*

(i) G is (η, f)-C-pseudomonotone;
(ii) the map $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex;
(iii) $u \mapsto \langle G(w), \eta(v, u)\rangle$ is C-upper semicontinuous;
(iv) f is C-lower semicontinuous;
(v) $\eta(tu + (1-t)v, w) = t\eta(u, w) + (1-t)\eta(v, w)$ for all $u, v, w \in K$;
(vi) $\eta(u, u) = 0$.

Then the problem VMVI(1.1) *has a solution.*

Proof. Since G is densely (η, f)-C-pseudomonotone on K, it follows that there exists a segment-dense subset $K_0 \subset K$ such that G is (η, f)-C-pseudomonotone at every point of K_0. For any $y \in K_0$, define a set valued mapping $\sigma : K_0 \to 2^K$ by

$$\sigma(v) = \{u \in K : \langle G(u), \eta(v, u)\rangle + f(v) - f(u) \notin -intC\}.$$

Clearly $\sigma(v) \neq \emptyset$ as $v \in \sigma(v)$. Now, we claim that σ is a KKM mapping. Suppose σ is not a KKM mapping. Then there exist a set $\{u_1, ..., u_n\} \subset K_0$ such that $co\{u_1, ..., u_n\} \nsubseteq \bigcup_{i=1}^{i=n} \sigma(u_i)$ i.e. there exist $\bar{u} \in co\{u_1, ..., u_n\}, \bar{u} = \sum_{i=1}^{i=n} \lambda_i u_i$ with $\sum_{i=1}^{i=n} \lambda_i = 1$ where $\lambda_i \geq 0$, but $\bar{u} \notin \bigcup_{i=1}^{i=n} \sigma(u_i)$ i.e $\bar{u} \notin \sigma(u_i)$ for any $i \in \{1, ..., n\}$. Thus,

$$\langle G(\bar{u}), \eta(u_i, \bar{u})\rangle + f(u_i) - f(\bar{u}) \in -intC,\ \forall i \in \{1, ..., n\}.$$

Multiplying by λ_i and taking sum

$$\sum_{i=1}^{i=n} \lambda_i \Big\langle G(\bar{u}), \eta(u_i, \bar{u})\Big\rangle + \sum_{i=1}^{i=n} \lambda_i f(u_i) - f(\bar{u}) \in -intC. \tag{3.9}$$

Since map $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex, we have

$$\Big\langle G(\bar{u}), \eta\Big(\sum_{i=1}^{i=n} \lambda_i u_i, \bar{u}\Big)\Big\rangle - \sum_{i=1}^{i=n} \lambda_i \Big\langle G(\bar{u}), \eta(u_i, \bar{u})\Big\rangle \in -C. \tag{3.10}$$

Again since f is C- convex, we get

$$f\left(\sum_{i=1}^{i=n} \lambda_i u_i\right) - \sum_{i=1}^{i=n} \lambda_i f(u_i) \in -C. \tag{3.11}$$

Taking account of relations (3.9)-(3.11), we deduce that

$$0 = \langle G(\bar{u}), \eta(\bar{u}, \bar{u})\rangle \in -intC.$$

Which contradicts the assumption that C is a pointed cone. Hence σ is a KKM mapping. Define another map $\rho : K_0 \to 2^K$ by

$$\rho(v) = \{u \in K : \langle G(v), \eta(v,u)\rangle + f(v) - f(u) \notin -intC\}.$$

Clearly $\sigma(v) \subset \rho(v)$, for each $v \in K_0$. Indeed, for given $v \in K_0$, let $u \in \sigma(v)$, then

$$\langle G(u), \eta(v,u)\rangle + f(v) - f(u) \notin -intC.$$

By (η, f)-C-pseudomonotonicity of G, we have

$$\langle G(v), \eta(v,u)\rangle + f(v) - f(u) \notin -intC.$$

Thus, $u \in \rho(v)$. Therefore we conclude that ρ is also KKM mapping. Since K is closed, by the same argument as in Theorem 3.3, we conclude that for each $v \in K_0$, $\rho(v)$ is closed in K and hence weakly compact. From Lemma 2.2, it follows that

$$\bigcap_{v \in K_0} \rho(v) \neq \emptyset.$$

Let $u^* \in \bigcap_{v \in K_0} \rho(v)$ then,

$$\langle G(v), \eta(v,u^*)\rangle + f(v) - f(u^*) \notin -intC, \ \forall v \in K_0 \tag{3.12}$$

We are going to show that u^* is the solution of the problem. For this, let $w \in K$ be any arbitrary element. Since K_0 is segment dense in K, then there exist an element $w_0 \in K_0$ such that w is a cluster point of $[w, w_0] \cap K_0$. Therefore there is a sequence $\{w_n\} \in [w, w_0] \cap K_0$ such that $\{w_n\}$ converges to w. Put $w_n = (1 - t_n)w + t_n w_0 = w + t_n(w_0 - w) \in K_0$, where $t_n \in [0, 1]$ and $t_n \to 0$. Substituting $v = w_n$ in (3.12), we have

$$\langle G(w_n), \eta(w_n, u^*)\rangle + f(w_n) - f(u^*) \notin -intC.$$

It follows from condition (ii) that

$$(1 - t_n)\langle G(w_n), \eta(w, u^*)\rangle + t_n\langle G(w_n), \eta(w_0, u^*)\rangle + f(w_n) - f(u^*) \notin -intC.$$

Taking $t_n \to 0$, and using η-C-hemicontinuity of G and C-hemicontinuity of f, we have

$$\langle G(w), \eta(w, u^*)\rangle + f(w) - f(u^*) \notin -intC.$$

This shows that u^* is a solution of VMVI(1.1). □

4. Approximation of solution through Auxiliary Principle Technique

In this section,we find the approximate solution to vector mixed variational inequality problem through iterative technique of Glowinski et al. [15,16,19] which is known as a auxiliary principle technique. For this purpose we construct the following auxiliary problem in order to approximate the solution of VMVI(1.1). For a given $u \in K$ and a real $\theta > 0$, the auxiliary problem is to find a $w \in K$ such that

$$\theta\left[\langle G(w), \eta(v,w)\rangle + f(v) - f(w)\right] + \langle Tv - Tw, \eta(w,u)\rangle \notin -intC, \ \forall v \in K. \tag{4.1}$$

where $T : K \to L(E, B)$ is a map. If $w = u$, then w coincides with the solution of VMVI(1.1).
Now, we present the following iterative scheme for the approximate solution of VMVI(1.1).

Algorithm 1. Let $\{\theta_n\}$ be a decreasing sequence of positive numbers that converges to 0 and $\epsilon > 0$ be a fixed given number.

(i) For $n = 0$, let u_0 be the initial solution.

(ii) At nth step solve the auxiliary problem (4.1) with $u = u_n$. Let the solution be $w = u_{n+1}$. That is, find $w = u_{n+1}$ of the problem:

$$\theta_{n+1}\left[\langle G(u_{n+1}), \eta(v, u_{n+1})\rangle + f(v) - f(u_{n+1})\right] + \langle Tv - Tu_{n+1}, \eta(u_{n+1}, u_n)\rangle \notin -intC.$$

(iii) stop when $||u_{n+1} - u_n|| \leq \epsilon$ otherwise move to (ii).

Theorem 4.1. *Let K be any non-empty convex subset of a reflexive Banach space E and B be a real Banach space. Let $G : K \to L(E, B)$ and $f : K \to B$ be a C-convex and C-upper semi-continuous function. Let $T : K \to L(E, B)$ be an operator and $\eta : K \times K \to E$ be a mapping. Suppose the following conditions hold:*

(i) $u \mapsto T(u)$ *is continuous from weak topology to strong topology;*
(ii) $\eta(u, w) = \eta(u, v) + \eta(v, w)$ *and* $\eta(u, v) + \eta(v, u) = 0\ \forall u, v, w \in K$*;*
(iii) $\eta(u, u) = 0$*;*
(iv) G *is* (η, C)*- monotone;*
(v) T *is* (η, C)*- strongly monotone with modulus* $\mathbf{a} \in intC$ *;*
(vi) for a nonempty, compact and convex subset D *of* K *there exist* $u_0 \in D$ *such that*

$$\langle G(u), \eta(u_0, u)\rangle + f(u_0) - f(u) \in -intC,\ \forall u \in K\backslash D.$$

Then the auxiliary problem (4.1) *has a unique solution.*

Proof. First we will show that there exists a nonempty, compact and convex subset D of K and $u_0 \in K$ such that

$$\langle G(u), \eta(u_0, u)\rangle + f(u_0) - f(u) + \langle Tu_0 - Tu, \eta(u, w)\rangle \in -intC.$$

For this, take $\mathbf{b} := -\mathbf{a} \in -intC$ and let $u_0 \in K$ be arbitrary but fixed .
For $\lambda > 0$, by condition (ii) we have

$$\lambda||u - u_0||\mathbf{b} - \langle Tu_0 - Tu, \eta(u, w)\rangle = \lambda||u - u_0||\mathbf{b} + \langle Tu - Tu_0, \eta(u, u_0)\rangle + \langle Tu - Tu_0, \eta(u_0, w)\rangle$$

It follows from condition (v),

$$\lambda||u - u_0||\mathbf{b} - \langle Tu_0 - Tu, \eta(u, w)\rangle \in C + \mathbf{a}||u - u_0||^2 - \lambda||u - u_0||\mathbf{a} + \langle Tu - Tu_0, \eta(u_0, w)\rangle. \tag{4.2}$$

Since u_0 and w are some arbitrary fixed points so $\langle Tu - Tu_0, \eta(u_0, w)\rangle$ is a function of u. From condition (i) it follows that T is C- lower semicontinuous. Hence for $r > 0$ there exists $\mathbf{r} \in B$ such that

$$\langle Tz - Tu_0, \eta(u_0, w)\rangle - \mathbf{r} \in intC\ \forall z \in K_0 = \{u \in K : ||u - u_0|| \leq r\} \tag{4.3}$$

Now, let $u \in K\backslash K_0$ and $z := \frac{r}{||u-u_0||}u + \big(1 - \frac{r}{||u-u_0||}\big)u_0 \in K_0$ and putting the value of z in (4.3) we have,

$$\frac{r}{||u - u_0||}\langle Tu - Tu_0, \eta(u_0, w)\rangle - \mathbf{r} \in intC,\ \forall u \in K\backslash K_0,$$

$$\langle Tu - Tu_0, \eta(u_0, w)\rangle \in \frac{||u - u_0||}{r}\mathbf{r} + intC,\ \forall u \in K\backslash K_0. \tag{4.4}$$

Since $\mathbf{a} \in intC$ and $\mathbf{r} \in B$ therefore $0 \in -\mathbf{a} + intC$ and $0 \in -\mathbf{a}r + intC$. Then there exists $\beta \in (0, 1)$ such that, for all $\gamma \in (0, \beta)$ we have $\gamma\mathbf{r} \in -\mathbf{a}r + intC$.

$$\mathbf{r} \in -r\lambda\mathbf{a} + intC,\ \text{for } \lambda > \frac{1}{\beta}, \tag{4.5}$$

from (4.2) and (4.4),we obtain

$$\lambda||u-u_0||\mathbf{b}-\langle Tu_0-Tu,\eta(u,w)\rangle$$

$$\in intC+\mathbf{a}||u-u_0||^2-\lambda||u-u_0||\mathbf{a}+\tfrac{||u-u_0||}{r}\mathbf{r}$$

$$\in intC+\mathbf{a}||u-u_0||^2-2\lambda||u-u_0||\mathbf{a}+||u-u_0||\tfrac{\mathbf{r}}{r}+\lambda||u-u_0||\mathbf{a},$$

$$\lambda||u-u_0||\mathbf{b}-\langle Tu_0-Tu,\eta(u,w)\rangle \in intC+[||u-u_0||-2\lambda]||u-u_0||\mathbf{a}+\frac{||u-u_0||}{r}[\mathbf{r}+r\lambda\mathbf{a}]. \tag{4.6}$$

Choose $R_\lambda=\max\{\alpha,2\lambda\}$, from (4.5) and (4.6) we obtain that for $u\in K$ such that $||u-u_0||\geq R_\lambda$ we have

$$\lambda||u-u_0||\mathbf{b}-\langle Tu_0-Tu,\eta(u,w)\rangle \in intC,$$

$$\text{i.e } \langle Tu_0-Tu,\eta(u,w)\rangle \in -intC\ \forall u\in K\backslash K_0. \tag{4.7}$$

By combining condition (v) and (4.7),we can state that there exist a compact convex subset D of K and $u_0\in K$, such that

$$\langle G(u),\eta(u_0,u)\rangle+f(u_0)-f(u)+\langle Tu_0-Tu,\eta(u,w)\rangle\in -intC. \tag{4.8}$$

Now (4.8) serves as coercivity condition, we are to show the existence of solution of the auxiliary problem (4.1).

Let $W=\{u_1,u_2,...,u_n\}$ be any finite subset of K and setting $\tilde{D}=co\{W\cup D\}$ which is a convex and compact set.
Let us consider maps $\tilde{\sigma}:\tilde{D}\to 2^{\tilde{D}}$ defined as

$$\tilde{\sigma}(v)=\{u\in\tilde{D}:\langle G(u),\eta(v,u)\rangle+f(v)-f(u)+\langle Tv-Tu,\eta(u,w)\rangle\notin -intC\}$$

and $\tilde{\rho}:\tilde{D}\to 2^{\tilde{D}}$ defined as

$$\tilde{\rho}(v)=\{u\in\tilde{D}:\langle G(v),\eta(u,v)\rangle+f(v)-f(u)+\langle Tv-Tu,\eta(u,w)\rangle\notin -intC\}$$

Then by moving in the same direction as we have done in Theorem 3.5, existence of solution of (4.1) is guaranteed. Next remaining part is to establish the uniqueness of solution. By contradiction we assume u and u^* are to be the solutions, then we have

$$\langle G(u),\eta(v,u)\rangle+f(v)-f(u)+\langle Tv-Tu,\eta(u,w)\rangle\notin -intC,\ \forall v\in K \tag{4.9}$$

and

$$\langle G(u^*),\eta(v,u^*)\rangle+f(v)-f(u^*)+\langle Tv-Tu^*,\eta(u^*,w)\rangle\notin -intC,\ \forall v\in K \tag{4.10}$$

putting $v=u^*$ in (4.9) and $v=u$ in (4.10) and by adding, we get

$$\langle G(u)-G(u^*),\eta(u^*,u)\rangle+\langle Tu^*-Tu,\eta(u,w)-\eta(u^*,w)\rangle\notin -intC$$

$$\langle G(u)-G(u^*),\eta(u^*,u)\rangle+\langle Tu^*-Tu,\eta(u,w)+\eta(w,u^*)\rangle\notin -intC$$

$$\langle Tu^*-Tu,\eta(u,u^*)\rangle\notin -intC+\langle G(u^*)-G(u),\eta(u^*,u)\rangle$$

Since G is (η,C)- monotone, it follows that

$$\langle Tu^*-Tu,\eta(u,u^*)\rangle\notin -intC$$

$$\langle Tu-Tu^*,\eta(u,u^*)\rangle-||u-v||^2\mathbf{a}\notin -intC$$

Adding above two inequalities, we obtain

$$-||u-v||^2\mathbf{a} \notin -intC,$$

i.e

$$-\mathbf{a} \notin -intC.$$

which is a contradiction. Hence the solution is unique. □

Theorem 4.2. *Suppose all conditions of Theorem* 4.1 *hold and in addition assume that there exists* $l \in intC^*$ *such that* $\langle l, u\rangle \geq 0$ *for all* $u \notin -intC$. *Let the map* $\eta : K \times K \to E$ *be Lipschitz continuous. Assume that there exist* $\epsilon \in (0,1)$ *such that* $\tau\frac{\theta_{n+1}}{\theta_n\langle l,\mathbf{a}\rangle}||T_l|| < \epsilon$. *Then the iterative scheme converges.*

Proof. From algorithm 1, we have

$$\begin{aligned}&\langle G(u_n), \eta(v,u_n)\rangle + f(v) - f(u_n) + \tfrac{1}{\theta_n}\langle Tv - Tu_n, \eta(u_n,u_{n-1})\rangle\\ &\notin -intC,\ \forall v \in K.\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}&\langle G(u_{n+1}), \eta(v,u_{n+1})\rangle + f(v) - f(u_{n+1}) + \tfrac{1}{\theta_{n+1}}\langle Tv - Tu_{n+1}, \eta(u_{n+1},u_n)\rangle\\ &\notin -intC,\ \forall v \in K.\end{aligned} \tag{4.12}$$

Putting $v = u_{n+1}$ in (4.11) and $v = u_n$ in (4.12) then adding both, we get

$$\begin{aligned}&\langle G(u_n) - G(u_{n+1}), \eta(u_{n+1},u_n)\rangle + \tfrac{1}{\theta_n}\langle Tu_{n+1} - Tu_n, \eta(u_n,u_{n-1})\rangle\\ &+\tfrac{1}{\theta_{n+1}}\langle Tu_n - Tu_{n+1}, \eta(u_{n+1},u_n)\rangle \notin -intC,\end{aligned}$$

which implies

$$\begin{aligned}&\tfrac{1}{\theta_n}\langle Tu_{n+1} - Tu_n, \eta(u_n,u_{n-1})\rangle + \tfrac{1}{\theta_{n+1}}\langle Tu_n - Tu_{n+1}, \eta(u_{n+1},u_n)\rangle\\ &\notin -intC + \langle G(u_{n+1}) - G(u_n), \eta(u_{n+1},u_n)\rangle\\ &\notin -intC.\end{aligned}$$

Since G is (η, C)- monotone

$$\frac{\theta_{n+1}}{\theta_n}\langle Tu_{n+1} - Tu_n, \eta(u_n,u_{n-1})\rangle - \langle Tu_{n+1} - Tu_n, \eta(u_{n+1},u_n)\rangle \notin -intC.$$

Since T is (η, C)- strongly monotone, we obtain

$$\frac{\theta_{n+1}}{\theta_n}\langle Tu_{n+1} - Tu_n, \eta(u_n,u_{n-1})\rangle - \mathbf{a}||u_{n+1} - u_n||^2 \notin -intC.$$

Making use of the condition that there exist $l \in intC^*$ such that $\langle l, u\rangle \geq 0$ for all $u \notin -intC$ therefore we have,

$$\frac{\theta_{n+1}}{\theta_n}\langle l, \langle Tu_{n+1} - Tu_n, \eta(u_n,u_{n-1})\rangle\rangle - \langle l, \mathbf{a}\rangle||u_{n+1} - u_n||^2 \geq 0,$$

which gives

$$\langle l, \mathbf{a}\rangle||u_{n+1} - u_n||^2 \leq \frac{\theta_{n+1}}{\theta_n}\langle l, \langle Tu_{n+1} - Tu_n, \eta(u_n,u_{n-1})\rangle\rangle,$$

which implies

$$\langle l, \mathbf{a}\rangle||u_{n+1} - u_n||^2\frac{\theta_{n+1}}{\theta_n} \leq ||T_l||\ ||u_{n+1} - u_n||\ ||\eta(u_n,u_{n-1})||,$$

It implies

$$||u_{n+1} - u_n|| \leq \frac{\theta_{n+1}}{\theta_n\langle l, \mathbf{a}\rangle}||T_l||\ ||\eta(u_n,u_{n-1})||.$$

Since η is Lipschitz continuous,

$$||u_{n+1} - u_n|| \leq \tau\frac{\theta_{n+1}}{\theta_n\langle l, \mathbf{a}\rangle}||T_l||\ ||u_n - u_{n-1}||,$$

$$||u_{n+1} - u_n|| \leq \epsilon ||u_n - u_{n-1}||.$$

Hence u_n converges to $w \in K$. □

5. Concluding Remarks

In this paper, we discussed the existence and uniqueness of solution for vector mixed variational inequality in reflexive Banach spaces under (η, C)-pseudo-monotonicity and densely (η, f)-C-pseudo-monotonocity, where the practiced partial ordering is not a usual partial ordering on n-dimensional Euclidean space E^n rather a partial ordering induced by a closed convex pointed cone in Banach space.

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