

SET DOMINATION IN FUZZY GRAPHS USING STRONG ARCS

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ABSTRACT. Set domination in fuzzy graphs is very useful for solving traffic problems during communication in computer networks and travelling networks. In this article, the concept of set domination in fuzzy graphs using strong arcs is introduced. The strong set domination number of complete fuzzy graph and complete bipartite fuzzy graph is determined. It is obtained the properties of the new parameter and related it to some other known domination parameters of fuzzy graphs. An upper bound for the strong set domination number of fuzzy graphs is also obtained.

INTRODUCTION

Fuzzy graphs were introduced by Rosenfeld, who has described the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness [22]. Bhutani and Rosenfeld have introduced the concept of strong arcs [7].

The work on fuzzy graphs was also done by Mordeson, Pradip, Talebi, and Yeh [16, 21, 32, 33]. It was during 1850s, a study of dominating sets in graphs started purely as a problem in the game of chess. Chess enthusiasts in Europe considered the problem of determining the minimum number of queens that can be placed on a chess board so that all the squares are either attacked by a queen or occupied by a queen. The concept of domination in graphs was introduced by Ore and Berge in 1962, the domination number and independent domination number are introduced by Cockayne and Hedetniemi [11]. Connected domination in graphs was discussed by Sampathkumar and Walikar [23]. Somasundaram and Somasundaram discussed domination in fuzzy graphs. They defined domination using effective edges in fuzzy graph [26, 27]. Nagoorgani and Chandrasekharan defined domination in fuzzy graphs using strong arcs [18]. Manjusha and Sunitha discussed some concepts in domination and total domination in fuzzy graphs using strong arcs [12, 13]. In this paper it is discussed set domination in fuzzy graphs using strong arcs.

PRELIMINARIES

It is quite known that graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations, by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a 'fuzzy graph model'. We summarize briefly some basic definitions in fuzzy graphs which are presented in [6, 7, 12, 17–19, 22, 26, 28].

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A fuzzy graph is denoted by $G : (V, \sigma, \mu)$, where V is a node set, σ and μ are mappings defined as $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$, where σ and μ represent the membership values of a node and an arc respectively. For any fuzzy graph, $\mu(x, y) \leq \min\{\sigma(x), \sigma(y)\}$. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e., $\mu(x, x) = \sigma(x)$, for all x) and symmetric (i.e., $\mu(x, y) = \mu(y, x)$, for all (x, y)). In all the examples σ is chosen suitably. Also, we denote the underlying crisp graph by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. Throughout we assume that $\sigma^* = V$. The fuzzy graph $H : (\tau, \nu)$ is said to be a partial fuzzy subgraph of $G : (V, \sigma, \mu)$ if $\nu \subseteq \mu$ and $\tau \subseteq \sigma$. In particular we call $H : (\tau, \nu)$ a fuzzy subgraph of $G : (V, \sigma, \mu)$ if $\tau(u) = \sigma(u)$ for all $u \in \tau^*$ and $\nu(u, v) = \mu(u, v)$ for all $(u, v) \in \nu^*$. A fuzzy graph $G : (V, \sigma, \mu)$ is called trivial if $|\sigma^*| = 1$. Two nodes u and v in a fuzzy graph G are said to be adjacent (neighbors) if $\mu(u, v) > 0$. The set of all neighbors of u is denoted by $N(u)$.

An arc (u, v) of a fuzzy graph $G : (V, \sigma, \mu)$ with $\mu(u, v) > 0$ is called a weakest arc of G if (u, v) is an arc with minimum $\mu(u, v)$.

A path P of length n is a sequence of distinct nodes u_0, u_1, \dots, u_n such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, \dots, n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a cycle and P is called a fuzzy cycle, if it contains more than one weakest arc. The strength of a cycle is the strength of the weakest arc in it. The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x, y)$.

A fuzzy graph $G : (V, \sigma, \mu)$ is connected if for every x, y in σ^* , $CONN_G(x, y) > 0$.

An arc (u, v) of a fuzzy graph is called an effective arc if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called effective neighborhood of u and is denoted by $EN(u)$.

A fuzzy graph $G : (V, \sigma, \mu)$ is said to be complete if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, for all $u, v \in \sigma^*$.

The order p and size q of a fuzzy graph $G : (V, \sigma, \mu)$ are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{(x, y) \in V \times V} \mu(x, y)$.

Let $G : (V, \sigma, \mu)$ be a fuzzy graph and $S \subseteq V$. Then the scalar cardinality of S is defined to be $\sum_{v \in S} \sigma(v)$ and it is denoted by $|S|_s$. Let p denotes the scalar cardinality of V , also called the order of G .

The complement of a fuzzy graph $G : (V, \sigma, \mu)$, denoted by \bar{G} is defined to be $\bar{G} = (V, \sigma, \bar{\mu})$ where $\bar{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y)$ for all $x, y \in V$ [30].

An arc of a fuzzy graph $G : (V, \sigma, \mu)$ is called strong if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted. A fuzzy graph G is called a strong fuzzy graph if each arc in G is a strong arc. Depending on $CONN_G(x, y)$ of an arc (x, y) in a fuzzy graph G , Mathew and Sunitha, [28] defined three different types of arcs. Note that $CONN_{G-(x, y)}(x, y)$ is the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x, y) . An arc (x, y) in G is α -strong if $\mu(x, y) > CONN_{G-(x, y)}(x, y)$. An arc (x, y) in G is β -strong if $\mu(x, y) = CONN_{G-(x, y)}(x, y)$. An arc (x, y) in G is δ -arc if $\mu(x, y) < CONN_{G-(x, y)}(x, y)$.

Thus an arc (x, y) is a strong arc if it is either α -strong or β -strong. Also y is called strong neighbor of x if arc (x, y) is strong. The set of all strong neighbors of x is called the strong neighborhood of x and is denoted by $N_s(x)$. The closed strong neighborhood $N_s[x]$ is defined as $N_s[x] = N_s(x) \cup \{x\}$. A path P is called strong path if P contains only strong arcs.

A fuzzy graph $G : (V, \sigma, \mu)$ is said to be bipartite [26] if the vertex set V can be partitioned into two non empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$ then G is called a complete bipartite graph and is denoted by K_{σ_1, σ_2} , where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2 .

A node u is said to be isolated if $\mu(u, v) = 0$ for all $v \neq u$.

1. STRONG SET DOMINATION IN FUZZY GRAPHS

The concept of domination in graphs was introduced by Ore and Berge in 1962, the domination number and independent domination number are introduced by Cockayne and Hedetniemi [11]. Set domination in graphs was discussed by Sampathkumar and Pushpalatha [24]. For the terminology of domination and set domination in crisp graphs we refer to [10, 24].

For a node v of a graph $G : (V, E)$, recall that a neighbor of v is a node adjacent to v in G . Also the neighborhood $N(v)$ of v is the set of neighbors of v . The closed neighborhood $N[v]$ is defined as $N[v] = N(v) \cup \{v\}$. A node v in a graph G is said to dominate itself and each of its neighbors, that is v dominates the nodes in $N[v]$. A set S of nodes of G is a dominating set of G if every node of $V(G) - S$ is adjacent to some node in S . A minimum dominating set in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set $S \subset V$ is a set dominating set of G if for every set $D \subset V - S$, there exists a non empty subset $E \subset S$ such that the sub graph $\langle D \cup E \rangle$ is connected. The minimum of the cardinalities of the set dominating sets of G is termed as the set domination number of G , and is denoted as $\gamma_s(G)$.

Nagoorgani and Chandrasekaran [18] introduced the concept of domination using strong arcs in fuzzy graphs. According to Nagoorgani a node v in a fuzzy graph G is said to strongly dominate itself and each of its strong neighbors, i.e., v strongly dominates the nodes in $N_s[v]$. A set D of nodes of G is a strong dominating set of G if every node of $V(G) - D$ is a strong neighbor of some node in D . They defined a minimum strong dominating set in a fuzzy graph G as a strong dominating set with minimum number of nodes [18]. An equivalent definition which provides much motivation for this paper as follows A set $D \subset V$ is a strong dominating set of G if for every singleton subset $\{u\}$ of $V - D$, there exists a singleton subset $\{v\}$ of D such that $\langle \{u, v\} \rangle$ is connected.

Manjusha and Sunitha [14] defined strong domination number using membership values (weights) of arcs in fuzzy graphs as follows.

Definition 1.1. [14] The weight of a strong dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values(weights) of strong arcs incident on u . The strong domination number of a fuzzy graph G is defined as the minimum weight of strong dominating sets of G and it is denoted by $\gamma_s(G)$ or simply γ_s . A minimum strong dominating set in a fuzzy graph G is a strong dominating set of minimum weight.

Let $\gamma_s(\overline{G})$ or $\overline{\gamma_s}$ denote the strong domination number of the complement of a fuzzy graph G .

Now the set domination in fuzzy graphs using strong arcs is defined as follows.

Definition 1.2. A set $D \subset V$ is a strong set dominating set of $G : (V, \sigma, \mu)$ if for every set $T \subset V - D$ of $V - D$, there exists a non empty subset $S \subset D$ such that the subgraph $\langle S \cup T \rangle$ is connected.

Definition 1.3. The weight of a strong set dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values(weights) of strong arcs incident on u . The strong set domination number of a fuzzy graph G is defined as the minimum weight of strong set dominating sets of G and it is denoted by $\gamma_{ss}(G)$ or simply γ_{ss} . A minimum strong set dominating set in a fuzzy graph G is a strong set dominating set of minimum weight.

Let $\gamma_{ss}(\overline{G})$ or $\overline{\gamma_{ss}}$ denote the strong set domination number of the complement of a fuzzy graph G .

Apart from obtaining bounds for $\gamma_{ss}(G)$, we relate to some other known parameters of G .

Throughout this article, by a fuzzy graph we mean a connected fuzzy graph.

A strong dominating set D of a fuzzy graph $G : (V, \sigma, \mu)$ is a strong connected dominating set of G if the induced fuzzy subgraph $\langle D \rangle$ is connected. The weight of a strong connected dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values(weights) of strong arcs incident on u . The strong connected domination number of a fuzzy graph G is defined as the minimum weight of strong connected dominating sets of G and it is denoted by $\gamma_{sc}(G)$ or simply γ_{sc} . A minimum strong connected dominating set in a fuzzy graph G is a strong connected dominating set of minimum weight.

Remark 1.4. Clearly any strong connected dominating set is a strong set dominating set.

Example 1.5. Consider the fuzzy graph given in Fig. 1.

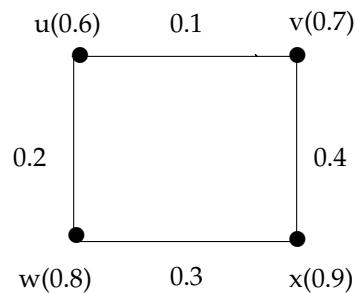


Fig. 1 Illustration of strong set domination in fuzzy graphs

In this fuzzy graph, strong arcs are (u, w) , (w, x) and (x, v) . The strong set dominating sets are $D_1 = \{u, x\}$, $D_2 = \{u, v\}$, $D_3 = \{w, x\}$, and $D_4 = \{v, w\}$. Among these the minimum strong set dominating sets are D_1 and D_3 where

$$W(D_1) = 0.2 + 0.3 = 0.5 \text{ and } W(D_3) = 0.2 + 0.3 = 0.5.$$

Hence

$$\gamma_{ss} = 0.5.$$

Theorem 1.6. Let $G : (V, \sigma, \mu)$ be any connected fuzzy graph and $H : (V, \nu, \tau)$ be any maximum spanning tree of G . Then every strong set dominating set of H is also a strong set dominating set of G and consequently $\gamma_{ss}(G) \leq \gamma_{ss}(H)$.

Proof. Let D be a strong set dominating set of H . Since H is a maximum spanning tree of G we have $\sigma = \nu$. Hence the nodes in D strongly set dominates all the nodes in $V \setminus D$. Hence D is a strong set dominating set of G . Hence $\gamma_{ss}(G) \leq \gamma_{ss}(H)$. \square

2. STRONG SET DOMINATION IN CLASSES OF FUZZY GRAPHS

In this section, it is determined the strong set domination number of complete fuzzy graph, complete bipartite fuzzy graph, fuzzy cycles and join of a fuzzy graph with a complete fuzzy graph.

Proposition 2.1. If $G : (V, \sigma, \mu)$ is a complete fuzzy graph, then $\gamma_{ss}(G) = \wedge \{\mu(u, v) : u, v \in \sigma^*\}$.

Proof: Since G is a complete fuzzy graph, all arcs are strong [29] and each node is adjacent to all other nodes. Hence $D = \{u\}$ is a strong set dominating set for each $u \in \sigma^*$. Hence the result follows.

Proposition 2.2.

$$\gamma_{ss}(K_{\sigma_1, \sigma_2}) = \begin{cases} \mu(u, v), & \text{if } |V_1| = 1 \text{ or } |V_2| = 1 \\ 2\mu(u, v), & \text{if } |V_1| \text{ and } |V_2| \geq 2 \end{cases}$$

where $\mu(u, v)$ is the weight of a weakest arc in K_{σ_1, σ_2} .

Proof: In K_{σ_1, σ_2} , all arcs are strong. Also each node in V_1 is adjacent with all nodes in V_2 . Hence in K_{σ_1, σ_2} , the strong set dominating sets are V_1, V_2 and any set containing at least 2 nodes, one in V_1 and other in V_2 . Among this if V_1 or V_2 contains only one element say u , then $D = \{u\}$ is the minimum strong set dominating set in G . Hence $\gamma_{ss}(K_{\sigma_1, \sigma_2}) = \mu(u, v)$ where $\mu(u, v)$ is the minimum weight of arcs incident on u . If both V_1 and V_2 contains more than one element then the set $\{u, v\}$ of nodes of any weakest arc (u, v) in K_{σ_1, σ_2} forms a strong set dominating set.

Hence $\gamma_{ss}(K_{\sigma_1, \sigma_2}) = \mu(u, v) + \mu(u, v) = 2\mu(u, v)$. Hence the result.

Theorem 2.3. Let $G : (V, \sigma, \mu)$ be a fuzzy cycle where G^* is a cycle. Then, $\gamma_{ss}(G) = \bigwedge \{W(D) : D \text{ is a strong set dominating set in } G \text{ with } |D| \geq (n - 3)\}$, where n is the number of nodes in G .

Proof: In a fuzzy cycle every arc is strong. Also, the number of nodes in a strong set dominating set of both G and G^* are same because each arc in both graphs are strong. In graph G^* , the strong set domination number of G^* is obtained as $(n - 3)$ [24]. Hence the minimum number of nodes in a strong set dominating set of G is $(n - 3)$. Hence the result follows.

Definition 2.4. [16,17] Union of two fuzzy graphs: Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ with $V_1 \cap V_2 = \phi$ and let $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$ be the union of G_1^* and G_2^* .

Then the union of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ defined by

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & \text{if } u \in V_1 \setminus V_2 \\ \sigma_2(u) & \text{if } u \in V_2 \setminus V_1 \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(u, v) = \begin{cases} \mu_1(u, v) & \text{if } (u, v) \in E_1 \setminus E_2 \\ \mu_2(u, v) & \text{if } (u, v) \in E_2 \setminus E_1 \end{cases}$$

Definition 2.5. [16,17] Join of two fuzzy graphs: Consider the join $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs where E' is the set of all arcs joining

the nodes of V_1 and V_2 where we assume that $V_1 \cap V_2 = \phi$. Then the

join of two fuzzy graphs G_1 and G_2 is a fuzzy graph

$G = G_1 + G_2 : (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ defined by

$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u), u \in V_1 \cup V_2$$

and

$$(\mu_1 + \mu_2)(u, v) = \begin{cases} (\mu_1 \cup \mu_2)(u, v) & \text{if } (u, v) \in E_1 \cup E_2 \text{ and} \\ \sigma_1(u) \wedge \sigma_2(v) & \text{if } (u, v) \in E' \end{cases}$$

Theorem 2.6. For any fuzzy graph $G : (V, \sigma, \mu)$, $\gamma_{ss}(K_\sigma + G) = \mu(u, v)$ where $\mu(u, v)$ is the weight of a weakest arc incident on u for any node $u \in K_\sigma$.

Proof: For any fuzzy graph G , any node in K_σ is adjacent to all other nodes in K_σ and G and note that all such arcs are strong arcs. Hence any singleton set $D = \{u\}$ for each node u in K_σ , is a strong set dominating set of $K_\sigma + G$. Hence $\gamma_{ss}(K_\sigma + G) = \mu(u, v)$ where $\mu(u, v)$ is the weight of a weakest arc incident on u for any node $u \in K_\sigma$.

3. MINIMAL STRONG SET DOMINATION IN FUZZY GRAPHS

In this section it is defined minimal strong set dominating sets and discussed some properties.

Definition 3.1. A strong set dominating set D of a connected fuzzy graph $G : (V, \sigma, \mu)$ is called a minimal strong set dominating set if no proper subset of D is a strong set dominating set of G .

Remark 3.2. Every minimum strong set dominating set is minimal but not conversely.

Example 3.3. Consider the fuzzy graph in Figure 4.

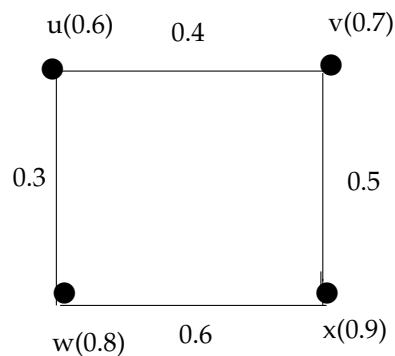


Figure 4: Illustration of minimal strong set domination

In the fuzzy graph of Figure 4, (u, v) , (v, x) , (x, w) are strong arcs and (u, w) is a δ -arc. $D = \{u, w\}$ is a minimal strong set dominating set but not minimum strong set dominating set since the set $\{u, x\}$ forms a minimum strong set dominating set with $\gamma_{ss}(G) = 0.9$, but $W(D) = 1$.

Note that in a complete fuzzy graph the minimum and minimal strong set dominated sets are same and any singleton set of nodes is the minimum strong set dominating set. Hence the following theorems are obvious.

Theorem 3.4. Every non trivial complete fuzzy graph G has a strong set dominating set D whose complement $V \setminus D$ is also a strong set dominating set.

Theorem 3.5. Let G be a complete fuzzy graph. If D is a minimal strong set dominating set then $V \setminus D$ is a strong set dominating set.

Note that in a complete bipartite fuzzy graph the end nodes of any weakest arc forms a minimal strong set dominating set. Hence the following theorems are obvious.

Theorem 3.6. Every non trivial complete bipartite fuzzy graph G has a strong set dominating set D of two elements whose complement $V \setminus D$ is also a strong set dominating set.

Theorem 3.7. Let G be a complete bipartite fuzzy graph. If D is a minimal strong set dominating set of two elements then $V \setminus D$ is a strong set dominating set.

Remark 3.8. Theorems 5.4 to 5.7 are not true in general connected fuzzy graphs as seen in the following example.

Example 3.9. Consider the fuzzy graph given in Figure 5.

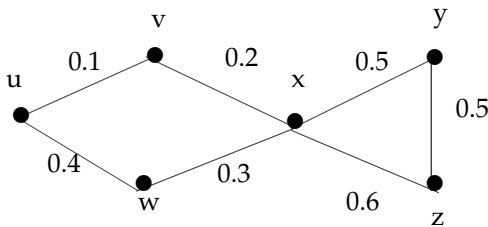


Figure 5: Example of a strong set dominating set D

such that $V \setminus D$ is not a strong set dominating set

In this fuzzy graph all node weights are taken as 1. $D = \{u, v, w, y\}$ is a strong set dominating set. But $V \setminus D = \{x, z\}$ is not a strong set dominating set.

4. STRONG SET DOMINATION IN FUZZY TREES

Note that in the definition of a fuzzy tree, F is the unique maximum spanning tree (MST) of G [31]. An arc is called a fuzzy bridge of a fuzzy graph $G : (V, \sigma, \mu)$ if its removal reduces the strength of connectedness between some pair of nodes in G [22].

Similarly a fuzzy cut node w is a node in G whose removal from G reduces the strength of connectedness between some pair of nodes other than w [22].

A node z is called a fuzzy end node if it has exactly one strong neighbor in G [8].

A non trivial fuzzy tree G contains at least two fuzzy end nodes and every node in G is either a fuzzy cut node or a fuzzy end node [8].

In a fuzzy tree G an arc is strong if and only if it is an arc of F where F is the associated unique maximum spanning tree of G [7, 31]. Note that these strong arcs are α -strong and there are no β -strong arcs in a fuzzy tree [28]. Also note that in a fuzzy tree G an arc (x, y) is α -strong if and only if (x, y) is a fuzzy bridge of G [28].

Theorem 4.1. In a non trivial fuzzy tree $G : (V, \sigma, \mu)$, each node of a strong set dominating set is incident on an α -strong arc (fuzzy bridge) of G .

Proof: Let D be a strong set dominating set of G . Let $u \in D$. Since D is a strong dominating set, there exists $v \in V \setminus D$ such that (u, v) is a strong arc. Then (u, v) is an arc of the unique MST F of G [7, 31]. Hence (u, v) is an α -strong arc or a fuzzy bridge of G [22]. Since u is arbitrary, this is true for every node of the strong set dominating set of G . This completes the proof.

Proposition 4.2. In a non trivial fuzzy tree $G : (V, \sigma, \mu)$, no node of a strong set dominating set is an end node of a β -strong arc.

Proof: Note that a fuzzy graph is a fuzzy tree if and only if it has no β -strong arcs [28]. Hence the proposition.

Theorem 4.3. *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$, except K_2 , the set of all fuzzy cut nodes is a strong set dominating set.*

Proof: Let D be the set of all fuzzy cut nodes of a non trivial fuzzy tree $G : (V, \sigma, \mu)$. Then D is a strong dominating set in G [12] and induced fuzzy subgraph $\langle D \rangle$ is connected [15]. Note that the strong neighbour of a fuzzy end node is a fuzzy cut node Hence for every proper subset S of $V - D$ there exists a nonempty proper subset T of D such that $S \cup T$ is connected. Hence D is a strong set dominating set of G .

Remark 4.4. The set of all fuzzy end nodes need not be a strong set dominating set in a non trivial fuzzy tree $G : (V, \sigma, \mu)$ except K_2 .

Theorem 4.5. *In a fuzzy tree $G : (V, \sigma, \mu)$, each node of every strong set dominating set is contained in the unique maximum spanning tree of G .*

Proof: Since G is a fuzzy tree, G has a unique maximum spanning tree F which contains all the nodes of G . In particular, F contains all nodes of every strong set dominating set of G . This completes the proof.

Theorem 4.6. *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$ except K_2 , $\gamma_{ss}(G) = W(S)$ where S is the set of all fuzzy cut nodes of G .*

Proof: Note that the set S of all fuzzy cut nodes of G is a strong set dominating set of G (Theorem 4.3). Here we have to prove that S is a minimum strong set dominating set. Suppose if possible S is not a minimum strong set dominating set. Then there exists a strong set dominating set S' such that $W(S') < W(S)$. Then S' has 4 choices.

- 1]. S' contains all fuzzy cut nodes and at least one fuzzy end node.
- 2]. At least one fuzzy cut node say w is not contained in S' and S' contains no fuzzy end node.
- 3]. S' is a combination of fuzzy cut nodes and fuzzy end nodes.
- 4]. S' contains only fuzzy end nodes.

In case 1 it is obvious that $W(S') > W(S)$.

In case 2 $\langle S' \rangle$ (the fuzzy sub graph induced by S') is not connected if w is an internal node of $\langle S \rangle$ (the fuzzy sub graph induced by S) or S' is not a strong dominating set if w is an end node of the fuzzy subgraph $\langle S \rangle$ for,

A fuzzy tree contains at least 2 fuzzy end nodes. If w is an end node of $\langle S \rangle$ then one neighboring node of w is a fuzzy end node say u in G and w is the only strong neighbor of u in G . Therefore, if w is not contained in $\langle S' \rangle$ then u is not strongly dominated by any node in G . Hence S' is not a strong dominating set of G .

Case 3 has 3 possibilities.

a]. G has a unique maximum weighted arc adjacent to any fuzzy end node, then $W(S') > W(S)$ since weight of maximum arc is contributed to $W(S')$ but not to $W(S)$

b]. The unique maximum weighted arc is adjacent to any fuzzy cut node then $W(S') \geq W(S)$

c]. G has more than one maximum weighted arc and one of these is adjacent to a fuzzy cut node and other is adjacent to a fuzzy end node then $W(S') > W(S)$.

In case 4 we can consider the cases a, b, c as in case 3, we get similar results.

Therefore in all the cases we get a contradiction. Hence the minimum strong set dominating set of G is the set of all fuzzy cut nodes of G .

$$\text{Hence } \gamma_{ss}(G) = W(S)$$

Theorem 4.7. Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph and S be the set of all internal nodes of any maximum spanning tree of G . Then, $\gamma_{ss}(G) \leq W(S)$ and equality holds if G is a fuzzy tree.

Proof: Every connected fuzzy graph has at least one maximum spanning tree T and $\gamma_{ss}(T) = W(S)$ [Theorem 4.6]. By Theorem 1.6, every strong set dominating set of T is also a strong set dominating set of G and hence $\gamma_{ss}(G) \leq \gamma_{ss}(T)$. Hence

$$\gamma_{ss}(G) \leq W(S)$$

and equality holds if G is a fuzzy tree by theorem 6.7. Hence the result.

5. PRACTICAL APPLICATION

Consider a group of students who have close relationship with each other. Depending upon the knowledge (k) they possess, fuzzy values are assigned in the range of $0 < k < 1$ for poor to intelligent student respectively. Assume the students as nodes and their relationship as arcs. A student who possesses intrinsic knowledge has relationship with other students with less knowledge than him or her, may try to disseminate his or her entire knowledge to all of them. However, his or her friends could imbibe as much as they can and act accordingly. Such a scenario can be framed as a fuzzy graph. Set domination on fuzzy graph concept can be applied for assigning group mentors in the class. Among the group of students, some could not directly interact with the teachers and may hesitate for clearing their doubts. In such situations, class can be formed into groups and a leader is assigned to each group of students who have higher knowledge than others. This will help to improve overall performance of the students. To select a leader in each group, set domination concept can be applied.

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REFERENCES

- [1] M. Akram, Bipolar fuzzy graphs, Inf. Sci. 181 (2011), 5548-5564.
- [2] M. Akram, W.A. Dudek, Regular bipolar fuzzy graphs, Neural Comput. Appl. 21 (2012), 197-205.
- [3] H. Rashmanlou, S. Samanta, M. Pal, R.A. Borzooei, A study on bipolar fuzzy graphs, J. Intell. Fuzzy Syst. 28 (2015), 571-580.
- [4] O. T. Manjusha and M. S. Sunitha, Strong domination in fuzzy graphs, Fuzzy Inf. Eng. 7 (2015), 369-377.
- [5] E. Sampathkumar, The global domination number of a graph, J. Math. Phys. Sci. 23 (1989), 377-385.
- [6] K. R. Bhutani, On automorphisms of fuzzy graphs, Pattern Recogn. Lett. 9 (1989), 159-162.
- [7] K. R. Bhutani and A. Rosenfeld, Strong arcs in fuzzy graphs, Inform. Sci. 152 (2003), 319-322.
- [8] K. R. Bhutani and A. Rosenfeld, Fuzzy end nodes in fuzzy graphs, Inform. Sci. 152 (2003), 323-326.
- [9] K. R. Bhutani and Abdullah Batton, On M-strong fuzzy graphs, Inform. Sci. 1559 (2003), 103-109.
- [10] G. Chartrand and P. Zang, Introduction to graph theory, Tata McGraw-Hill Edition, 2006.
- [11] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of domination in graphs, Marcel Dekker, Inc., New York, 1998.
- [12] O. T. Manjusha and M. S. Sunitha, Notes on domination in fuzzy graphs, J. Intell. Fuzzy Syst. 27 (2014), 3205-3212.

- [13] O. T. Manjusha and M. S. Sunitha, Total domination in fuzzy graphs using strong arcs, *Ann. Pure Appl. Math.* 9 (2014), 23–33.
- [14] O. T. Manjusha and M. S. Sunitha, Strong domination in fuzzy graphs, *Fuzzy Inf. Eng.* 7 (2015), 369–377.
- [15] O. T. Manjusha and M. S. Sunitha, Connected domination in fuzzy graphs using strong arcs, *Ann. Fuzzy Math. Inf.* 10 (2015), 979–994.
- [16] J. N. Mordeson and Peng Chang-Shyh, Operations on fuzzy graphs, *Inform. Sci.* 79 (1994), 159–170.
- [17] J. N. Mordeson and P. S. Nair, *Fuzzy graphs and fuzzy hypergraphs*, Physica - Verlag, 2000.
- [18] A. Nagoorgani and V. T. Chandrasekaran, Domination in fuzzy graph, *Adv. Fuzzy Sets Syst. I* (2006), 17–26.
- [19] A. Nagoorgani and P. Vijayalakshmi, Insensitive arc in domination of fuzzy graph, *Int. J. Contemp. Math. Sci.* 6 (2011), 1303–1309.
- [20] A. Nagoorgani and R. Jahir Hussain, Connected and global domination of fuzzy graph, *Bull. Pure Appl. Sci.* 27 (2008), 1–11.
- [21] P. Debnath, Domination in interval-valued fuzzy graphs, *Ann. Fuzzy Math. Inform.* 6 (2013), 363–370.
- [22] A. Rosenfeld, *Fuzzy graphs, in fuzzy sets and their application to cognitive and decision processes*, Academic Press, (1975), 77–95.
- [23] E. Sampathkumar and H. B. Walikar, The connected domination number of a Graph, *J. Math. Phys. Sci.* 13 (1979), 607–613.
- [24] E. Sampathkumar and L. Pushpalatha, Set domination in graphs, *J. Graph Theory* 18 (1994), 489–495.
- [25] K. R. Sandeep Narayan and M. S. Sunitha, Connectivity in a fuzzy graph and its complement, *Gen. Math. Notes* 9 (2012), 38–43.
- [26] A. Somasundaram and S. Somasundaram, Domination in fuzzy graphs-I, *Pattern Recognit. Lett.* 19 (1998), 787–791.
- [27] A. Somasundaram, Domination in fuzzy graphs-II, *J. Fuzzy Math.* 13 (2005), 281–288.
- [28] S. Mathew and M. S. Sunitha, Types of arcs in a fuzzy graph, *Inform. Sci.* 179 (2009), 1760–1768.
- [29] S. Mathew and M. S. Sunitha, Node connectivity and arc connectivity of a fuzzy graph, *Inform. Sci.* 180 (2010), 519–531.
- [30] M. S. Sunitha and A. Vijayakumar, Complement of a fuzzy graph, *Indian J. Pure Appl. Math.* 33 (2002), 1451–1464.
- [31] M. S. Sunitha and A. Vijayakumar, A characterization of fuzzy trees, *Inform. Sci.* 113 (1999), 293–300.
- [32] A. A. Talebi and H. Rashmanlou, Isomorphism on interval valued fuzzy graphs, *Ann. Fuzzy Math. Inform.* 6 (2013), 47–58.
- [33] R. T. Yeh and S. Y. Bang, Fuzzy relations, fuzzy graphs and their applications to clustering analysis, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), *Fuzzy sets and Their Applications*, Academic Press, (1975), 125–149.