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EQUIVALENCE OF K-FUNCTIONALS AND MODULUS OF SMOOTHNESS CONSTRUCTED BY THE MEHLER-FOCK-CLIFFORD TRANSFORM

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ABSTRACT. Using the Mehelt-Fock-Clifford transform, we define generalized modulus of smoothness in the space $L^2(J; x^{-\frac{1}{2}}dx)$. Based on the kernel $P_{i\sqrt{\lambda}-\frac{1}{2}}$ and the operator A_x^m we define Sobolev-type and K-functionals. The main result of the paper is the proof of the equivalence theorem for a K-functional and a modulus of smoothness for the Mehler-Fock-Clifford transform.

1. INTRODUCTION

The Mehler–Fock–Clifford transform is a generalization of the Fourier transform, adapted to hyperbolic geometry and Clifford algebra. Fourier corresponds to the flat case (Euclidean), while Mehler–Fock is its natural analogue for hyperbolic space and its extensions.

The Fourier transform has a very important role in solving many problems in physics and computer science:

Wave Propagation and Optics

The Fourier transform allows us to represent waves as superpositions of plane waves.

In optics, diffraction and interference patterns are described using Fourier analysis. For instance, the image formation in lenses and microscopes is modeled as a Fourier transform of the incoming light wave.

Quantum Mechanics

The Fourier transform links position space and momentum space:

$$\Psi(p) = \frac{1}{\sqrt{2\pi h}} \int_{-\infty}^{+\infty} \Psi(x) e^{\frac{-ipx}{h}} dx,$$

This means the probability distribution of a particle in momentum space is the Fourier transform of its wave function in position space.

Image Processing

Images are treated as 2D signals.

Fourier transforms are used in:

Filtering (removing noise, blurring, sharpening).

Image compression (JPEG uses Discrete Cosine Transform, closely related to Fourier).

Pattern recognition (detecting periodic textures).

Machine Learning and AI

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Fourier features are sometimes used to represent data in a way that captures periodicity or oscillatory behavior, improving models for time series and signals.

In deep learning, the Fourier domain helps accelerate convolutions (via the convolution theorem).

The Mehler–Fock–Clifford transform reduces to the Bessel transform (and to the radial Fourier transform) when passing from the hyperbolic (Clifford) to the Euclidean (Clifford) setting.

The Bessel integral transforms and their inverses are widely used to solve various problems in calculus, mechanics, mathematical physics and computational mathematics (see [17], [18], [19], [20], [21], [22]).

This clearly shows the importance of the Mehler-Fock-Clifford transformation in solving several problems in physics and computer science, which will push us to look for approximations that will be very useful in solving these problems (see [23]).

In [11], we proved the equivalence theorem for a K-functional constructed from Sobolevtype spaces and a modulus of smoothness for the Hankel-Clifford transform on $(0;+\infty)$, using a generalized translation operator.

In this work, we prove the analog of this result (see [11]) for the Mehler-Fock-Clifford transform, For this purpose, we use the translation operator.

We point out that similar results have been established in the context of: Fourier transform, Dunkl transfom and q-Bessel Fourier transform, see [12], [13], [14], [16] and [15].

2. PRELIMINARIES

We define the Mehler-Fock-Clifford (MFC) transform as:

$$F(\lambda) = M(f)(\lambda) = \int_{\frac{1}{4}}^{\infty} f(x) P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) \frac{dx}{\sqrt{x}}\ ,\ \ \lambda > 0,$$

and its inversion:

$$f(x) = M^{-1}(F)(x) = \frac{1}{2}\int_0^{\infty} \tanh(\pi\lambda) P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) F(\lambda) d\lambda\ ,\ \ x > \frac{1}{4}.$$

Where $P_{i\sqrt{\lambda}-\frac{1}{2}}(x)$ is cone function (associated Legendre function of zero order) and it is represented in terms of Gaussian hypergeometric function ${}_2F_1$ as:

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(x) ={}_2 F_1\Big(\frac{1}{2}+i\sqrt{\lambda}, \frac{1}{2}-i\sqrt{\lambda}; 1; \frac{(1-x)}{2}\Big).$$

The theory and properties of Hankel-Clifford transform have already been studied by the several researchers via [3], [4] and [5]. As per this argument the Legendre-Clifford function according to [2] is defined as:

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) = \frac{\sqrt{2}}{\pi}\cosh(\pi\sqrt{\lambda}) \int_0^{\infty} \frac{\cos(\sqrt{\lambda}t)}{\sqrt{2\sqrt{x}+\cosh t}} dt.$$

We are now going to see some properties of kernel $P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})$ cited in [6]:

Proposition 2.1. *For every positive integer m there exists $M > 0$ such that:*

$$\frac{d^m}{dx^m} P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) \leq M\cosh(\pi\sqrt{\lambda}).$$

According to the properties cited in , we get:

Proposition 2.2. *$P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})$ can also be represented as:*

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) = \frac{1}{\pi}\int_0^{\pi} \Big[2\sqrt{x} + (4x-1)^{\frac{1}{2}}\cos(\xi)\Big]^{-\frac{1}{2}+i\sqrt{\lambda}} d\xi,$$

and:

$$|P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})| \leq P_{-\frac{1}{2}}(2\sqrt{x}).$$

Using asymptotic behaviours of $P_{-\frac{1}{2}}(2\sqrt{x})$ cited in [1], [9] and [10] ,we obtain:

Proposition 2.3.

$$|P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})| \leq C.$$

Where $C > 0$ is a constant.

By $L^p(J;\omega(x)dx)$, $J =]\frac{1}{4}, \infty[$, $1 \leq p < \infty$, we denote the weighted L^p-space with the norm

$$\|f\|_{L^p(J;\omega(x)dx)} = \left(\int_{\frac{1}{4}}^{\infty} |f(x)|^p \omega(x)dx\right)^{\frac{1}{2}}.$$

Plancherel's and Parseval's relations have been obtained as:

$$\int_{\frac{1}{4}}^{\infty} f(x)\overline{g(x)}\frac{dx}{\sqrt{x}} = \frac{1}{2}\int_0^{\infty} \tanh(\pi\sqrt{\lambda})M(f)(\lambda)\overline{(Mg)(\lambda)}d\lambda.$$

i.e.,

$$\int_{\frac{1}{4}}^{\infty} |f(x)|^2 \frac{dx}{\sqrt{x}} = \frac{1}{2}\int_0^{\infty} \tanh(\pi\sqrt{\lambda})|M(f)(\lambda)|^2 d\lambda.$$

Or

$$\|f\|_{L^2(J;x^{-\frac{1}{2}}dx)} = \|M(f)\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

Thus, the MFC-transform is isometrically-isomorphism operator from $L^2(J; x^{-\frac{1}{2}}dx)$ to $L^2\big(\mathbb{R}^+; \frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda\big)$.
From [8], we define the symmetric function $D(x, y, z) \geq 0$ as:

$$D(x, y, z) = \int_0^{\infty} \tanh(\pi\sqrt{\lambda})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{y})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{z})d\lambda,$$

where $D(x, y, y) = \frac{1}{\pi}\big(16\sqrt{xyz} + 1 - 4x - 4y - 4z\big)^{-\frac{1}{2}}$ for $z \in I_{x,y}$ and $D(x, y, z) = 0$ otherwise,
and

$$I_{x,y} = \left]4\sqrt{xy} - \Big[(4x-1)(4y-1)\Big]^{\frac{1}{2}}; 4\sqrt{xy} + \Big[(4x-1)(4y-1)\Big]^{\frac{1}{2}}\right[.$$

Now using the inversion of MFC-transform, then we have:

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{y}) = \int_{\frac{1}{4}}^{\infty} D(x, y, z)P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{z})\frac{dz}{\sqrt{z}}.$$

and

$$\int_{\frac{1}{4}}^{\infty} D(x, y, z)\frac{dz}{\sqrt{z}} = 1.$$

The translation operator is defined as:

$$(T_h f)(y) = \int_{\frac{1}{4}}^{\infty} D(x, y, z)f(z)\frac{dz}{\sqrt{z}},$$

and we see that:

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{y}) = \Big(T_h\big(P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{z})\big)\Big)(y).$$

Theorem 2.4. *[6] If $f \in L^1(J; x^{-\frac{1}{2}}dx)$, then the MFC-transform of the translation operator checks the following property:*

$$M(T_h f)(\lambda) = P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})M(f)(\lambda).$$

$$M(A_x^m f)(\lambda) = (-1)^m(\lambda + \frac{1}{4})^m M(f)(\lambda).$$

3. THE MAIN RESULT

Let the function $f(x) \in L^2(J, x^{-\frac{1}{2}}dx)$ such that $J =]\frac{1}{4}; +\infty[$. We define differences of the order m ($m \in 1, 2, ...$) with a step $h > \frac{1}{4}$:

$$\Delta_h^m f(x) = \left[\left(1 + P_{-\frac{1}{2}}(2\sqrt{h})\right)I + T_h\right]^m f(x),$$

where I is the unit operator.
For any positive integer m, we define the generalized module of smoothness of the mth order by the formula:

$$\omega_m(f, \delta) = \sup_{0<h\leq\delta} \|\Delta_h^m f(x)\|_{L^2(J, x^{-\frac{1}{2}}dx)}.$$

Such that $\delta > 0$.

Let W_2^m be the Sobolev space constructed by the operator A_x^k,i.e.,

$$W_2^m = \{f \in L^2(J, x^{-\frac{1}{2}}dx) : A_x^k \in L^2(J, x^{-\frac{1}{2}}dx); k = 1, 2, ..., m\},$$

with:

$$A_x^k = \sum_{j=1}^{2k} q_j^k(x) A_x^j.$$

Let us define the K-functional constructed by the space $L^2(J, x^{-\frac{1}{2}}dx)$ and W_2^m by:

$$K_m(f, t) = \inf\{\|f - g\|_{L^2(J, x^{-\frac{1}{2}}dx)} + t\|A_x^k g\|_{L^2(J, x^{-\frac{1}{2}}dx)}; g \in W_2^m\},$$

where $f \in L^2(J, x^{-\frac{1}{2}}dx)$ and $t > \frac{1}{4}$.

Lemma 3.1. *We have:*

$$1 \leq 1 + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) + P_{-\frac{1}{2}}(2\sqrt{h}) \leq 1 + 2P_{-\frac{1}{2}}(2\sqrt{h}).$$

Proof. We know that:

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) \leq P_{-\frac{1}{2}}(2\sqrt{h}).$$

So

$$-P_{-\frac{1}{2}}(2\sqrt{h}) \leq P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) \leq P_{-\frac{1}{2}}(2\sqrt{h})$$

Then:

$$1 \leq 1 + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) + P_{-\frac{1}{2}}(2\sqrt{h}) \leq 1 + 2P_{-\frac{1}{2}}(2\sqrt{h}).$$

■

Lemma 3.2. *If $f \in L^2(J, x^{-\frac{1}{2}}dx)$, then there exits $c_1 > 0$ such that:*

$$\|\Delta_h^m f(x)\|_{L^2(J, x^{-\frac{1}{2}}dx)} \leq c_1\|f\|_{L^2(J, x^{-\frac{1}{2}}dx)}.$$

Proof. We have that:

$$\|\Delta_h^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}dx)} = \left\|\left[\left(1+P_{-\frac{1}{2}}(2\sqrt{h})\right)I + T_h\right]^m f(x)\right\|_{L^2(J,x^{-\frac{1}{2}}dx)}.$$

Using the Plancherel's and Parseval's relations we get that:

$$\|\Delta_h^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}dx)} = \left\|\left[1+P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) + P_{-\frac{1}{2}}(2\sqrt{h})\right]^m M(f)(\lambda)\right\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

and by Lemma 3.1 we have that:

$$1+P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) + P_{-\frac{1}{2}}(2\sqrt{h}) \leq 1+2P_{-\frac{1}{2}}(2\sqrt{h}).$$

Then there exists $M>0$ such that:

$$|P_{-\frac{1}{2}}(2\sqrt{h})| \leq M.$$

So:

$$|1+P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) + P_{-\frac{1}{2}}(2\sqrt{h})| \leq (1+2M)^m.$$

Then:

$$\|\Delta_h^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}dx)} \leq (1+2M)^m \|M(f)(\lambda)\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

Since by the Plancherel's and Parseval's relations we have:

$$\|f(x)\|_{L^2(J,x^{-\frac{1}{2}}dx)} = \|M(f)(\lambda)\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

Then:

$$\|\Delta_h^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}dx)} \leq c_1 \|M(f)(\lambda)\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)},$$

with $c_1 = (1+2M)^m > 0$. ■

Lemma 3.3. *Let* $f \in W_2^m$, *then:*

$$\omega_m(f,\delta) \leq c_2 \delta^{2n} \|A_x^k f\|_{L^2(J,x^{-\frac{1}{2}}dx)}.$$

Proof. If $h \in]\frac{1}{4}, \delta]$, we have

$$\|\Delta_h^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}dx)} = \left\|\left[\left(1+P_{-\frac{1}{2}}(2\sqrt{h})\right)I + T_h\right]^m M(f)(\lambda)\right\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)},$$

and we have that:

$$M(A_x^k(f))(\lambda) = (-1)^n(\lambda+\frac{1}{4})^m M(f)(\lambda).$$

By using the Plancherel's and Parseval's relations we have:

$$\|M(A_x^k f)(\lambda)\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)} = \|A_x^k f\|_{L^2(J,x^{-\frac{1}{2}}dx)}.$$

Then:

$$\|A_x^k f\|_{L^2(J,x^{-\frac{1}{2}}dx)} = \left\|(\lambda+\frac{1}{4})^m M(f)(\lambda)\right\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

We have:

$$\|\Delta_h^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}dx)} = \left\|\left[\left(1+P_{-\frac{1}{2}}(2\sqrt{h})\right)I + T_h\right]^m M(f)(\lambda)\right\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

Then:

$$\|\Delta_h^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}dx)} = h^{2m}\left\|\frac{\left[\left(1+P_{-\frac{1}{2}}(2\sqrt{h})\right)I + T_h\right]^m}{h^{2m}(\lambda+\frac{1}{4})^m}(\lambda+\frac{1}{4})^m M(f)(\lambda)\right\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

We have $h > \frac{1}{4}$, so $\frac{1}{h^{2m}} < 4^{2m}$, and we have $\lambda > 0$ then $\frac{1}{(\lambda+\frac{1}{4})^m} < 4^m$, we have also

$$|1 + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) + P_{-\frac{1}{2}}(2\sqrt{h})| \leq c_1.$$

It follows that:

$$\|\Delta_h^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq h^{2m}4^{3m}c_1 \left\|(\lambda + \frac{1}{4})^m M(f)(\lambda)\right\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

Then:

$$\|\Delta_h^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq c_2 h^{2m} \|A_x^k f\|_{L^2(J,x^{-\frac{1}{2}}\,dx)},$$

with $c_2 = c_1 4^{3m} > 0$. So

$$\omega_m(f,\delta) \leq c_2 \delta^{2m} \|A_x^k f\|_{L^2(J,x^{-\frac{1}{2}}\,dx)}.$$

■ For any $f \in L^2(J, x^{-\frac{1}{2}}\,dx)$ and any number $v > 0$, let us define the function

$$P_v(f)(x) = F^{-1}\big(M(f)(\lambda)\chi_v(\lambda)\big),$$

where $\chi_v(\lambda)$ is the function defined by $\chi_v(\lambda) = 1$, for $|\lambda| \leq v$ and $\chi_v(\lambda) = 0$, for $|\lambda| > v$, F^{-1} is the inverse Mehler Fock-Cliffaurd transform. One can easily prove that the function $P_v(f)$ is infinitely differentiable and belongs to all classes W_2^m.

Lemma 3.4. *For any fuction $f \in L^2(J, x^{-\frac{1}{2}}\,dx)$, then:*

$$\|f - p_v(f)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq c_3 \|\Delta_{\frac{1}{v}}^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)}.$$

Proof. By using the Plancherel's and Parseval's relations we get that:

$$\|f - p_v(f)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} = \|(1-\chi_v(\lambda))M(f)(\lambda)\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

so:

$$\|f - p_v(f)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} = \left\|\frac{(1-\chi_v(\lambda))}{\left[1 + P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}}) + P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}})\right]^m}\left[1 + P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{h}}) + P_{-\frac{1}{2}}(\frac{2}{\sqrt{h}})\right]^m M(f)(\lambda)\right\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

We have that

$$\sup_{\lambda>0} \frac{(1-\chi_v(\lambda))}{\left[1 + P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}}) + P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}})\right]^m} \leq 1.$$

Because for every $\lambda > 0$ we have that $(1-\chi_v(\lambda)) \leq 1$ and by Lemma 3.1 we have that:

$$1 + P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}}) + P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}}) \geq 1.$$

Then:

$$\|f - p_v(f)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq \| \left[1 + P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{h}}) + P_{-\frac{1}{2}}(\frac{2}{\sqrt{h}})\right]^m M(f)(\lambda)\right\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

So:

$$\|f - p_v(f)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq c_3 \|\Delta_{\frac{1}{v}}^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)},$$

with $c_3 = 1$. ■

Corollary 3.5.

$$\|f - p_v(f)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq c_3 \omega_m(f, \frac{1}{v}).$$

Lemma 3.6.

$$\|A_x^m\big(P_v(f)\big)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq c_4 v^{2m} \|\Delta_{\frac{1}{v}}^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)}.$$

Proof. By the Plancherel's and Parseval's relations we get that:

$$\|A_x^m\big(P_v(f)\big)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} = \|M(A_x^m\big(P_v(f)\big))(\lambda)\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

So

$$\|A_x^m\big(P_v(f)\big)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} = \|(\lambda+\frac{1}{4})^m M(P_v(f))(\lambda)\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

It follows that:

$$\|A_x^m\big(P_v(f)\big)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} = \|(\lambda+\frac{1}{4})^m \chi_v(\lambda) M(f)(\lambda)\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

Then:

$$\|A_x^m\big(P_v(f)\big)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} = \left\|\frac{(\lambda+\frac{1}{4})^m\chi_v(\lambda}{\Big[1+P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}})+P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}})\Big]^m}\Big[1+P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}})+P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}})\Big]^m \chi_v(\lambda)M(f)(\lambda)\right\|_{L^2(\mathbb{R}^+;\frac{1}{2}\tanh(\pi\sqrt{\lambda})d\lambda)}.$$

We have that:

$$\sup_{\lambda>0}\frac{(\lambda+\frac{1}{4})^m\chi_v(\lambda)}{\Big[1+P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}})+P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}})\Big]^m} = \sup_{\lambda<v}\frac{(\lambda+\frac{1}{4})^m}{\Big[1+P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}})+P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}})\Big]^m}.$$

Then:

$$\sup_{\lambda>0}\frac{(\lambda+\frac{1}{4})^m\chi_v(\lambda)}{\Big[1+P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}})+P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}})\Big]^m} = v^{2m}\sup_{\lambda<v}\frac{\frac{(\lambda+\frac{1}{4})^m}{v^{2m}}}{\Big[1+P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}})+P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}})\Big]^m}.$$

Since

$$1+P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}})+P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}}) \geq 1.$$

Then:

$$\sup_{\lambda>0}\frac{(\lambda+\frac{1}{4})^m\chi_v(\lambda)}{\Big[1+P_{i\sqrt{\lambda}-\frac{1}{2}}(\frac{2}{\sqrt{v}})+P_{-\frac{1}{2}}(\frac{2}{\sqrt{v}})\Big]^m} \leq v^{2m}\sup_{\lambda<v}\frac{(\lambda+\frac{1}{4})^m}{v^{2m}}.$$

Let

$$c_4 = \sup_{\lambda<v}\frac{(\lambda+\frac{1}{4})^m}{v^{2m}}.$$

Then

$$\|A_x^m\big(P_v(f)\big)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq c_4 v^{2m}\|\Delta_{\frac{1}{v}}^m f(x)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)}.$$

■

Corollary 3.7.

$$\|A_x^m\big(P_v(f)\big)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq c_4 v^{2m}\omega_m(f,\frac{1}{v}).$$

Theorem 3.8. *One can find positive numbers c_5 and c_6 which the inequality*

$$c_5\omega_m(f,\delta) \leq K_m(f,\delta^{2m}) \leq c_6\omega_m(f,\delta).$$

$f \in L^2(J,x^{-\frac{1}{2}}dx)$ and $\delta > \frac{1}{4}$.

Proof. Firstly prove of the inequality

$$c_5\omega_m(f,\delta) \leq K_m(f,\delta^{2m}).$$

Let $h \in]\frac{1}{4},\delta]$ and $g \in W_2^m$, using Lemma 3.2 and Lemma 3.3 we have:

$$\|\Delta_h^m f\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq \|\Delta_h^m f - \Delta_h^m g\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} + \|\Delta_h^m g\|_{L^2(J,x^{-\frac{1}{2}}\,dx)}.$$

So

$$\|\Delta_h^m f\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \leq c_1\|f-g\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} + c_2 h^{2m}\|A_x^m g\|_{L^2(J,x^{-\frac{1}{2}}\,dx)}.$$

It follows that:

$$\|\Delta_h^m f\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} \le c_7\Big(\|f-g\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} + \delta^{2m}\|A_x^m g\|_{L^2(J,x^{-\frac{1}{2}}\,dx)}\Big),$$

with $c_7 = max(c_1, c_2)$. Calculating the supremum with rspect to $h \in]\frac{1}{4}, \delta]$ and the infimum with respact all possible functions $g \in W_2^m$, we obtain

$$\omega_m(f,\delta) \le c_7 K_m(f, \delta^{2m}).$$

Which shows that

$$c_5\omega_m(f,\delta) \le K_m(f,\delta^{2m}),$$

with $c_5 = c_7^{-1}$.
Now, we prove the inequality:

$$K_m(f,\delta^{2m}) \le c_6\omega_m(f,\delta).$$

Since $P_v(f) \in W_2^m$, by the definition of a K-functional we have:

$$K_m(f,\delta^{2m}) \le \|f - P_v(f)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)} + c_2 h^{2m}\|A_x^m P_v(f)\|_{L^2(J,x^{-\frac{1}{2}}\,dx)}.$$

Using collaires 3.5 and 3.7 we obtain

$$K_m(f,\delta^{2m}) \le c_3\omega_m(f,\frac{1}{v}) + c_4 v^{2m}\delta^{2m}\omega_m(f,\frac{1}{v}).$$

Then

$$K_m(f,\delta^{2m}) \le c_3\omega_m(f,\frac{1}{v}) + c_4(v\delta)^{2m}\omega_m(f,\frac{1}{v}).$$

Since v is ana arbitrary positive value, choosing $v = \frac{1}{\delta}$, we obtain:

$$K_m(f,\delta^{2m}) \le c_6\omega_m(f,\delta).$$

with $c_6 = c_3 + c_4$,
Which shows the theorem. ■

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