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ON BIVARIATE FUBINI-FIBONACCI POLYNOMIALS AND NUMBERS

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ABSTRACT. In this paper, we introduce a new type of Fubini polynomials called bivariate Fubini-Fibonacci polynomials, denoted by $F_n^f(x,y)$ using golden exponential function, via generating function

$$\frac{e_f^{xt}}{1-y(e_f^t-1)}=\sum_{n=0}^{\infty}F_n^f(x,y)\frac{t^n}{f_n!}.$$

We then derive some fundamental properties of these polynomials including addition formula, explicit formula, recurrence relations, and derivative and integral formulas. Moreover, we establish the relationships between Fubini-Fibonacci polynomials and other Fibonacci polynomials and obtain the harmonic-based f-exponential generating functions of Fubini-Fibonacci polynomials and numbers.

1. INTRODUCTION

The study of special polynomials such as the Fubini polynomials has garnered significant interest among numerous researchers, with some investigating their properties in relation to other types of polynomials. The classical Fubini polynomials or geometric polynomials $F_n(y)$ are defined in [2] by

$$F_n(y)=\sum_{k=0}^{n}S_2(n,k)k!y^k, \tag{1.1}$$

where $S_2(n,k)$ is the Stirling numbers of the second kind [3,4]. The polynomials $F_n(y)$ satisfy the generating function:

$$\frac{1}{1-y(e^t-1)}=\sum_{n=0}^{\infty}F_n(y)\frac{t^n}{n!}, \tag{1.2}$$

and the recurrence relation

$$F_{n+1}(y)=y\frac{d}{dy}[F_n(y)+yF_n(y)] \quad \text{(see [5])}.$$

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By setting $y = 1$ in (1.1), we obtain the n^{th} *Fubini number* (sometimes called *ordered Bell number*) F_n, defined by

$$F_n(1) := F_n = \sum_{k=0}^{n} S_2(n,k)k!$$

Combinatorically, the number F_n counts all the possible set partitions of an n-element set such that the order of the blocks matters.

In [6], Kargin introduced the bivariate Fubini polynomials, denoted by $F_n(x,y)$, as a generalization of the classical Fubini polynomials, which are defined by means of generating function

$$\frac{e^{xt}}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!}. \quad \text{(see [7,8])} \tag{1.3}$$

When $x = 0$, (1.3) reduces to classical Fubini polynomials.

Recent studies on new families of Fubini numbers and polynomials have introduced several classes including the q-class Fubini polynomials [10], higher-order central Fubini polynomials of two variables [11], bivariate Apostol-Fubini polynomials of higher order [9], poly-fubini polynomials of two variables [16], w-torsion Fubini polynomials [12], Fubini- type numbers and polynomials via generating functions and p-adic integral approach [13], degenerate Fubini polynomials [8, 14], and degenerate central Fubini polynomials [15].

Now, we will present some necessary information about golden calculus [17,20] that we will use throughout the article. Golden calculus explores concepts parallel to the classical calculus. It is based on Fibonacci numbers and Binet's formula of Fibonacci numbers. The Fibonacci numbers satisfy the recurrence relation [21]:

$$f_n = f_{n-1} + f_{n-2}, \qquad \text{for } n{\geq}2$$

where the first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$. These numbers can be expressed explicitly using Binet's formula [21]:

$$f_n = \frac{\varphi^n - {\varphi'}^n}{\varphi - \varphi'},$$

where φ, φ' are positive and negative roots of equation $x^2 - x - 1 = 0$. These roots explicitly are

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \varphi' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi}.$$

Number φ is known as the *golden ratio* or the *golden section*. The *golden derivative* is defined as the operator

$$f_{x\frac{d}{dx}} = \frac{\varphi^{x\frac{d}{dx}} - {\varphi'}^{x\frac{d}{dx}}}{\varphi - \varphi'} = \left[x\frac{d}{dx}\right]_f.$$

The golden derivative for any function $f(x)$ is given as

$$f_{x\frac{d}{dx}}f(x) = D_f f(x) = \frac{f(\varphi x) - f(-\frac{x}{\varphi})}{(\varphi + \frac{1}{\varphi})x}.$$

The golden derivative operator is a linear operator since for every pair of functions f and g and scalar c, the following properties hold;

$$D_f^x(f(x) + g(x)) = D_f^x(f(x)) + D_f^x(g(x))$$
$$D_f^x(cf(x)) = cD_f^x(f(x)).$$

The golden binomial and its expansion in terms of fibonomial coefficients is derived. Let us first introduce the f-factorial in which defined as

$$f_n! = f_n \cdot f_{n-1} \cdot f_{n-2} \cdots f_1,$$

where $f_0! = 1$. The *golden binomial coefficients* are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_f = \frac{[n]_f!}{[n-k]_f![k]_f!} = \frac{f_n!}{f_{n-k}!f_k!}. \tag{1.4}$$

with n and k being nonnegative integers, $n \geq k$ and are called the *fibonomials*. The golden binomial can be expanded

$$(x+y)_f^n \equiv (x+y)_{\varphi,-\frac{1}{\varphi}}^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_f (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k. \tag{1.5}$$

The golden exponential functions are defined as

$$e_f^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{f_n!}; \qquad E_f^x \equiv \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{x^n}{f_n!},$$

These golden exponential functions have golden derivatives given as

$$D_f e_f^{kx} = k e_f^{kx},$$
$$D_f E_f^{kx} = k E_f^{-kx},$$

for arbitrary constant k (or f-periodic function). The following identity holds:

$$e_f^x E_f^y = e_f^{(x+y)_f},$$

where,

$$e_f^{(x+y)_f} = \sum_{n=0}^{\infty} \frac{(x+y)_f^n}{f_n!}.$$

Recently, Pashaev and Ozvatan [1, 17] introduced generating functions for Bernoulli-Fibonacci polynomials using golden exponential function. The generating functions for Bernoulli- Fibonacci polynomials, denoted by $B_n^f(x)$, are defined by

$$\frac{t e_f^{xt}}{e_f^t - 1} = \sum_{n=0}^{\infty} B_n^f(x) \frac{t^n}{f_n!}. \tag{1.6}$$

When $x = 0$, (1.6) reduces to the generating function of Bernoulli-Fibonacci numbers $B_n^f := B_n^f(0)$ given by

$$\frac{t}{e_f^t - 1} = \sum_{n=0}^{\infty} B_n^f \frac{t^n}{f_n!}.$$

On the other hand, Kus et al. [18] introduced generating functions for Euler- Fibonacci polynomials $E_n^f(x)$ given by

$$\frac{2 e_f^{xt}}{e_f^t + 1} = \sum_{n=0}^{\infty} E_n^f(x) \frac{t^n}{f_n!}. \tag{1.7}$$

Setting $x = 0$, equation (1.7) reduces to the generating function of Euler-Fibonacci numbers $E_n^f := E_n^f(0)$ given by

$$\frac{2}{e_f^t + 1} = \sum_{n=0}^{\infty} E_n^f \frac{t^n}{t_n!}.$$

In parallel to equations (1.6) and (1.7), the generating functions of Genocchi-Fibonacci polynomials $G_n^f(x)$ are defined as

$$\frac{2te_f^{xt}}{e_f^t+1}=\sum_{n=0}^{\infty}G_n^f(x)\frac{t^n}{f_n!}. \tag{1.8}$$

If we take $x=0$ in (1.8), we get the generating function of Genocchi-Fibonacci numbers $G_n^f := G_n^f(0)$ given by

$$\frac{2t}{e_f^t+1}=\sum_{n=0}^{\infty}G_n^f\frac{t^n}{t_n!}.$$

In this paper, we will introduce a new class of Fubini polynomials and numbers using the golden exponential function, and establish various fundamental properties and identities.

2. BIVARIATE FUBINI-FIBONACCI POLYNOMIALS AND NUMBERS

We now introduce the bivariate Fubini-Fibonacci polynomials, univariate Fubini-Fibonacci polynomials and Fubini-Fibonacci numbers using golden exponential functions.

Definition 2.1. The *bivariate Fubini-Fibonacci polynomials*, denoted by $F_n^f(x,y)$, are defined through the generating function:

$$\frac{e_f^{xt}}{1-y(e_f^t-1)}=\sum_{n=0}^{\infty}F_n^f(x,y)\frac{t^n}{f_n!}. \tag{2.1}$$

When $x=0$ in (2.1), we obtain the *univariate Fubini- Fibonacci polynomials* $F_n^f(y)=F_n^f(0,y)$. That is,

$$\frac{1}{1-y(e_f^t-1)}=\sum_{n=0}^{\infty}F_n^f(y)\frac{t^n}{f_n!}.$$

Also, setting $x=0$ and $y=1$ in (2.1), the *Fubini-Fibonacci numbers* $F_n^f=F_n^f(0,1)$ are obtained with generating function,

$$\frac{1}{2-e_f^t}=\sum_{n=0}^{\infty}F_n^f\frac{t^n}{f_n!}.$$

The bivariate Fubini-Fibonacci polynomials can be expressed in terms of univariate Fubini-Fibonacci polynomials.

Theorem 2.2. *For $n\geq 0$,*

$$F_n^f(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_f F_k^f(y)x^{n-k}. \tag{2.2}$$

Proof. Using Definition 2.1, we have

$$\begin{aligned}\sum_{n=0}^{\infty}F_n^f(x,y)\frac{t^n}{f_n!} &= \frac{e_f^{xt}}{1-y(e_f^t-1)}\\ &= \sum_{n=0}^{\infty}F_n^f(y)\frac{t^n}{f_n!}\sum_{n=0}^{\infty}x^n\frac{t^n}{f_n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_f F_k^f(y)x^{n-k}\frac{t^n}{f_n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{f_n!}$ gives (2.2). □

Theorem 2.3. *For $n \geq 1$, bivariate Fubini-Fibonacci polynomials can be calculated recursively by*

$$F_n^f(x,y) = x^n + y \sum_{l=0}^{n-1} \left[\begin{array}{c} n \\ l \end{array} \right]_f F_l^f(x,y). \tag{2.3}$$

Proof. Starting from the generating function of bivariate Fubini-fibonacci polynomials,

$$\frac{e_f^{xt}}{1-y(e_f^t-1)} = \sum_{n=0}^{\infty} F_n^f(x,y) \frac{t^n}{f_n!}.$$

If we multiply this equality by $(1-y(e_f^t-1))$, we have

$$\begin{aligned} \frac{e_f^{xt}}{1-y(e_f^t-1)} \cdot (1-y(e_f^t-1)) &= \sum_{n=0}^{\infty} F_n^f(x,y) \frac{t^n}{f_n!} \cdot (1-y(e_f^t-1)) \\ &= (y+1) \sum_{n=0}^{\infty} F_n^f(x,y) \frac{t^n}{f_n!} - y(e_f^t) \sum_{n=0}^{\infty} F_n^f(x,y) \frac{t^n}{f_n!}. \end{aligned}$$

Thus,

$$e_f^{xt} = \sum_{n=0}^{\infty} \left[(y+1) F_n^f(x,y) - y(e_f^t) F_n^f(x,y) \right] \frac{t^n}{f_n!}.$$

But

$$\begin{aligned} y \sum_{n=0}^{\infty} (e_f^t) F_n^f(x,y) \frac{t^n}{f_n!} &= y \sum_{n=0}^{\infty} F_n^f(x,y) \frac{t^n}{f_n!} \sum_{n=0}^{\infty} \frac{t^n}{f_n!} \\ &= y \sum_{n=0}^{\infty} \sum_{l=0}^{n} \left[\begin{array}{c} n \\ l \end{array} \right]_f F_l^f(x,y) \frac{t^n}{f_n!}. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} x^n \frac{t^n}{f_n!} = \sum_{n=0}^{\infty} \left[(y+1) F_n^f(x,y) - y \sum_{l=0}^{n} \left[\begin{array}{c} n \\ l \end{array} \right]_f F_l^f(x,y) \right] \frac{t^n}{f_n!}.$$

Comparing the coefficients of $\frac{t^n}{f_n!}$, we obtain

$$(y+1) F_n^f(x,y) = x^n + y \sum_{l=0}^{n} \left[\begin{array}{c} n \\ l \end{array} \right]_f F_l^f(x,y),$$

or

$$F_n^f(x,y) = x^n + y \sum_{l=0}^{n-1} \left[\begin{array}{c} n \\ l \end{array} \right]_f F_l^f(x,y).$$

□

Example 2.4. Applying Theorem 2.3, we obtain the first few terms of bivariate Fubini-Fibonacci polynomials. The first few terms of bivariate Fubini-Fibonacci polynomials are as follows:

$$\begin{aligned} &F_0^f(x,y) = 1, \\ &F_1^f(x,y) = x + y, \\ &F_2^f(x,y) = x^2 + yx + (y + y^2), \\ &F_3^f(x,y) = x^3 + (2y)x^2 + (2y + 2y^2)x + (y + 4y^2 + 2y^3), \\ &F_4^f(x,y) = x^4 + (3y)x^3 + (6y + 6y^2)x^2 + (3y + 12y^2 + 6y^3)x + (y + 12y^2 + 12y^3 + 6y^4). \end{aligned}$$

Setting $x = 0$ as special case, we obtain the first few terms of univariate Fubini- Fibonacci polynomials. For the first few terms of polynomials we have

$$\begin{aligned}
F_0^f(y) &= 1,\\
F_1^f(y) &= y,\\
F_2^f(y) &= y + y^2,\\
F_3^f(y) &= y + 4y^2 + 2y^3,\\
F_4^f(y) &= y + 12y^2 + 12y^3 + 6y^4.
\end{aligned}$$

Also, setting $x = 0$ and $y = 1$, we obtain the first few Fubini-Fibonaci numbers

$$F_0^f = 1, \qquad F_1^f = 1, \qquad F_2^f = 2, \qquad F_3^f = 7, \qquad F_4^f = 31.$$

The polynomials $F_n^f(x, y)$ satisfy the addition formula in the next theorem.

Theorem 2.5. *For $n \geq 0$,*

$$\begin{aligned}
F_n^f((x+z)_f, y) &= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_f F_k^f(x, y)(-1)^{\frac{(n-k)(n-k-1)}{2}} z^{n-k} && (2.4)\\
&= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_f F_k^f(z, y)(-1)^{\frac{(n-k)(n-k-1)}{2}} x^{n-k}. && (2.5)
\end{aligned}$$

Proof. Using Definition 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n^f((x+z)_f, y)\frac{t^n}{f_n!} &= \frac{e_f^{((x+z)_f)t}}{1 - y(e_f^t - 1)}\\
&= \sum_{n=0}^{\infty} F_n^f(x, y)\frac{t^n}{f_n!} \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} z^n \frac{t^n}{f_n!}\\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_f F_k^f(x, y)(-1)^{\frac{(n-k)(n-k-1)}{2}} z^{n-k} \frac{t^n}{f_n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{f_n!}$ gives (2.4), and interchanging the roles of x and z in (2.4) gives (2.5). □

Setting $z = 1$ in (2.5), we get a recurrence relation of the bivariate Fubini-Fibonacci polynomials.

Corollary 2.6. For $n \geq 0$,

$$F_n^f((x+1)_f, y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_f F_k^f(1, y)(-1)^{\frac{(n-k)(n-k-1)}{2}} x^{n-k}. \tag{2.6}$$

The bivariate Fubini-Fibonacci polynomials satisfy the following derivative and integral properties.

Theorem 2.7. *For $n \geq 1$,*

$$D_f^x F_n^f(x, y) = f_n F_{n-1}^f(x, y). \tag{2.7}$$

$$\int F_n^f(x, y) d_f x = \frac{1}{f_{n+1}} F_{n+1}^f(x, y) + C, \tag{2.8}$$

for any constant C.

Proof. It follows from Theorem 2.2 that

$$\begin{aligned} D_f^x F_n^f(x,y) &= D_f^x \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_f F_k^f(y) x^{n-k} \\ &= \sum_{k=0}^{n-1} \left[\begin{array}{c} n \\ k \end{array} \right]_f f_{n-k} F_k^f(y) x^{n-k-1} \\ &= f_n \sum_{k=0}^{n-1} \left[\begin{array}{c} n-1 \\ k \end{array} \right]_f F_k^f(y) x^{(n-1)-k} \\ &= f_n F_{n-1}^f(x,y). \end{aligned}$$

For the integral property, we apply the derivative property,

$$D_f^x F_{n+1}^f(x,y) = f_{n+1} F_n^f(x,y). \tag{2.9}$$

Integrating both sides of equation (2.9), we obtain

$$F_{n+1}^f(x,y) = \int f_{n+1} F_n^f(x,y) d_f x.$$

Hence,

$$\int F_n^f(x,y) d_f x = \frac{1}{f_{n+1}} F_{n+1}^f(x,y) + C.$$

□

3. RELATIONS INVOLVING FUBINI-FIBONACCI POLYNOMIALS AND OTHER FIBONACCI TYPE POLYNOMIALS

The next theorem gives the relationships involving Fubini-Fibonacci polynomials and other Fibonacci polynomials.

Theorem 3.1. *The following relations hold:*

$$F_{n-1}^f(x,y) = \frac{1}{f_n} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_f B_k^f(x) \left[F_{n-k}^f(1,y) - F_{n-k}^f(y) \right], \qquad \forall n \geq 1. \tag{3.1}$$

$$F_n^f(x,y) = \frac{1}{2} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_f E_k^f(x) \left[F_{n-k}^f(1,y) + F_{n-k}^f(y) \right], \qquad \forall n \geq 0. \tag{3.2}$$

$$F_{n-1}^f(x,y) = \frac{1}{2f_n} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_f G_k^f(x) \left[F_{n-k}^f(1,y) + F_{n-k}^f(y) \right], \qquad \forall n \geq 1. \tag{3.3}$$

Proof. Using the Definition 2.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} F_n^f(x,y) \frac{t^n}{f_n!} &= \frac{e_f^{xt}}{1 - y(e_f^t - 1)} \\ &= \frac{1}{t} \left[\frac{t e_f^{xt}}{(e_f^t - 1)} \left(\frac{e_f^t}{1 - y(e_f^t - 1)} - \frac{1}{1 - y(e_f^t - 1)} \right) \right]. \end{aligned}$$

This gives

$$\begin{aligned}\sum_{n=0}^{\infty}F_n^f(x,y)\frac{t^{n+1}}{f_n!}&=\sum_{n=0}^{\infty}B_n^f(x)\frac{t^n}{f_n!}\left[\sum_{n=0}^{\infty}\left(F_n^f(1,y)-F_n^f(y)\right)\frac{t^n}{f_n!}\right]\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_f B_k^f(x)\left(F_{n-k}^f(1,y)-F_{n-k}^f(y)\right)\frac{t^n}{f_n!}.\end{aligned}\tag{3.4}$$

Taking the golden derivative with respect to t on both sides of (3.4), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}f_{n+1}F_n^f(x,y)\frac{t^n}{f_n!}&=\sum_{n=1}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_f f_nB_k^f(x)\left(F_{n-k}^f(1,y)-F_{n-k}^f(y)\right)\frac{t^{n-1}}{f_n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_f B_k^f(x)\left(F_{n+1-k}^f(1,y)-F_{n+1-k}^f(y)\right)\frac{t^n}{f_n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{f_n!}$, we get

$$f_{n+1}F_n^f(x,y)=\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_f B_k^f(x)\left(F_{n+1-k}^f(1,y)-F_{n+1-k}^f(y)\right),\qquad\forall n\geq 0,$$

or equivalently,

$$F_{n-1}^f(x,y)=\frac{1}{f_n}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_f B_k^f(x)\left(F_{n-k}^f(1,y)-F_{n-k}^f(y)\right),\qquad\forall n\geq 1.$$

Relations (3.2) and (3.3) can be derived analogously. □

4. HARMONIC-BASED f-EXPONENTIAL GENERATING FUNCTION

In this section, we obtain harmonic-based f-exponential generating function of bivariate Fubini-Fibonacci numbers, univariate Fubini-Fibonacci polynomials and Fubini-Fibonacci numbers. Let us first present some definitions that we will use to prove the next theorem. In [19] Tuglu et al. defined the harmonic Fibonacci numbers as follows

$$\mathbb{F}_n=\sum_{k=1}^{n}\frac{1}{f_k},$$

where $\mathbb{F}_0=0$.

Furthermore, Kus et al. [18] defined the harmonic-based f-exponential generating function as follows

$$e^{f_{\mathbb{F}}t}=1+\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^n}{f_n!},\tag{4.1}$$

where

$$\partial_{f,t}e^{f_{\mathbb{F}}t}=1+\sum_{n=1}^{\infty}\mathbb{F}_{n+1}\frac{t^n}{f_n!}.\tag{4.2}$$

In addition to these, Kus et al. [18] obtained the harmonic-based f- exponential generating function of Bernoulli-Fibonacci and Euler-Fibonacci polynomials.

Theorem 4.1. *The bivariate Fubini-Fibonacci polynomials satisfy the harmonic-based f-exponential generating function*

$$\sum_{n=0}^{\infty}F_n^f(x,y)\frac{t^n}{f_n!}=\frac{e_f^{xt}}{1-yt(\partial_{f,t}e^{f_{\mathbb{F}}t}-e^{f_{\mathbb{F}}t}+1)}.\tag{4.3}$$

Proof. By using the harmonic-based f-exponential function $e^{f_{\mathbb{F}}t}$ and the exponential generating functions of bivariate Fubini-Fibonacci polynomials, we get

$$\begin{aligned}
1-y(e_f^t-1) &= 1-y\left(\sum_{n=0}^{\infty}\frac{t^n}{f_n!}-1\right)\\
&= 1-y\left(1+\sum_{n=1}^{\infty}\frac{t^n}{f_n!}-1\right)\\
&= 1-y\left(t\sum_{n=0}^{\infty}\frac{1}{f_{n+1}}\frac{t^n}{f_n!}\right)\\
&= 1-yt\left(\sum_{n=0}^{\infty}(\mathbb{F}_{n+1}-\mathbb{F}_n)\frac{t^n}{f_n!}\right)\\
&= 1-yt\left(1+\sum_{n=1}^{\infty}\mathbb{F}_{n+1}\frac{t^n}{f_n!}-\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^n}{f_n!}\right)\\
&= 1-yt(\partial_{f,t}e^{f_{\mathbb{F}}t}-e^{f_{\mathbb{F}}t}+1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty}F_n^f(x,y)\frac{t^n}{f_n!} &= \frac{e_f^{xt}}{1-y(e_f^t-1)}\\
&= \frac{e_f^{xt}}{1-yt(\partial_{f,t}e^{f_{\mathbb{F}}t}-e^{f_{\mathbb{F}}t}+1)}.
\end{aligned}$$

□

Taking $x=0$ in (4.3), we obtain the harmonic-based f-exponential generating function of univariate Fubini-Fibonacci polynomials in the next corollary.

Corollary 4.2. The harmonic-based f-exponential generating function of univariate Fubini-Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty}F_n^f(y)\frac{t^n}{f_n!} = \frac{1}{1-yt(\partial_{f,t}e^{f_{\mathbb{F}}t}-e^{f_{\mathbb{F}}t}+1)}. \tag{4.4}$$

Also, setting $x=0$ and $y=1$ in (4.3), we obtain the harmonic-based f-exponential generating function of Fubini-Fibonacci numbers in the next corollary.

Corollary 4.3. The harmonic-based f-exponential generating function of Fubini-Fibonacci numbers is given by

$$\sum_{n=0}^{\infty}F_n^f\frac{t^n}{f_n!} = \frac{1}{1-t(\partial_{f,t}e^{f_{\mathbb{F}}t}-e^{f_{\mathbb{F}}t}+1)}.$$

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