

FUZZY QUASI-UNIFORM ENTROPY: EXTENDING TOPOLOGICAL ENTROPY TO FUZZY QUASI-METRIC SPACES

JOHN PAMBA*, HOPE SABAO, AND ISAAC DANIEL TEMBO

ABSTRACT. In this paper, we extend the notions of uniform and quasi-uniform entropy for uniformly continuous self-maps to the setting of fuzzy quasi-metric spaces. We introduce a new dynamical invariant, called *fuzzy quasi-uniform entropy*, which simultaneously accounts for asymmetry inherent in quasi-metrics and imprecision arising from fuzzy structures. Working within the framework of uniformly continuous self-maps on fuzzy quasi-metric spaces, we define this entropy using Bowen-type fuzzy quasi-balls and establish its fundamental properties. Several classical results from uniform and quasi-uniform entropy theory are shown to extend naturally to this fuzzy quasi-metric context. In particular, under a suitable restriction on the fuzziness scale parameter, the fuzzy quasi-uniform entropy dominates the classical quasi-uniform entropy, thereby capturing finer distinctions in orbit complexity. We further prove that this entropy is monotone with respect to the fuzziness parameter, invariant under topological conjugacy, and converges to classical topological entropy in the metric case. Illustrative examples demonstrate the effectiveness of the theory, including situations where fuzzy quasi-uniform entropy detects nontrivial dynamical behavior even when classical quasi-uniform entropy vanishes. These results establish fuzzy quasi-uniform entropy as a robust extension of existing entropy notions for dynamical systems exhibiting asymmetry, uncertainty, or lack of metric symmetry.

1. INTRODUCTION

Entropy is a fundamental concept used to quantify uncertainty across various fields such as Mathematics, Statistical mechanics, Physics, Engineering, Computer Science, Economics and Finance. It quantifies how unpredictable or chaotic the system is and it is often defined by use of open covers or spanning sets or separated sets, depending on the space. Its notion first appeared in the context of thermodynamics in 1865 by Rudolf Clausius as a measure of the irreversible dissipation of energy in thermodynamic processes which was part of his work on the second law of thermodynamics. Claude Shannon introduced the concept of entropy in Mathematics as a fundamental measure of information, uncertainty and randomness in his 1948 paper "A Mathematical Theory of Communication" [18]. Since then, a number of researchers have been motivated to extend a number of results in many fields of mathematics like topology, probability and statistics, functional analysis and measure theory, fuzzy mathematics and metric spaces.

In 1971, Bowen [1] extended the concept of entropy to topological entropy by defining the notion of entropy by using uniformly continuous self-maps on metric spaces. This entropy by Bowen was called uniform entropy or Bowen's entropy, denoted by h_u . Independently, in 1970, Dinaburg also defined the notion of entropy for uniformly continuous self-maps of metric spaces [2]. In particular, see [19] for fundamentals on

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCES, UNIVERSITY OF ZAMBIA, LUSAKA, ZAMBIA
 E-mail addresses: jpamba@zut.edu.zm, hope.sabao@unza.zm, isaactembo@unza.zm.

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*Corresponding author.

topological entropy. See also [3] [4] [9] [12] for more recent works on the concept of topological entropy. Recently, in 2023, Haihambo and Olela-Otafudu extended the work of Bowen [1] from uniform entropy to quasi-uniform entropy for a uniformly continuous self-map ψ of a quasi-metric space (X, q) [9]. It was shown that the quasi-uniform entropy $(h_{QU}(\psi, q))$ is less or equal to the Bowen's uniform entropy $(h_U(\psi, q^s))$ of uniformly continuous self-maps, ψ , and a number of useful results about uniform entropy on metric spaces were extended to quasi-metric spaces.

In 1975, Kramosi and Michálek [11] introduced the concept of fuzzy metric spaces as a generaliaation of statistical(probabilistic) metric spaces. Since then, a lot of concepts have been extended on fuzzy metric spaces. George and Veeramani [6], gave a modified definition of a fuzzy metric, which is usually denoted by GV-fuzzy metric. In 2012, Karamil and Molaei [10] introduced the concept of fuzzy entropy which has helped to explain the complexity of systems created by the self-maps of fuzzy metric spaces.

In 2004, Gregori and Romagosa [8] introduced the concept of fuzzy quasi-metric spaces and proved a number of results in this setting. Recently, in 2025, Pamba et al. [16] investigated the concept of uniformly continuous maps on fuzzy quasi metric spaces and proved a number of important properties.

In this article, we are motivated to extend the concept of uniform entropy $h_{U,q^s}(\psi)$, originally introduced by Bowen [1], and the quasi-uniform entropy $h_{QU}(\psi, q)$, developed by Haihambo and Olela-Otafudu [9], to the setting of fuzzy quasi-metric spaces. This leads us to define a new invariant, the *fuzzy quasi-uniform entropy* $h_{FQU}(\psi, M)$, which generalizes both prior notions by incorporating both asymmetry and fuzziness in the underlying geometry. We adopt the framework of uniformly continuous self-maps on fuzzy quasi-metric spaces and build upon it to define the new entropy. Several classical results from the theories of uniform and quasi-uniform entropy are adapted and extended. In particular, we show that under the condition $t \leq 1 - \epsilon$ for any $\epsilon \in (0, 1)$ and any $t > 0$, we have that $h_{FQU}(\psi) \geq h_{QU}(\psi)$, establishing that the fuzzy extension captures at least as much complexity as its classical counterpart. Additionally, we prove several new properties of the fuzzy quasi-uniform entropy, including its monotonicity with respect to the fuzziness scale parameter t , invariance under topological conjugacy, and convergence to classical topological entropy in the metric case as $t \rightarrow 0$. These results help to situate h_{FQU} within the broader entropy landscape and demonstrate its compatibility with established notions. Illustrative examples are provided to validate the theory and to highlight cases where the fuzzy entropy detects nontrivial dynamics such as positive entropy even in settings where the classical quasi-uniform entropy may be zero.

2. PELIMINARIES

In this section, we present some definitions and concepts from the literature which will be needed in the sequel.

Definition 2.1. [20] Let X be a set and let $q : X \times X \rightarrow [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the non-negative reals. Then, q is called a quasi-metric on X if for all $x, y, z \in X$:

- (i) $q(x, y) = 0$ if and only if $x = y$,
- (ii) $q(x, z) \leq q(x, y) + q(y, z)$ whenever $x, y, z \in X$.

We say that the pair (X, q) is a quasi-metric space.

If q is a quasi-metric on X , then the function q^t defined on $X \times X$ by $q^t(x, y) = q(y, x)$ is also a quasi-metric on X , called the conjugate of q . We define the function q^s on $X \times X$ by $q^s(x, y) = \max\{q(x, y), q^t(x, y)\}$ which is a metric on X . Each quasi-metric on X generates a topology τ_q on X which has as a base the family of open balls $\{B_q(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_q(x, \epsilon) = \{y \in X : q(x, y) < \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. This is a T_0 -topology.

We also recall from [13] that a binary operation $\star : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t -norm if \star satisfies the following conditions: (i) \star is commutative and associative; (ii) \star is continuous; (iii) $x \star 1 = x$; for every $x \in [0, 1]$; (iv) $x \star y \leq w \star z$ whenever $x \leq w$ and $y \leq z$, with $w, x, y, z \in [0, 1]$.

We now define a fuzzy quasi-metric by generalising the notion of a fuzzy metric introduced by George and Veeramani [6].

Definition 2.2. Let X be a non-empty set and let \star be a continuous t -norm. A mapping $M : X \times X \times (0, \infty) \rightarrow (0, 1]$ is called a *GV-fuzzy quasi-metric* on X if for all $x, y, z \in X$ and $t, s > 0$, the following conditions hold:

- (1) $M(x, y, t) > 0$;
- (2) $M(x, x, t) = 1$ for all $t > 0$, and

$$x = y \iff M(x, y, t) = M(y, x, t) = 1 \text{ for all } t > 0;$$

- (3) $M(x, z, t + s) \geq M(x, y, t) \star M(y, z, s)$;
- (4) for all $x, y \in X$, the mapping $t \mapsto M(x, y, t)$ is continuous on $(0, \infty)$.

The triple (X, M, \star) is called a *GV-fuzzy quasi-metric space*. Throughout this paper, a GV-fuzzy quasi-metric space will simply be referred to as a *fuzzy quasi-metric space*.

If (M, \star) is a fuzzy quasi-metric on X , then (M^{-1}, \star) is also a fuzzy quasi-metric on X , where $M^{-1} : X \times X \times (0, \infty) \rightarrow (0, 1]$ is defined by

$$M^{-1}(x, y, t) = M(y, x, t).$$

Moreover, define the mapping $M^i : X \times X \times (0, \infty) \rightarrow (0, 1]$ by

$$M^i(x, y, t) = \min\{M(x, y, t), M(y, x, t)\}.$$

Then (M^i, \star) is a fuzzy metric on X in the sense of George and Veeramani.

Example 2.3. Let (X, q) be a quasi-metric space. Let \star be the product t -norm, defined by $a \star b = ab$ for all $a, b \in [0, 1]$. Define $M_q : X \times X \times (0, \infty) \rightarrow (0, 1]$ by

$$M_q(x, y, t) = \frac{t}{t + q(x, y)}.$$

Then (X, M_q, \star) is a fuzzy quasi-metric space. It is called the *standard fuzzy quasi-metric space* induced by the quasi-metric q .

Example 2.4. Let $X = \mathbb{R}$ and let \star be the product t -norm, $a \star b = a \cdot b$ for each $a, b \in [0, 1]$. Define the function

$$M(x, y, t) = e^{-2 \frac{\max\{y-x, 0\}}{t}}, \quad \text{for all } x, y \in X, t > 0.$$

Then one can easily verify that (X, M, \star) is a fuzzy quasi-metric space. It is asymmetric whenever $x \neq y$ and $x > y$, making it genuinely quasi-metric.

We now define some topological concepts on a fuzzy quasi-metric space as follows:

Definition 2.5. [10] Given a fuzzy quasi-metric space (X, M, \star) , we define the open ball $B_M(x, \epsilon, t)$ for $x \in X, 0 < \epsilon < 1$ and $t > 0$ as the set

$$B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}.$$

For each $x \in X, 0 < \epsilon_1 \leq \epsilon_2 < 1$ and $0 < t_1 \leq t_2$, we have that

$$B_M(x, \epsilon_1, t_1) \subseteq B_M(x, \epsilon_2, t_2)$$

Definition 2.6. [10] We define the topology τ_M on (X, M, \star) as

$$\tau_M = \{A \subseteq X : \text{for each } x \in A \exists \epsilon \in (0, 1), t > 0, \text{ with } B_M(x, \epsilon, t) \subseteq A\}$$

Note that if (M, \star) is a fuzzy quasi-metric, then τ_M is a T_0 topology.

The following definitions can easily be seen as extensions from the fuzzy metric spaces.

Definition 2.7. A fuzzy quasi-metric space (X, M, \star) is bounded if there exists a constant m such that $M(x, y, t) \geq m$ for all $x, y \in X$ and $t > 0$.

Definition 2.8. A fuzzy quasi-metric space (X, M, \star) is said to be totally bounded if for all $\epsilon \in (0, 1)$, there exists a finite set $S = \{x_1, x_2, \dots, x_n\}$ such that for all $t > 0$,

$$X \subseteq \bigcup_{x \in S} B_M(x, \epsilon, t)$$

We now present the concept of uniformly continuous mappings on the fuzzy quasi metric space as follows:

Definition 2.9. [16] The function ψ is uniformly continuous on X if for every $\epsilon \in (0, 1)$ and $t > 0$, there exists $\delta \in (0, 1)$ such that for all $x, y \in X$,

$$M(\psi(x), \psi(y), t) > 1 - \epsilon \text{ whenever } M(x, y, t) > 1 - \delta$$

Definition 2.10. [16] Let (X_1, M_1, \star_1) and (X_2, M_2, \star_2) be fuzzy quasi-metric spaces. A mapping $\psi : X_1 \rightarrow X_2$ is said to be *uniformly continuous* if for every $\epsilon \in (0, 1)$ and $t_\epsilon > 0$, there exist $\delta \in (0, 1)$ and $t_\delta > 0$ such that for all $x, y \in X_1$,

$$M_2(\psi(x), \psi(y), t_\epsilon) > 1 - \epsilon \text{ whenever } M_1(x, y, t_\delta) > 1 - \delta.$$

Lemma 2.11. [16] Let (X, M, \star) be a fuzzy quasi-metric space. Then

- (1) $\psi : (X, M, \star) \rightarrow (X, M, \star)$ is uniformly continuous if and only if $\psi : (X, M^{-1}, \star) \rightarrow (X, M^{-1}, \star)$ is uniformly continuous.
- (2) If $\psi : (X, M, \star) \rightarrow (X, M, \star)$ is uniformly continuous, then $\psi : (X, M^i, \star) \rightarrow (X, M^i, \star)$ is uniformly continuous. The converse does not hold in general.

Follow [16] for the proof of this Lemma and for more results on uniformly continuous maps on fuzzy quasi metric spaces.

We now present our main results in the next section.

3. MAIN RESULTS

In this section, we extend the work of Bowen in [1] and that of Haihambo and Olela-Otafudu in [9] on the notion of uniform entropy for self-maps of metric spaces and quasi-metric spaces, respectively, to fuzzy quasi-metric spaces (X, M, \star) . We shall use the topology τ_M , which is the topology induced on X by the fuzzy quasi-metric M .

3.1. Fuzzy Quasi-Uniform Entropy.

Definition 3.1. Let (X, M, \star) be a fuzzy quasi-metric space and let $\psi : X \rightarrow X$ be a self-map. For $x \in X$, $n \in \mathbb{N}$, $\epsilon \in (0, 1)$, and $t > 0$, the *Bowen fuzzy quasi-ball of order n* centered at x is defined by

$$B_n^M(x, \epsilon, t) = \{y \in X : M(\psi^k(x), \psi^k(y), t) > 1 - \epsilon \text{ for all } k = 0, 1, \dots, n - 1\}.$$

If one works with the reverse (conjugate) fuzzy quasi-metric defined by

$$M^{-1}(x, y, t) = M(y, x, t),$$

then the corresponding Bowen fuzzy quasi-ball is given by

$$B_n^{M^{-1}}(x, \epsilon, t) = \{y \in X : M(\psi^k(y), \psi^k(x), t) > 1 - \epsilon \text{ for all } k = 0, 1, \dots, n-1\}.$$

Let

$$B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}.$$

Define

$$D_n^M(x, \epsilon, t, \psi) = \bigcap_{k=0}^{n-1} \psi^{-k}(B_M(\psi^k(x), \epsilon, t))$$

and

$$\begin{aligned} D_n^{M^i}(x, \epsilon, t, \psi) &= \bigcap_{k=0}^{n-1} \psi^{-k}(B_M(\psi^k(x), \epsilon, t)) \cap \bigcap_{k=0}^{n-1} \psi^{-k}(B_{M^{-1}}(\psi^k(x), \epsilon, t)) \\ &= D_n^M(x, \epsilon, t, \psi) \cap D_n^{M^{-1}}(x, \epsilon, t, \psi). \end{aligned}$$

Clearly, if M is a metric, then $D_n^{M^i}(x, \epsilon, t, \psi)$ coincides with the classical Bowen ball introduced in [1].

Remark 3.2. The sets $D_n^M(x, \epsilon, t, \psi)$ and $D_n^{M^i}(x, \epsilon, t, \psi)$ defined above generalize several well-known notions of dynamical balls appearing in metric, quasi-metric, and fuzzy metric entropy theory. This can be verified as follows.

- (1) Reduction to fuzzy metric balls.

If (X, M, \star) is a fuzzy metric space, then M is symmetric, that is,

$$M(x, y, t) = M(y, x, t) \quad \text{for all } x, y \in X, t > 0.$$

Consequently, $M^{-1} = M$, and hence

$$D_n^{M^i}(x, \epsilon, t, \psi) = D_n^M(x, \epsilon, t, \psi).$$

Thus, in the fuzzy metric setting, $D_n^{M^i}(x, \epsilon, t, \psi)$ coincides with the standard fuzzy Bowen ball used in fuzzy entropy theory.

- (2) Reduction to quasi-metric dynamical balls.

If M is induced by a quasi-metric q , that is,

$$M(x, y, t) = \frac{t}{t + q(x, y)},$$

then the condition

$$M(\psi^k(y), \psi^k(x), t) > 1 - \epsilon$$

is equivalent to

$$q(\psi^k(y), \psi^k(x)) < \frac{\epsilon t}{1 - \epsilon}.$$

Hence $D_n^M(x, \epsilon, t, \psi)$ coincides with the usual (n, ϵ) -quasi-metric dynamical ball defined using q . The set $D_n^{M^i}(x, \epsilon, t, \psi)$ then enforces separation in both forward and backward quasi-metric directions, which is standard in quasi-metric entropy theory.

(3) Reduction to Bowen's topological balls.

If M is induced by a metric d , then $d(x, y) = d(y, x)$ and $M^{-1} = M$. In this case,

$$D_n^{M^i}(x, \epsilon, t, \psi) = \bigcap_{k=0}^{n-1} \left\{ y \in X : d(\psi^k(y), \psi^k(x)) < \frac{\epsilon t}{1 - \epsilon} \right\},$$

which is precisely Bowen's (n, ϵ) -ball as introduced in [1]. Therefore, $D_n^{M^i}(x, \epsilon, t, \psi)$ coincides with the classical dynamical ball used in topological entropy.

Thus, the fuzzy quasi-dynamical balls $D_n^M(x, \epsilon, t, \psi)$ and $D_n^{M^i}(x, \epsilon, t, \psi)$ provide a unified framework that simultaneously extends fuzzy metric balls, quasi-metric balls, and Bowen's topological balls.

Definition 3.3. A fuzzy quasi-metric space (X, M, \star) is said to be compact if for every $t > 0$ and $\epsilon \in (0, 1)$ there is $\{x_1, x_2, \dots, x_n\} \in X$ such that

$$X = \bigcup_{i=1}^n B_M(x_i, \epsilon, t)$$

where

$$B_M(x_i, \epsilon, t) = \{x : M(x, y, t) > 1 - \epsilon\}.$$

Note that compactness implies totally boundedness, but the converse needs completeness. Throughout this paper, we shall denote by $K^*(X)$ the collection of all the non-empty τ_M -compact subsets of X .

Definition 3.4. Let $K \in K^*(X)$. A subset F of X is said to be (n, ϵ, t) -span K with respect to ψ if

$$K \subseteq \bigcup_{x \in F} D_n^M(x, \epsilon, t, \psi).$$

Definition 3.5. Let $K \in K^*(X)$. Then for $\epsilon \in (0, 1)$ and $t > 0$ we define the minimum cardinality of the (n, ϵ, t) -spanning set $F \subseteq X$ by

$$r_n(\epsilon, t, K, \psi) = \min\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^M(x, \epsilon, t, \psi)\}$$

whenever $K \in K^*(X)$, where $|F|$ is the cardinality of F .

Definition 3.6. Let $K \in K^*(X)$. A subset F of X is said to be (n, ϵ, t) -separated with respect to ψ if

$$D_n^{M^i}(x, \epsilon, t, \psi) \cap D_n^{M^i}(y, \epsilon, t, \psi) = \emptyset,$$

for any $x, y \in F$ with $x \neq y$, $\epsilon \in (0, 1)$ and $t > 0$.

For $K \in K^*(X)$, we set

$$s_n(\epsilon, t, K, \psi) = \max\{|F| : F \subseteq X \text{ and } F \text{ is } (n, \epsilon, t)\text{-separated with respect to } \psi\}$$

Since K is τ_M -compact, then the quantities $r_n(\epsilon, t, K, \psi)$ and $s_n(\epsilon, t, K, \psi)$ are finite and well defined.

We now define the fuzzy quasi-uniform entropy of ψ with respect to K .

Definition 3.7. Let (X, M, \star) be a fuzzy quasi-metric space and $\psi : (X, M, \star) \rightarrow (X, M, \star)$ be a uniformly continuous self-map. Then the fuzzy quasi-uniform entropy of ψ with respect to $K \in K^*(X)$ is defined by

$$h_r(\epsilon, t, K, \psi) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup \frac{\log r_n(\epsilon, t, K, \psi)}{n}$$

and

$$h_s(\epsilon, t, K, \psi) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup \frac{\log s_n(\epsilon, t, K, \psi)}{n}$$

for $x, y \in F$ where $F \subseteq X$.

We shall also write

$$h_r(\epsilon, t, K, \psi) = \lim_{\epsilon \rightarrow 0} r(\epsilon, t, K, \psi) \text{ and } h_s(\epsilon, t, K, \psi) = \lim_{\epsilon \rightarrow 0} s(\epsilon, t, K, \psi)$$

where

$$r(\epsilon, t, K, \psi) = \lim_{n \rightarrow \infty} \sup \frac{\log r_n(\epsilon, t, K, \psi)}{n} \text{ and } s(\epsilon, t, K, \psi) = \lim_{n \rightarrow \infty} \sup \frac{\log s_n(\epsilon, t, K, \psi)}{n},$$

respectively.

Definition 3.8. Let (X, M, \star) be a fuzzy quasi-metric space and

$\psi : (X, M, \star) \rightarrow (X, M, \star)$ be a uniformly continuous self-map. Let $K \in K^*(X)$. Then

$$h_{FQU}(K, t, \psi) = h_r(K, t, \psi) = h_s(K, t, \psi),$$

for $t > 0$, is the fuzzy quasi-uniform entropy of ψ with respect to K .

One can easily check that if (X, M, \star) is a quasi-metric space, or just a metric space, then the above definition of entropy coincides with that of quasi-uniform entropy, h_{QU} , or uniform entropy, h_U , given by Haihambo and Olera-Otafudu [9] and Bowen [1], respectively. Thus, this definition naturally extends the classical metric entropy when M is induced by a crisp metric. The fuzziness and asymmetry of M allow for more refined descriptions of dynamical behavior in systems where imprecision or directionality of proximity matters.

We now establish that the above definition is well-posed, i.e meaning it does not depend on arbitrary choices and behaves consistently with respect to the fuzzy quasi-metric structure.

Proposition 3.9. Let (X, M, \star) be a fuzzy quasi-metric space, and let $\psi : X \rightarrow X$ be a uniformly continuous self-map. Let $K \in K^*(X)$. Then, for any fixed $t > 0$, the fuzzy quasi-uniform entropy $h_{FQU}(K, t, \psi)$ defined via (n, ϵ, t) -separated sets is well-defined.

Proof. Let $\epsilon \in (0, 1)$ and $n \in \mathbb{N}$. A finite subset $E \subseteq K$ is called (n, ϵ, t) -separated if for all distinct $x, y \in E$, there exists $0 \leq k < n$ such that

$$M(\psi^k x, \psi^k y, t) < 1 - \epsilon.$$

This condition captures the fuzzy distinguishability of orbits within K at precision level t .

Define $s_n(\epsilon, t, K, \psi)$ to be the maximal cardinality of an (n, ϵ, t) -separated subset of K . Since K is compact and ψ is uniformly continuous, it follows that

$$s_n(\epsilon, t, K, \psi) < \infty$$

for all $n \in \mathbb{N}$. An infinite separated set would contradict compactness, as it would imply the existence of infinitely many mutually "separated" points, violating total boundedness.

We now claim that the sequence $\log s_n(\epsilon, t, K, \psi)$ is subadditive. Specifically,

$$s_{m+n}(\epsilon, t, K, \psi) \leq s_m(\epsilon, t, K, \psi) \cdot s_n(\epsilon, t, K, \psi).$$

Let $E \subseteq K$ be a maximal $(m+n, \epsilon, t)$ -separated set. For any two distinct $x, y \in E$, there exists $0 \leq k < m+n$ such that

$$M(\psi^k x, \psi^k y, t) < 1 - \epsilon.$$

Now consider two cases:

- (1) If $0 \leq k < m$, then x and y are (m, ϵ, t) -separated.
- (2) If $m \leq k < m + n$, let $k' = k - m$. Then

$$M(\psi^{k'}(\psi^m x), \psi^{k'}(\psi^m y), t) < 1 - \epsilon,$$

so $\psi^m x$ and $\psi^m y$ are (n, ϵ, t) -separated.

This implies that the projection of E onto its first m iterates and the projection after applying ψ^m form separated sets of sizes at most $s_m(\epsilon, t, K, \psi)$ and $s_n(\epsilon, t, K, \psi)$, respectively. Hence,

$$s_{m+n}(\epsilon, t, K, \psi) \leq s_m(\epsilon, t, K, \psi) \cdot s_n(\epsilon, t, K, \psi).$$

Taking logarithms gives the subadditivity:

$$\log s_{m+n}(\epsilon, t, K, \psi) \leq \log s_m(\epsilon, t, K, \psi) + \log s_n(\epsilon, t, K, \psi).$$

By Fekete's lemma [5], the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon, t, K, \psi)$$

exists. Define

$$h_{FQU}(\epsilon, t, K, \psi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon, t, K, \psi).$$

Next, observe that if $0 < \epsilon_1 < \epsilon_2 < 1$, then any (n, ϵ_2, t) -separated set is also (n, ϵ_1, t) -separated, since

$$1 - \epsilon_2 < 1 - \epsilon_1 \quad \Rightarrow \quad M(\psi^k x, \psi^k y, t) < 1 - \epsilon_2 \Rightarrow M(\psi^k x, \psi^k y, t) < 1 - \epsilon_1.$$

Hence,

$$s_n(\epsilon_2, t, K, \psi) \leq s_n(\epsilon_1, t, K, \psi) \quad \Rightarrow \quad h_{FQU}(\epsilon_2, t, K, \psi) \leq h_{FQU}(\epsilon_1, t, K, \psi).$$

Thus, $h_{FQU}(\epsilon, t, K, \psi)$ is monotone decreasing in ϵ and bounded below by 0. Therefore, the limit

$$h_{FQU}(K, t, \psi) := \lim_{\epsilon \rightarrow 0} h_{FQU}(\epsilon, t, K, \psi)$$

exists in $[0, \infty]$. This defines the fuzzy quasi-uniform entropy of ψ on K at precision level t , and proves the proposition.

The entropy $h(\psi)$ is thus well-defined as a limit of monotone decreasing functions of ϵ based on subadditive growth rates of separated sets. □

Lemma 3.10. *Let (X, M, \star) be a fuzzy quasi-metric space and $\psi : (X, M, \star) \rightarrow (X, M, \star)$ be a uniformly continuous self-map. Let $K \in K^*(X)$, $n \in \mathbb{N}$ and $\epsilon \in (0, 1)$. If $F \subseteq K$ is (n, ϵ, t) -separated with respect to ψ and $s_n(\epsilon, t, K, \psi) = |F|$, then*

$$K \subseteq \bigcup_{x \in F} D_n^{M^i}(x, \epsilon, t, \psi).$$

Proof. Let $\epsilon \in (0, 1)$, $t > 0$, and $n \in \mathbb{N}$ be fixed, and let $K \in K^*(X)$. Suppose that $F \subseteq K$ is an (n, ϵ, t) -separated set with respect to ψ such that

$$s_n(\epsilon, t, K, \psi) = |F|.$$

Assume, by way of contradiction, that

$$K \not\subseteq \bigcup_{x \in F} D_n^{M^i}(x, \epsilon, t, \psi).$$

Then there exists a point

$$y \in K \setminus \bigcup_{x \in F} D_n^{M^i}(x, \epsilon, t, \psi).$$

In particular, $y \notin F$.

Define $F_0 := F \cup \{y\} \subseteq K$. We claim that F_0 is (n, ϵ, t) -separated with respect to ψ .

Let $w, z \in F_0$ with $w \neq z$. If $w, z \in F$, then the claim follows from the fact that F is (n, ϵ, t) -separated.

Now assume without loss of generality that $w = y$ and $z \in F$. Since $y \notin D_n^{M^i}(z, \epsilon, t, \psi)$, by definition of the dynamical fuzzy ball there exists an integer $k \in \{0, 1, \dots, n-1\}$ such that

$$M^i(\psi^k(z), \psi^k(y), t) \leq 1 - \epsilon.$$

Hence, the pair (z, y) is (n, ϵ, t) -separated with respect to ψ .

Therefore, every pair of distinct points in F_0 is (n, ϵ, t) -separated, and so F_0 is an (n, ϵ, t) -separated subset of K with

$$|F_0| = |F| + 1,$$

which contradicts the maximality of F , since $s_n(\epsilon, t, K, \psi) = |F|$.

This contradiction shows that

$$K \subseteq \bigcup_{x \in F} D_n^{M^i}(x, \epsilon, t, \psi),$$

and the proof is complete. \square

Proposition 3.11. (Compare [[1] , Lemma 1 and [9], Proposition 1]) Let (X, M, \star) be a fuzzy quasi-metric space, $K \in K^*(X)$ and $\psi : (X, M, \star) \rightarrow (X, M, \star)$ be a uniformly continuous self-map. Then for all $\epsilon_1, \epsilon_2 \in (0, 1)$, $t > 0$ and each $n \in \mathbb{N}$, we have that:

- (1) $r_n(\epsilon, t, K, \psi) \leq s_n(\epsilon, t, K, \psi) \leq r_n(\frac{\epsilon}{2}, t, K, \psi)$
- (2) if $0 < \epsilon < \epsilon_1$, then $r_n(\epsilon, t, K, \psi) \geq r_n(\epsilon_1, t, K, \psi)$ and $s_n(\epsilon, t, K, \psi) \geq s_n(\epsilon_1, t, K, \psi)$

Proof.

- (1) Let $G \subseteq K \in K^*(X)$, such that G is (n, ϵ, t) -separated with respect to ψ , and for all $n \in \mathbb{N}$, $s_n(\epsilon, t, k, \psi) = |G|$. Then by lemma 3.10 above

$$K \subseteq \bigcup_{x \in G} D_n^{M^i}(x, \epsilon, t, \psi) \subseteq \bigcup_{x \in G} D_n^M(x, \epsilon, t, \psi)$$

Thus, $|G| \in \{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^M(x, \epsilon, t, \psi) \}$

Hence, $r_n(\epsilon, t, K, \psi) \leq s_n(\epsilon, t, K, \psi)$. Let us consider $G \subseteq K$ such that G is (n, ϵ, t) -separated and $F \subseteq X$ such that $K \subseteq \bigcup_{x \in F} D_n^{M^i}(x, t, \frac{\epsilon}{2}, \psi)$. Define a function $g : G \rightarrow F$ by choosing, for each $x \in G$, some point $g(x) \in F$ with

$g(x) \in D_n^{M^i}(\frac{\epsilon}{2}, t, K, \psi)$. Then g is injective as in [1] and therefore $|G| \leq |F|$ so that $s_n(\epsilon, t, K, \psi) \leq r_n(\frac{\epsilon}{2}, t, K, \psi)$.

Therefore, $r_n(\epsilon, t, K, \psi) \leq s_n(\epsilon, t, K, \psi) \leq r_n(\frac{\epsilon}{2}, t, K, \psi)$ as required.

- (2) If $0 < \epsilon < \epsilon_1$, then for $x \in X$, $D_n^M(x, t, \epsilon, \psi) \subseteq D_n^M(x, t, \epsilon_1, \psi)$. Hence,

$$\{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^{M^i}(x, t, \epsilon, \psi) \} \subseteq \{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^{M^i}(x, t, \epsilon_1, \psi) \}$$

so that

$$r_n(\epsilon, t, K, \psi) \geq r_n(\epsilon_1, t, K, \psi)$$

Finally, if $0 < \epsilon < \epsilon_1$, then $D_n^{M^i}(x, t, \epsilon, \psi) \subseteq D_n^{M^i}(x, t, \epsilon_1, \psi)$ for each $x \in X$. Similarly, if $F \subseteq X$ is (n, ϵ_1, t) -separated, then F is (n, ϵ, t) -separated since for any $x, y \in F$ with $x \neq y$, we have that

$$[D_n^{M^i}(x, t, \epsilon, \psi) \cap D_n^{M^i}(y, t, \epsilon, \psi)] \subseteq [D_n^{M^i}(x, t, \epsilon_1, \psi) \cap D_n^{M^i}(y, t, \epsilon_1, \psi)] = \emptyset.$$

Thus, $s_n(\epsilon, t, K, \psi) \geq s_n(\epsilon_1, t, K, \psi)$ as required. \square

Lemma 3.12. Let (X, M, \star) be a fuzzy quasi-metric space, $\psi : (X, M, \star) \rightarrow (X, M, \star)$ be a uniformly continuous self-map and $K \in K^*(X)$. Then for all $t > 0$,

$$h_r(K, t, \psi) = h_s(K, t, \psi)$$

The proof follows from proposition 3.11.

3.2. Fundamental properties of the fuzzy quasi-uniform entropy.

Proposition 3.13. (Logarithmic law)(Compare [[1] , Proposition 4 and [9], Proposition 3]) Let (X, M, \star) be a fuzzy quasi-metric space and $\psi : (X, M, \star) \rightarrow (X, M, \star)$ be a uniformly continuous self-map. Then

$$h_{FQU}(\psi^\lambda) = \lambda h_{FQU}(\psi)$$

for each $\lambda \in \mathbb{N}$.

Proof. Let $K \in K^*(X)$, $\epsilon \in (0, 1)$, $t > 0$ and $\lambda, n \in \mathbb{N}$. We consider two cases.

Case 1

If $\lambda = 1$, then $\psi^\lambda = \psi$, so that $h_{FQU}(\psi^\lambda) = \lambda h_{FQU}(\psi)$ holds.

Case 2

If $\lambda > 1$, let $x \in X$ and $y \in D_{\lambda n}^M(x, t, \epsilon, \psi)$. Then $\psi^k(y) \in B_M(\psi^j(x), \epsilon, t)$ for all $0 \leq k < \lambda n$.

Setting $k = \lambda i$ for $0 \leq i < n$, gives $\psi^{\lambda i}(y) \in B_M(\psi^{\lambda i}(x), \epsilon, t)$. Since $\psi^{\lambda i} = (\psi^\lambda)^i$ for all $0 \leq i < n$, then $y \in D_n^M(x, t, \epsilon, \psi^\lambda)$ so that

$$D_{\lambda n}^M(x, t, \epsilon, \psi) \subseteq D_n^M(x, t, \epsilon, \psi^\lambda).$$

Hence,

$$\{ | F | : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_{\lambda n}^M(x, t, \epsilon, \psi) \} \subseteq \{ | F | : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^M(x, t, \epsilon, \psi^\lambda) \}$$

Hence, $r_n(\epsilon, t, K, \psi^\lambda) \leq r_{\lambda n}(\epsilon, t, K, \psi)$. Taking logarithms on both sides and dividing throughout by n and λn gives

$$\frac{\log r_n(\epsilon, t, K, \psi^\lambda)}{n} \leq \frac{\log r_{\lambda n}(\epsilon, t, K, \psi)}{\lambda n} = \frac{\log r_s(\epsilon, t, K, \psi)}{s}, \text{ where } s = \lambda n.$$

Letting $n \rightarrow \infty$ and so, $s \rightarrow \infty$, we have that, $r(\epsilon, t, K, \psi^\lambda) \leq r(\epsilon, t, K, \psi)$. Thus, letting $\epsilon \rightarrow 0$, and after taking supremum over $K^*(X)$, we obtain

$$h_{FQU}(\psi^\lambda) \leq \lambda h_{FQU}(\psi) \tag{3.1}$$

Conversely, let $\epsilon \in (0, 1)$. Since $\psi : (X, M, \star) \rightarrow (X, M, \star)$ is uniformly continuous, there exists $\delta \in (0, 1)$ such that if $x, y \in X$ with $y \in B_M(x, t, \delta)$, then $\psi^j(y) \in B_M(\psi^j(x), t, \epsilon)$ for all $0 \leq j < \lambda$.

For each $x \in X$, let $y \in D_n^M(x, t, \delta, \psi^\lambda)$. Then $(\psi^\lambda)^k(y) \in B_M((\psi^\lambda)^k(x), t, \delta)$ for all $0 \leq k < \lambda$.

Since $\psi^{\lambda k} = (\psi^\lambda)^k$ for all $0 \leq k < \lambda$, then $\psi^{\lambda k}(y) \in B_M(\psi^{\lambda k}(x), t, \delta)$ for all $0 \leq k < \lambda$.

Let $i = \lambda k$, so that $0 \leq i < \lambda n$. Then $\psi^i(y) \in B_M(\psi^i(x), t, \delta)$ for all $0 \leq i < \lambda n$. Since $\psi^i(y) \in B_M(\psi^i(x), t, \delta)$, then $\psi^{ij}(y) \in B_M(\psi^{ij}(x), t, \delta)$ for all $0 \leq j < \lambda$ and $0 \leq i < \lambda n$. Letting $j = 1$, we obtain, $\psi^i(y) \in B_M(\psi^i(x), t, \delta)$ for all $0 \leq i < \lambda n$. Hence, $y \in D_{\lambda n}^M(x, t, \epsilon, \psi)$ and $D_n^M(x, t, \delta, \psi^\lambda) \subseteq D_{\lambda n}^M(x, t, \delta, \psi^\lambda)$. Thus,

$$\{ | F | : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^M(x, t, \delta, \psi^\lambda) \} \subseteq \{ | F | : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_{\lambda n}^M(x, t, \epsilon, \psi) \}.$$

Therefore, $r_{\lambda n}(\epsilon, t, K, \psi) \leq r_n(\epsilon, t, K, \psi^\lambda)$, so that similar to the above argument, we obtain

$$\lambda h_{FQU}(\psi) \leq h_{FQU}(\psi^\lambda) \quad (3.2)$$

From (3.1) and (3.2), we get $h_{FQU}(\psi^\lambda) = \lambda h_{FQU}(\psi)$, as required. \square

The proposition below shows that the fuzzy quasi uniform entropy is less or equal the fuzzy uniform entropy.

Proposition 3.14. (Compare [9] Proposition 6) Let (X, M, \star) be a fuzzy quasi-metric space and $\psi : (X, M, \star) \rightarrow (X, M, \star)$ be a uniformly continuous self-map. Let $K \subseteq K^\star(X)$. Then

- (1) $h_{FQU}(\psi, M) \leq h_{FQU}(\psi, M^i)$
- (2) $\max\{h_{FQU}(\psi, M), h_{FQU}(\psi, M^{-1})\} \leq h_{FQU}(\psi, M^i)$

Proof.

- (1) Let $F \subseteq K$ be (n, ϵ, t) -symmetrised fuzzy spanning set for X with respect to ψ , $n \in \mathbb{N}$, $t > 0$ and $\epsilon \in (0, 1)$. Then if $x \in X$, there exists $y \in F$ such that $M_n^i(x, y, t) > 1 - \epsilon$.

Now, from $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, it follows that

$M^i(x, y, t) \leq M(x, y, t)$. Thus, $M_n(x, y, t) > 1 - \epsilon$. Hence, $F \subseteq K$ is also (n, ϵ, t) -fuzzy quasi spanning set for X with respect to ψ , and this implies that $r_n(\epsilon, K, t, \psi, M) \leq r_n(\epsilon, K, t, \psi, M^i)$ and consequently that $r(\epsilon, K, t, \psi, M) \leq r(\epsilon, K, t, \psi, M^i)$. Therefore,

$$h_{FQU}(\psi, M) \leq h_{FQU}(\psi, M^i)$$

as required.

- (2) From (1) above, we can deduce that

$$h_{FQU}(\psi, M^{-1}) \leq h_{FQU}(\psi, (M^{-1})^i) \leq h_{FQU}(\psi, M^i),$$

since $(M^{-1})^i = M^i$. Therefore, both $h_{FQU}(\psi, M)$ and $h_{FQU}(\psi, M^{-1})$ are less or equal to $h_{FQU}(\psi, M^i)$, so that

$$\max\{h_{FQU}(\psi, M), h_{FQU}(\psi, M^{-1})\} \leq h_{FQU}(\psi, M^i),$$

as required. \square

We now compare the effective radius of fuzzy quasi-balls with that of classical quasi-metric balls. Let (X, q) be a quasi metric space such that $q(x, y)$ is a quasi metric for all $x, y \in X$, and let the fuzzy quasi metric be given by

$$M(x, y, t) = \frac{t}{t + q(x, y)},$$

where $t > 0$. Then, for a fixed tolerance parameter $\epsilon \in (0, 1)$, the fuzzy quasi-ball centered at $x \in X$ is defined by

$$B_{M_q}(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}.$$

We wish to determine how the fuzzy quasi ball radius compares to that of the classical quasi-ball

$$B_q(x, \epsilon) = \{y \in X : q(x, y) < \epsilon\}.$$

Starting from the fuzzy condition:

$$M(x, y, t) > 1 - \epsilon \iff \frac{t}{t + q(x, y)} > 1 - \epsilon,$$

we solve for $q(x, y)$:

$$t > (1 - \varepsilon)(t + q(x, y)) \iff \varepsilon t > (1 - \varepsilon)q(x, y),$$

which yields:

$$q(x, y) < \frac{\varepsilon t}{1 - \varepsilon}.$$

Therefore, the fuzzy quasi-ball of parameter ε and scale t is equivalent to the set:

$$B_{M_q}(x, \varepsilon, t) = \left\{ y \in X : q(x, y) < \frac{\varepsilon t}{1 - \varepsilon} \right\}.$$

To ensure that this fuzzy quasi-ball is strictly smaller than the classical quasi-ball $B_q(x, \varepsilon)$, we require:

$$\frac{\varepsilon t}{1 - \varepsilon} < \varepsilon,$$

which simplifies to:

$$t < 1 - \varepsilon.$$

Similarly, the fuzzy quasi-ball is strictly larger than the classical quasi-ball if

$$t > 1 - \varepsilon.$$

Thus, we have the following lemma:

Lemma 3.15. *Let $q(x, y)$ be a quasi-metric on a set X , and define the fuzzy quasi metric by*

$$M(x, y, t) = \frac{t}{t + q(x, y)} \quad \text{for all } x, y \in X, t > 0.$$

Given a tolerance parameter $\varepsilon \in (0, 1)$, the fuzzy quasi-ball

$$B_{M_q}(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$$

coincides with the set

$$\left\{ y \in X : q(x, y) < \frac{\varepsilon t}{1 - \varepsilon} \right\}.$$

In particular, the fuzzy quasi-ball has strictly smaller radius than the classical quasi-ball

$$B_q(x, \varepsilon) = \{y \in X : q(x, y) < \varepsilon\}$$

if and only if

$$t < 1 - \varepsilon.$$

Theorem 3.16. *Let (X, q) be a quasi-metric space and (X, M, \star) be a fuzzy quasi-metric space such that*

$$M_q(x, y, t) = \frac{t}{t + q(x, y)}, \quad \text{for all } x, y \in X, \text{ and } t > 0.$$

Let $\psi : X \rightarrow X$ be a uniformly continuous self-map on X , and let $K \in K^(X)$. Let $h_q(K, \psi)$ and $h_{M_q}(K, \psi)$ denote the quasi-uniform and fuzzy quasi-uniform entropy of ψ restricted to K , respectively. Assume that*

$$t \leq 1 - \varepsilon, \quad \text{for all } \varepsilon \in (0, 1).$$

Then

$$h_q(K, \psi) \leq h_{M_q}(K, \psi).$$

Proof. Let $F \subseteq K$ be an (n, ε) -spanning set for K with respect to the quasi-metric q and the map ψ . That is, for every $x \in K$, there exists $y \in F$ such that

$$q_n(x, y) = \max_{0 \leq k < n} q(\psi^k(x), \psi^k(y)) < \varepsilon.$$

Note that both $q_n(x, y)$ and the fuzzy version $M_{q_n}(x, y, t)$ are defined in terms of orbit segments $\{\psi^k(x)\}_{k=0}^{n-1}$ and $\{\psi^k(y)\}_{k=0}^{n-1}$. Therefore, the map ψ plays a central role in the entropy calculation, as the spanning conditions are determined entirely by the dynamics of ψ .

Now, observe that for such a pair (x, y) , we have:

$$M_q(\psi^k(x), \psi^k(y), t) = \frac{t}{t + q(\psi^k(x), \psi^k(y))} > \frac{t}{t + \varepsilon}, \quad \text{for all } 0 \leq k < n.$$

Therefore,

$$M_{q_n}(x, y, t) = \min_{0 \leq k < n} M_q(\psi^k(x), \psi^k(y), t) > \frac{t}{t + \varepsilon} = 1 - \frac{\varepsilon}{t + \varepsilon}.$$

Thus

$$M_{q_n}(x, y, t) > 1 - \frac{\varepsilon}{t + \varepsilon},$$

which implies that F is also an $(n, \frac{\varepsilon}{t + \varepsilon}, t)$ -spanning set in the fuzzy setting. From the assumption that $t \leq 1 - \varepsilon$, we have:

$$\frac{\varepsilon}{t + \varepsilon} \leq \varepsilon,$$

so that this gives the inequality between covering numbers:

$$r_n(q, \varepsilon, K, \psi) \leq r_n(M_q, \frac{\varepsilon}{t + \varepsilon}, t, K, \psi), \quad \text{for each } n.$$

As $n \rightarrow \infty$, we have that

$$r(q, \varepsilon, K, \psi) \leq r(M_q, \frac{\varepsilon}{t + \varepsilon}, t, K, \psi).$$

Taking logarithms, dividing by n , and passing to the limit as $n \rightarrow \infty$, we obtain:

$$h_q(K, \psi, \varepsilon) \leq h_{M_q}(K, \psi, \delta).$$

Finally, letting $\varepsilon \rightarrow 0$ (which implies $\frac{\varepsilon}{t + \varepsilon} \rightarrow 0$), we get:

$$h_q(K, \psi) \leq h_{M_q}(K, \psi),$$

as required. □

This result demonstrates that fuzzy quasi-uniform entropy not only generalizes classical quasi-uniform entropy but also provides a finer lens for analyzing orbit complexity. By incorporating both asymmetry and graded similarity, the fuzzy framework captures distinctions that may be invisible in the classical setting, especially in systems where the dynamics are non-symmetric, imprecise, or lack a traditional metric structure.

Proposition 3.17. Let (X, q) be a quasi-metric space and define the associated fuzzy quasi-metric

$$M_q(x, y, t) = \frac{t}{t + q(x, y)}, \quad x, y \in X, t > 0.$$

Fix $\varepsilon \in (0, 1)$ and $t > 1 - \varepsilon$, and set

$$r = \frac{t\varepsilon}{1 - \varepsilon}.$$

Then there exist a compact set $K \subset X$ and a uniformly continuous self-map $\psi : X \rightarrow X$ such that, for every $n \in \mathbb{N}$, the following hold:

- (1) K is an (n, δ) -spanning set for (X, q, ψ) for every $0 < \delta < r$;

(2) K is not an (n, ε, t) -spanning set for (X, M_q, ψ) .

Consequently, for $t > 1 - \varepsilon$, the quasi-metric and fuzzy quasi-metric notions of spanning sets are not equivalent.

Proof. Let $X = [0, 1]$ and define a quasi-metric $q : X \times X \rightarrow [0, 1]$ by

$$q(x, y) = \begin{cases} y - x, & \text{if } y \geq x, \\ 1, & \text{if } y < x. \end{cases}$$

It is straightforward to verify that $q(x, x) = 0$ for all $x \in X$, that q is non-symmetric, and that it satisfies the triangle inequality. Hence (X, q) is a quasi-metric space.

Let $\psi : X \rightarrow X$ be the identity map, $\psi(x) = x$. Clearly, ψ is uniformly continuous. For each $n \in \mathbb{N}$ and $x, y \in X$, we then have

$$q_n(x, y) = \max_{0 \leq k < n} q(\psi^k(x), \psi^k(y)) = q(x, y),$$

and similarly,

$$M_{q_n}(x, y, t) = \min_{0 \leq k < n} M_q(\psi^k(x), \psi^k(y), t) = M_q(x, y, t).$$

Fix $\varepsilon \in (0, 1)$ and $t > 1 - \varepsilon$, and define

$$r = \frac{t\varepsilon}{1 - \varepsilon}.$$

Let $0 < \delta < r$ and choose

$$K = \{x_i = i\delta \mid i = 0, 1, \dots, N\} \subset [0, 1], \quad N = \left\lfloor \frac{1}{\delta} \right\rfloor.$$

Then K is finite and hence compact.

Proof of (1). Let $x \in X$ be arbitrary. Choose $x_i \in K$ such that

$$x_i \leq x < x_i + \delta.$$

Then $x \geq x_i$, and by the definition of q ,

$$q(x_i, x) = x - x_i < \delta.$$

Since $q_n = q$ for all n , it follows that

$$q_n(x_i, x) < \delta.$$

Thus K is an (n, δ) -spanning set for (X, q, ψ) for every $n \in \mathbb{N}$.

Proof of (2). For K to be an (n, ε, t) -spanning set for (X, M_q, ψ) , it would be required that for every $x \in X$ there exists $x_i \in K$ such that

$$M_q(x, x_i, t) > 1 - \varepsilon.$$

By direct computation,

$$M_q(x, x_i, t) > 1 - \varepsilon \iff q(x, x_i) < r.$$

Fix $i \in \{0, 1, \dots, N - 1\}$ and choose $x \in (x_i, x_{i+1})$. Then $x < x_{i+1}$ and $x > x_i$. For x_i we have $x > x_i$, hence

$$q(x, x_i) = 1,$$

while for x_{i+1} we have $x < x_{i+1}$, so

$$q(x, x_{i+1}) = x_{i+1} - x \geq \delta.$$

Since $\delta < r < 1$, it follows that

$$q(x, x_i) \geq r \quad \text{and} \quad q(x, x_{i+1}) \geq \delta.$$

Thus for every $x_j \in K$,

$$q(x, x_j) \geq \delta,$$

and consequently,

$$M_q(x, x_j, t) \leq \frac{t}{t + \delta} < \frac{t}{t + r} = 1 - \varepsilon.$$

Hence no point of K satisfies the fuzzy spanning condition for this x , and K is not an (n, ε, t) -spanning set for (X, M_q, ψ) .

Therefore, for $t > 1 - \varepsilon$, there exist compact sets and uniformly continuous self-maps for which the quasi-metric and fuzzy quasi-metric notions of spanning sets are not equivalent. □

Remark 3.18. Let (X, q) be a quasi-metric space and let

$$M_q(x, y, t) = \frac{t}{t + q(x, y)}$$

be the associated fuzzy quasi-metric. For fixed $\varepsilon \in (0, 1)$ and $t > 0$, the inequality

$$M_q(x, y, t) > 1 - \varepsilon$$

is equivalent to the quasi-metric condition

$$q(x, y) < \frac{t\varepsilon}{1 - \varepsilon}.$$

Accordingly, the quantity

$$r = \frac{t\varepsilon}{1 - \varepsilon}$$

represents the exact quasi-metric threshold corresponding to the fuzzy level $1 - \varepsilon$, since

$$\frac{t}{t + r} = 1 - \varepsilon.$$

This parameter allows a precise comparison between (n, δ) -spanning conditions in the quasi-metric setting and (n, ε, t) -spanning conditions in the fuzzy quasi-metric setting.

Remark 3.19. Proposition 3.17 illustrates that the assumption $t \leq 1 - \varepsilon$ in the entropy comparison theorem is not merely technical, but essential. When $t > 1 - \varepsilon$, the fuzzy quasi-metric M_q becomes too permissive, assigning high similarity values even to pairs of points that are well-separated in the quasi-metric. As a result, fuzzy quasi-spanning sets may become too small to adequately capture the underlying complexity of the space. This can lead to an underestimation of the entropy, and thus a failure of the inequality $h_q \leq h_{M_q}$. It therefore emphasizes that fuzzy entropy can be strictly smaller than classical quasi-metric entropy when the parameter t is too large relative to the desired separation threshold ε .

Here is a detailed, illustrative example comparing Bowen entropy [1], uniform quasi-metric entropy (as in Haiambo and Olela-Otafudu) [9], and uniform fuzzy quasi-metric entropy (as proposed in this paper).

Example 3.20 (Comparing Bowen, Quasi-Metric, and Fuzzy Quasi-Metric Entropy). Let

$$X = \{0, \frac{1}{2}, 1, \frac{5}{2}, 3\}$$

and define the map $\psi : X \rightarrow X$ by

$$\psi(0) = 3, \quad \psi(\frac{1}{2}) = \frac{5}{2}, \quad \psi(1) = 1, \quad \psi(\frac{5}{2}) = \frac{1}{2}, \quad \psi(3) = 0.$$

We compare three notions of entropy: classical Bowen (topological) entropy, quasi-metric entropy, and fuzzy quasi-metric entropy. Fix $\epsilon = 0.8$.

Step 1: Bowen (Topological) Entropy

Using the standard metric

$$d(x, y) = |x - y|,$$

define the Bowen balls and separated sets.

- Orbit of 0 under $\psi: 0 \rightarrow 3 \rightarrow 0 \dots$
- Orbit of $\frac{1}{2}$: $\frac{1}{2} \rightarrow \frac{5}{2} \rightarrow \frac{1}{2} \dots$
- Orbit of 1: fixed
- Orbit of $\frac{5}{2}$: see orbit of $\frac{1}{2}$
- Orbit of 3: see orbit of 0

At time $n = 1$, distinguishable points using $\epsilon = 0.8$ give minimal covering balls:

$$\{0, 3\}, \quad \left\{\frac{1}{2}, \frac{5}{2}\right\}, \quad \{1\}.$$

Thus, the Bowen/topological entropy is

$$h_{\text{top}}(\psi) = \log 3 \approx 1.099.$$

Step 2: Quasi-Metric Entropy

Define the quasi-metric $q : X \times X \rightarrow [0, \infty)$ by

$$q(x, y) = \begin{cases} x - y, & x \geq y, \\ 2|x - y|, & x < y. \end{cases}$$

Quasi-balls are

$$B_q(x, \epsilon) = \{y \in X : q(x, y) < \epsilon\}.$$

Compute for $\epsilon = 0.8$:

$$B_q(0, 0.8) = \{0\},$$

$$B_q\left(\frac{1}{2}, 0.8\right) = \left\{0, \frac{1}{2}\right\},$$

$$B_q(1, 0.8) = \left\{\frac{1}{2}, 1\right\},$$

$$B_q\left(\frac{5}{2}, 0.8\right) = \left\{\frac{5}{2}\right\},$$

$$B_q(3, 0.8) = \left\{\frac{5}{2}, 3\right\}.$$

To cover all of X , we need 3 balls:

$$B_q\left(\frac{1}{2}\right), \quad B_q(1), \quad B_q(3).$$

Hence, the quasi-metric entropy is

$$h_q(\psi, 0.8) = \log 3 \approx 1.099.$$

Step 3: Fuzzy Quasi-Metric Entropy

Define the fuzzy quasi-metric

$$M_q(x, y, t) = \frac{t}{t + q(x, y)}, \quad t > 0,$$

and fuzzy quasi-balls

$$B_{M_q}(x, \epsilon, t) = \{y \in X : M_q(x, y, t) > 1 - \epsilon\}.$$

(1) $t = 0.1 < 1 - \epsilon$:

$$M_q(x, y, 0.1) > 0.2 \implies q(x, y) < 0.4$$

Compute fuzzy balls:

$$\begin{aligned} B_{M_q}(0, 0.8, 0.1) &= \{0\}, & B_{M_q}(\tfrac{1}{2}, 0.8, 0.1) &= \{\tfrac{1}{2}\}, \\ B_{M_q}(1, 0.8, 0.1) &= \{1\}, & B_{M_q}(\tfrac{5}{2}, 0.8, 0.1) &= \{\tfrac{5}{2}\}, \\ B_{M_q}(3, 0.8, 0.1) &= \{3\}. \end{aligned}$$

All 5 balls are needed:

$$h_{M_q}(\psi, 0.8) = \log 5 \approx 1.609 > h_q(\psi, 0.8)$$

(2) $t = 1 > 1 - \epsilon$:

$$M_q(x, y, 1) > 0.2 \implies q(x, y) < 4$$

Now, a single ball centered at 3 covers all of X :

$$h_{M_q}(\psi, 0.8) = \log 1 = 0 < h_q(\psi, 0.8)$$

(3) $t = 1 - \epsilon = 0.2$:

$$M_q(x, y, 0.2) > 0.2 \implies q(x, y) < 0.8$$

Fuzzy balls coincide with quasi-balls:

$$h_{M_q}(\psi, 0.8) = h_q(\psi, 0.8) = \log 3 \approx 1.099$$

Remark 3.21.

- (1) Bowen/topological entropy coincides with quasi-metric entropy when quasi-balls are suitably chosen.
- (2) Fuzzy quasi-metric entropy depends on the parameter t : it can overestimate ($t < 1 - \epsilon$), underestimate ($t > 1 - \epsilon$), or match ($t = 1 - \epsilon$).
- (3) This illustrates the flexibility of fuzzy quasi-metric entropy and its relation to classical notions. The differences in entropy values arise from the way "closeness" is measured: Bowen entropy uses symmetric distances, quasi-metric entropy accounts for asymmetry, and fuzzy quasi-metric entropy allows a gradual, adjustable notion of proximity. Each model provides a distinct perspective on orbit distinguishability. There is no single "most accurate" entropy; rather, the choice should reflect how one wants to capture the *closeness, uncertainty, and complexity* inherent in the system. Selecting the appropriate notion depends on which aspects of the system's dynamics are most relevant for the analysis.

Table 1 provides a summary of the most suitable entropy types for different classes of dynamical systems based on their structural characteristics and the nature of proximity involved.

TABLE 1. Recommended entropy types based on system characteristics

System Type or Goal	Most Appropriate Entropy Type
Crisp, symmetric dynamics	Bowen (Topological) Entropy (as in [1])
Crisp systems with asymmetric dynamics	Quasi-Metric Entropy (as in [9])
Systems with vagueness, uncertainty, or tolerance(fuzziness)	Fuzzy Metric Entropy (as in [10])
Systems with both fuzziness and asymmetric	Fuzzy Quasi-Metric Entropy(our proposed entropy)

Proposition 3.22. (Compare [[1], Proposition 3 and [9], Proposition 2]) If M_1 and M_2 are uniformly equivalent fuzzy quasi-metrics on X and $\psi : (X, M_1, \star) \rightarrow (X, M_1, \star)$ is uniformly continuous, then

$$h_{FQU}(\psi, M_1) = h_{FQU}(\psi, M_2).$$

Proof. Let $\epsilon_1 \in (0, 1)$. Then by uniform continuity of $id_X : (X, M_1, \star) \rightarrow (X, M_2, \star)$ and $id_X : (X, M_2, \star) \rightarrow (X, M_1, \star)$, we can choose $0 < \epsilon_3 \leq \epsilon_2 \leq \epsilon_1 < 1$ and $t > 0$, such that for all $x, y \in X$,

$$M_1(\psi(x), \psi(y), t) > 1 - \epsilon_1 \text{ whenever } M_2(x, y, t) > 1 - \epsilon_2$$

and

$$M_2(\psi(x), \psi(y), t) > 1 - \epsilon_2 \text{ whenever } M_1(x, y, t) > 1 - \epsilon_3.$$

Hence, for all $n \in \mathbb{N}$, we have that

$$D_n(x, t, \epsilon_3, M_1) \subseteq D_n(x, t, \epsilon_2, M_2) \subseteq D_n(x, t, \epsilon_1, M_1).$$

Let $K \in K^*(X)$. Then an (n, ϵ_2, t) -spanning set for K with respect to M_2 is an (n, ϵ_1, t) -spanning set for K with respect to M_1 . Therefore, for all $n \in \mathbb{N}$, we have that

$$r_n(\epsilon_1, t, K, \psi, M_1) \leq r_n(\epsilon_2, t, K, \psi, M_2)$$

and similarly,

$$r_n(\epsilon_2, t, K, \psi, M_2) \leq r_n(\epsilon_3, t, K, \psi, M_1)$$

so that

$$r_n(\epsilon_1, t, K, \psi, M_1) \leq r_n(\epsilon_2, t, K, \psi, M_2) \leq r_n(\epsilon_3, t, K, \psi, M_1)$$

Letting $n \rightarrow \infty$, we obtain

$$r(\epsilon_1, t, K, \psi, M_1) \leq r(\epsilon_2, t, K, \psi, M_2) \leq r(\epsilon_3, t, K, \psi, M_1)$$

so that if $\epsilon_1 \rightarrow 0$, and taking supremum over $K \in K^*(X)$ we obtain

$$h_{FQU}(\psi, M_1) = h_{FQU}(\psi, M_2)$$

as required. □

Theorem 3.23. (Weak Addition Theorem)(Compare [[3], Proposition 4 and [9], Proposition 4]) Let (X_1, M_1, \star) and (X_2, M_2, \star) be fuzzy quasi-metric spaces and let $\psi_1 : X_1 \rightarrow X_1$ and $\psi_2 : X_2 \rightarrow X_2$ be uniformly continuous self-maps. Define a fuzzy quasi-metric M on $X_1 \times X_2$ by

$$M((x_1, x_2), (y_1, y_2), t) = \min\{M_1(x_1, y_1, t), M_2(x_2, y_2, t)\}.$$

Then $\psi_1 \times \psi_2 : (X_1 \times X_2, M) \rightarrow (X_1 \times X_2, M)$ is uniformly continuous and

$$h_{FQU}(\psi_1 \times \psi_2, M) \leq h_{FQU}(\psi_1, M_1) + h_{FQU}(\psi_2, M_2).$$

Moreover, if either X_1 or X_2 is τ_M -compact, then equality holds.

Proof. Let $\epsilon \in (0, 1)$. Since ψ_i is uniformly continuous, there exists $\delta_i \in (0, 1)$ such that

$$M_i(x_i, y_i, t) > 1 - \delta_i \Rightarrow M_i(\psi_i(x_i), \psi_i(y_i), t) > 1 - \epsilon, \quad i = 1, 2.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. If $M((x_1, x_2), (y_1, y_2), t) > 1 - \delta$, then $M_i(x_i, y_i, t) > 1 - \delta$ for $i = 1, 2$, and hence

$$M((\psi_1 \times \psi_2)(x_1, x_2), (\psi_1 \times \psi_2)(y_1, y_2), t) > 1 - \epsilon.$$

Thus $\psi_1 \times \psi_2$ is uniformly continuous.

For every $\epsilon \in (0, 1)$, $t > 0$, $n \in \mathbb{N}$, and $(x_1, x_2) \in X_1 \times X_2$,

$$D_n^M((x_1, x_2), \epsilon, t, \psi_1 \times \psi_2) = D_n^{M_1}(x_1, \epsilon, t, \psi_1) \times D_n^{M_2}(x_2, \epsilon, t, \psi_2).$$

Indeed, for $0 \leq k < n$,

$$\begin{aligned} M((\psi_1 \times \psi_2)^k(x_1, x_2), (\psi_1 \times \psi_2)^k(a_1, a_2), t) &> 1 - \epsilon \\ \iff \min\{M_1(\psi_1^k(x_1), \psi_1^k(a_1), t), M_2(\psi_2^k(x_2), \psi_2^k(a_2), t)\} &> 1 - \epsilon, \end{aligned}$$

which is equivalent to both coordinate inequalities.

Let $K_i \in K^*(X_i)$ and choose finite $F_i \subseteq X_i$ such that

$$K_i \subseteq \bigcup_{x_i \in F_i} D_n^{M_i}(x_i, \epsilon, t, \psi_i), \quad i = 1, 2.$$

Then

$$K_1 \times K_2 \subseteq \bigcup_{(x_1, x_2) \in F_1 \times F_2} D_n^M((x_1, x_2), \epsilon, t, \psi_1 \times \psi_2),$$

and hence

$$r_n(\epsilon, K_1 \times K_2, t, \psi_1 \times \psi_2, M) \leq r_n(\epsilon, K_1, t, \psi_1, M_1) r_n(\epsilon, K_2, t, \psi_2, M_2).$$

Taking logarithms, dividing by n , and letting $\epsilon \rightarrow 0$ yields

$$h_{FQU}(K_1 \times K_2, \psi_1 \times \psi_2, M) \leq h_{FQU}(K_1, \psi_1, M_1) + h_{FQU}(K_2, \psi_2, M_2).$$

Let $F_i \subseteq K_i$ be (n, ϵ, t) -separated with respect to ψ_i . Then $F_1 \times F_2$ is (n, ϵ, t) -separated with respect to $\psi_1 \times \psi_2$, since distinct points differ in at least one coordinate. Therefore,

$$s_n(\epsilon, K_1 \times K_2, t, \psi_1 \times \psi_2, M) \geq s_n(\epsilon, K_1, t, \psi_1, M_1) s_n(\epsilon, K_2, t, \psi_2, M_2),$$

and letting $\epsilon \rightarrow 0$ gives

$$h_{FQU}(K_1 \times K_2, \psi_1 \times \psi_2, M) \geq h_{FQU}(K_1, \psi_1, M_1) + h_{FQU}(K_2, \psi_2, M_2).$$

Combining both inequalities,

$$h_{FQU}(K_1 \times K_2, \psi_1 \times \psi_2, M) = h_{FQU}(K_1, \psi_1, M_1) + h_{FQU}(K_2, \psi_2, M_2).$$

Let $\pi_i : X_1 \times X_2 \rightarrow X_i$ be the projections. For any $K \in K^*(X_1 \times X_2)$,

$$K \subseteq \pi_1(K) \times \pi_2(K), \quad \pi_i(K) \in K^*(X_i).$$

Hence,

$$h_{FQU}(\psi_1 \times \psi_2, M) \leq h_{FQU}(\psi_1, M_1) + h_{FQU}(\psi_2, M_2).$$

If X_1 (or X_2) is τ_M -compact, then

$$h_{FQU}(\psi_1 \times \psi_2, M) = h_{FQU}(\psi_1, M_1) + h_{FQU}(\psi_2, M_2).$$

□

We now show in the next theorem that uniform entropy is invariant under topological conjugacy via a homeomorphism.

Theorem 3.24. *Let (X_1, M_1, \star) and (X_2, M_2, \star) be fuzzy quasi-metric spaces. Let*

$\psi_1 : (X_1, M_1, \star) \rightarrow (X_1, M_1, \star)$ and $\psi_2 : (X_2, M_2, \star) \rightarrow (X_2, M_2, \star)$ be uniformly continuous self-maps. Assume that ψ_1 and ψ_2 are topologically conjugate via a homeomorphism $\psi : (X_1, M_1, \star) \rightarrow (X_2, M_2, \star)$ such that both ψ and ψ^{-1} are uniformly continuous and

$$\psi \circ \psi_1 = \psi_2 \circ \psi.$$

Then

$$h_{FQU}(M_1, \psi_1) = h_{FQU}(M_2, \psi_2).$$

Proof. Fix $t > 0$ and $n \in \mathbb{N}$. Let $\epsilon_2 \in (0, 1)$. Since ψ is uniformly continuous, there exists $\epsilon_1 \in (0, 1)$ such that

$$M_1(x, y, t) > 1 - \epsilon_1 \quad \Rightarrow \quad M_2(\psi(x), \psi(y), t) > 1 - \epsilon_2.$$

Let $E_1 \subseteq X_1$ be an (n, ϵ_1, t) -fuzzy quasi-spanning set for X_1 with respect to ψ_1 . Define $E_2 = \psi(E_1) \subseteq X_2$. For any $x_2 \in X_2$, choose $x_1 \in X_1$ with $\psi(x_1) = x_2$. There exists $y_1 \in E_1$ such that

$$M_1(\psi_1^k(x_1), \psi_1^k(y_1), t) > 1 - \epsilon_1, \quad 0 \leq k < n.$$

Using conjugacy,

$$\psi(\psi_1^k(x_1)) = \psi_2^k(x_2), \quad \psi(\psi_1^k(y_1)) = \psi_2^k(\psi(y_1)).$$

Hence,

$$M_2(\psi_2^k(x_2), \psi_2^k(\psi(y_1)), t) > 1 - \epsilon_2, \quad 0 \leq k < n,$$

showing that E_2 is an (n, ϵ_2, t) -fuzzy quasi-spanning set for X_2 with respect to ψ_2 . Therefore,

$$r_n(\epsilon_2, t, X_2, \psi_2) \leq r_n(\epsilon_1, t, X_1, \psi_1).$$

Taking limits,

$$h_{FQU}(M_1, \psi_1) \leq h_{FQU}(M_2, \psi_2).$$

Let $\epsilon_1 \in (0, 1)$. Since ψ^{-1} is uniformly continuous, there exists $\epsilon_2 \in (0, 1)$ such that

$$M_2(u, v, t) > 1 - \epsilon_2 \quad \Rightarrow \quad M_1(\psi^{-1}(u), \psi^{-1}(v), t) > 1 - \epsilon_1.$$

Let $E_2 \subseteq X_2$ be an (n, ϵ_2, t) -fuzzy quasi-spanning set for X_2 with respect to ψ_2 and define $E_1 = \psi^{-1}(E_2) \subseteq X_1$. Using the conjugacy relation and the above implication, one verifies that E_1 is an (n, ϵ_1, t) -fuzzy quasi-spanning set for X_1 with respect to ψ_1 . Thus,

$$r_n(\epsilon_1, t, X_1, \psi_1) \leq r_n(\epsilon_2, t, X_2, \psi_2),$$

which yields

$$h_{FQU}(M_2, \psi_2) \leq h_{FQU}(M_1, \psi_1).$$

Combining both inequalities,

$$h_{FQU}(M_1, \psi_1) = h_{FQU}(M_2, \psi_2).$$

□

In the setting of fuzzy quasi-metric spaces, the scale parameter $t > 0$ controls the degree of fuzziness in the metric $M(x, y, t)$. Intuitively, larger values of t correspond to more tolerance in distinguishing points, seemingly making the space “coarser” from the perspective of separation. As a result, the size of (n, ϵ, t) -separated sets may increase with t , potentially affecting the computed entropy. We now formalize this intuition in the proposition below and show that entropy is non-decreasing with respect to the fuzziness scale parameter.

Proposition 3.25. Let (X, M, \star) be a fuzzy quasi-metric space such that for all $x, y \in X$, the function $t \mapsto M(x, y, t)$ is non-decreasing. Let $\psi : X \rightarrow X$ be a continuous self-map. Then for each fixed $\epsilon \in (0, 1)$, the function

$$t \mapsto h_{FQU}(\psi, t, \epsilon) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon, t)$$

is non-increasing on $(0, \infty)$. Consequently,

$$h_{FQU}(\psi, t) := \lim_{\epsilon \rightarrow 0} h_{FQU}(\psi, t, \epsilon)$$

is also non-increasing in t .

Proof. Fix $\epsilon \in (0, 1)$ and let $0 < t_1 < t_2$. We show that

$$s_n(\epsilon, t_1) \geq s_n(\epsilon, t_2) \quad \text{for all } n \in \mathbb{N}.$$

Recall that

$$M_\psi^n(x, y, t) := \min_{0 \leq k < n} M(\psi^k x, \psi^k y, t).$$

Since $t \mapsto M(x, y, t)$ is non-decreasing for all x, y , it follows that $t \mapsto M_\psi^n(x, y, t)$ is also non-decreasing. Let $E \subseteq X$ be an (n, ϵ, t_2) -separated set. Then for all distinct $x, y \in E$,

$$M_\psi^n(x, y, t_2) < 1 - \epsilon.$$

Since $t_1 < t_2$, we have

$$M_\psi^n(x, y, t_1) \leq M_\psi^n(x, y, t_2) < 1 - \epsilon,$$

and hence E is also (n, ϵ, t_1) -separated. Therefore,

$$s_n(\epsilon, t_1) \geq s_n(\epsilon, t_2).$$

Taking logarithms, dividing by n , and passing to \limsup , we obtain

$$h_{FQU}(\psi, t_1, \epsilon) \geq h_{FQU}(\psi, t_2, \epsilon).$$

Finally, letting $\epsilon \rightarrow 0$ preserves monotonicity, yielding

$$h_{FQU}(\psi, t_1) \geq h_{FQU}(\psi, t_2).$$

□

We now show that entropy should remain unchanged under transformations that preserve the fuzzy quasi-metric structure.

Proposition 3.26 (Invariance of Entropy under Isometric Isomorphism). Let $f : X \rightarrow Y$ be a homeomorphism between two fuzzy quasi-metric spaces (X, M_X, \star) and (Y, M_Y, \star) such that

$$M_Y(f(x), f(y), t) = M_X(x, y, t) \quad \text{for all } x, y \in X, t > 0.$$

Let $T : X \rightarrow X$ be a continuous map and define

$$\tilde{T} := f \circ T \circ f^{-1}.$$

Then, for all $t > 0$,

$$h_{FQU}(M_X, T, t) = h_{FQU}(M_Y, \tilde{T}, t).$$

Proof. Fix $t > 0$ and $\epsilon \in (0, 1)$. Let $E \subseteq X$ be an (n, ϵ, t) -separated set for T , i.e., for all distinct $x, y \in E$,

$$M_{X,T}^n(x, y, t) := \min_{0 \leq k < n} M_X(T^k x, T^k y, t) < 1 - \epsilon.$$

Define $f(E) = \{f(x) : x \in E\} \subseteq Y$. Since f is a bijection, $|f(E)| = |E|$. For $x, y \in E$ and $0 \leq k < n$, we have

$$\tilde{T}^k(f(x)) = f(T^k x),$$

and hence,

$$M_Y(\tilde{T}^k f(x), \tilde{T}^k f(y), t) = M_X(T^k x, T^k y, t).$$

Therefore,

$$M_{Y,\tilde{T}}^n(f(x), f(y), t) = M_{X,T}^n(x, y, t) < 1 - \epsilon,$$

showing that $f(E)$ is (n, ϵ, t) -separated in Y . Thus,

$$s_n^{M_X}(T, \epsilon, t) \leq s_n^{M_Y}(\tilde{T}, \epsilon, t).$$

Applying the same argument to f^{-1} yields the reverse inequality, and hence

$$s_n^{M_X}(T, \epsilon, t) = s_n^{M_Y}(\tilde{T}, \epsilon, t).$$

Taking logarithms, dividing by n , passing to $\limsup_{n \rightarrow \infty}$, and letting $\epsilon \rightarrow 0$, we obtain

$$h_{FQU}(M_X, T, t) = h_{FQU}(M_Y, \tilde{T}, t).$$

□

To allow for comparison with standard topological entropy, we now examine the case when the fuzzy quasi-metric is induced by a classical metric.

Proposition 3.27. Let (X, d) be a compact metric space, and define a fuzzy quasi metric $M : X \times X \times (0, \infty) \rightarrow [0, 1]$ by

$$M(x, y, t) := \exp\left(-\frac{d(x, y)}{t}\right), \quad \text{with } \star = \min.$$

Let $T : X \rightarrow X$ be a continuous map. Then the fuzzy entropy of T converges to the classical topological entropy as $t \rightarrow 0$:

$$\lim_{t \rightarrow 0} h(T, t) = h_{\text{top}}(T).$$

Proof. We begin by recalling that fuzzy entropy is defined as:

$$h(T, t) := \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon, t),$$

where $s_n(\epsilon, t)$ denotes the maximal cardinality of an (n, ϵ, t) -separated set with respect to the fuzzy metric M , defined as:

$$M_T^n(x, y, t) := \min_{0 \leq k < n} M(T^k x, T^k y, t) = \min_{0 \leq k < n} \exp\left(-\frac{d(T^k x, T^k y)}{t}\right).$$

Note that the condition $M_T^n(x, y, t) < 1 - \epsilon$ is equivalent to:

$$\min_{0 \leq k < n} \exp\left(-\frac{d(T^k x, T^k y)}{t}\right) < 1 - \epsilon,$$

which in turn is equivalent to:

$$\exists 0 \leq k < n \text{ such that } d(T^k x, T^k y) > -t \log(1 - \epsilon).$$

Define $\delta := -t \log(1 - \epsilon)$. Then an (n, ϵ, t) -separated set in the fuzzy setting corresponds exactly to an (n, δ) -separated set in the classical metric setting. As $t \rightarrow 0$, for fixed ϵ , we have:

$$\delta = -t \log(1 - \epsilon) \rightarrow 0^+.$$

Now recall the classical topological entropy of T is given by:

$$h_{\text{top}}(T) := \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n^{\text{top}}(\delta),$$

where $s_n^{\text{top}}(\delta)$ denotes the maximal cardinality of (n, δ) -separated sets with respect to the metric d . From the above equivalence between fuzzy (n, ϵ, t) -separated sets and classical (n, δ) -separated sets, we can relate the counts:

$$s_n(\epsilon, t) = s_n^{\text{top}}(\delta), \quad \text{with } \delta = -t \log(1 - \epsilon).$$

Hence,

$$h(T, t) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon, t) = \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n^{\text{top}}(\delta) = h_{\text{top}}(T),$$

where the limit $\delta \rightarrow 0$ occurs via $\delta = -t \log(1 - \epsilon)$ as $t \rightarrow 0$ and $\epsilon \rightarrow 0$. Therefore,

$$\lim_{t \rightarrow 0} h(T, t) = h_{\text{top}}(T).$$

□

Entropy provides a quantitative measure of how quickly distinct orbits in a dynamical system separate over time. In the context of fuzzy quasi-metric spaces, entropy captures the growth rate of distinguishable behaviors at a given fuzziness level. If the number of distinguishable orbits grows exponentially, the entropy must be positive. The following lemma formalizes this observation.

Lemma 3.28. (*Exponential Growth Implies Positive Entropy*) Let (X, M, \star) be a compact fuzzy quasi-metric space, and let $T : X \rightarrow X$ be a continuous map. Suppose there exist constants $C > 0$, $\lambda > 1$, and $\epsilon > 0$ such that

$$s_n(\epsilon) \geq C\lambda^n \quad \text{for all sufficiently large } n,$$

where $s_n(\epsilon)$ denotes the maximal cardinality of an (n, ϵ) -separated set. Then the fuzzy entropy $h(T)$ satisfies

$$h(T) \geq \log \lambda > 0.$$

Proof. Recall that the fuzzy entropy of T is defined as:

$$h(T) := \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon).$$

From the hypothesis, for the fixed $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$s_n(\epsilon) \geq C\lambda^n.$$

Taking logarithms and dividing by n , we obtain:

$$\frac{1}{n} \log s_n(\epsilon) \geq \frac{1}{n} \log(C\lambda^n) = \frac{1}{n} \log C + \log \lambda.$$

As $n \rightarrow \infty$, the term $\frac{1}{n} \log C \rightarrow 0$, so

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon) \geq \log \lambda.$$

Since this holds for a fixed $\epsilon > 0$, and the entropy is the outer limit as $\epsilon \rightarrow 0$, we get:

$$h(T) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon) \geq \log \lambda > 0.$$

Therefore, the entropy is strictly positive, reflecting the exponential complexity of orbit separation. \square

We finally construct a detailed example illustrating some of the key concepts discussed in this paper.

Example 3.29. Let $X = [0, 1]$ and define a quasi-metric $q : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ by

$$q(x, y) = \begin{cases} y - x, & \text{if } y \geq x, \\ \alpha(x - y), & \text{if } y < x, \end{cases} \quad \text{where } \alpha > 1.$$

This is a valid quasi-metric as it satisfies:

- $q(x, x) = 0$,
- $q(x, y) \geq 0$,
- The triangle inequality holds.

We define the fuzzy quasi-metric $M : X \times X \times (0, \infty) \rightarrow [0, 1]$ by

$$M(x, y, t) = \exp\left(-\frac{q(x, y)}{t}\right), \quad t > 0.$$

This satisfies:

- $M(x, x, t) = 1$,
- $M(x, y, t) \in (0, 1]$,
- $M(x, y, t) \neq M(y, x, t)$ in general (asymmetry),
- $M(x, y, \cdot)$ is non-decreasing.

Let $\psi_1 : X_1 \rightarrow X_1$ be the logistic map:

$$\psi_1(x) = 4x(1 - x), \quad x \in [0, 1],$$

and $\psi_2 : X_2 \rightarrow X_2$ be

$$\psi_2(x) = 1 - 2x^2, \quad x \in [-1, 1].$$

Define a homeomorphism $\psi : [0, 1] \rightarrow [-1, 1]$ by

$$\psi(x) = \cos(\pi x).$$

Then we have the following properties:

(1) We have the **conjugacy**:

$$\psi_2 \circ \psi = \psi \circ \psi_1.$$

(2) Since:

- ψ is a homeomorphism,
- ψ is uniformly continuous under M ,
- ψ_1 and ψ_2 are conjugate via ψ ,

the fuzzy quasi-uniform entropy is **invariant**:

$$h_{FQU}(M, \psi_1) = h_{FQU}(M, \psi_2).$$

(3) The function $t \mapsto M(x, y, t)$ is **non-decreasing** for fixed x, y . Therefore, the entropy

$$t \mapsto h_{FQU}(M, \psi_1, t)$$

is also **non-decreasing**. That is,

$$t_1 < t_2 \quad \Rightarrow \quad h_{FQU}(\psi_1, t_1) \leq h_{FQU}(\psi_1, t_2).$$

(4) Note that

$$M(x, y, t) < 1 - \epsilon \iff q(x, y) > -t \log(1 - \epsilon).$$

So an (n, ϵ, t) -separated set in fuzzy entropy corresponds to a classical (n, δ) -separated set with

$$\delta = -t \log(1 - \epsilon).$$

Hence, as $t \rightarrow 0$, $\delta \rightarrow 0$, and

$$\lim_{t \rightarrow 0} h_{FQU}(\psi_1, t) = h_{\text{top}}(\psi_1) = \log 2.$$

(5) The logistic map ψ_1 is chaotic, and we have:

$$s_n(\epsilon) \geq C \cdot 2^n, \quad \text{for some } C > 0.$$

Taking logarithms,

$$\frac{1}{n} \log s_n(\epsilon) \geq \log 2 + \frac{1}{n} \log C \quad \Rightarrow \quad h_{FQU}(\psi_1) \geq \log 2 > 0.$$

This example illustrates that fuzzy quasi-uniform entropy is invariant under topological conjugacy, monotone in the parameter t , and consistent with classical topological entropy as $t \rightarrow 0$. It also shows that the proposed entropy notion detects chaotic behavior in asymmetric fuzzy quasi-metric settings.

4. CONCLUSION

In this paper, we extended Bowen's notion of uniform entropy and the quasi-uniform entropy of Haihambo and Olela-Otafudu to the broader setting of fuzzy quasi-metric spaces by introducing the *fuzzy quasi-uniform entropy* $h_{FQU}(\psi, M)$. Working within the framework of uniformly continuous self-maps, we developed this entropy using Bowen-type fuzzy quasi-balls and established its fundamental properties in the presence of asymmetry and fuzziness.

We showed that, under the condition $t \leq 1 - \epsilon$ for $\epsilon \in (0, 1)$, the fuzzy quasi-uniform entropy dominates the classical quasi-uniform entropy, namely $h_{FQU}(\psi) \geq h_{QU}(\psi)$, thereby demonstrating that the fuzzy quasi-metric framework captures finer distinctions in orbit complexity than its quasi-metric counterpart. In addition, we proved that h_{FQU} is monotone with respect to the fuzziness scale parameter t , invariant under topological conjugacy, and converges to classical topological entropy in the metric case as $t \rightarrow 0$.

The examples presented illustrate the effectiveness of the proposed theory, including situations in which fuzzy quasi-uniform entropy detects nontrivial dynamical behavior even when classical quasi-uniform entropy vanishes. Overall, these results deepen the theoretical understanding of entropy in fuzzy dynamical systems and provide a robust framework for analyzing complex dynamical phenomena characterized by asymmetry, uncertainty, or imprecision. Future work may address computational aspects, sharper entropy bounds, and applications to concrete fuzzy and non-symmetric dynamical systems.

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