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NUMERICAL SOLUTIONS FOR TURNING POINT CONVECTION-DIFFUSION PROBLEMS WITH TIME DELAY ON MODIFIED GRADED MESHES

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ABSTRACT. This paper investigates the numerical solutions of convection-diffusion problems with turning points and time delays, using modified graded meshes. Turning point problems, characterized by sharp gradients in solutions, possess significant challenges to numerical methods, particularly in terms of stability and convergence. To address these challenges, we use upwind difference scheme on a modified graded mesh to handle the steep gradients at turning points as well as stability and convergence more effectively. Numerical experiments are conducted to demonstrate the efficiency of the modified graded meshes. Results shows significant improvements and accuracy in the solution, especially near turning points. The paper concludes with a discussion on the implications of our findings.

1. INTRODUCTION

Consider the following singularly perturbed, time-delayed, turning point convection-diffusion equation on the domain $Q = (-1, 1) \times (0, T]$:

$$L_\varepsilon^s w(s, \theta) = c(s, \theta) w(s, \theta - \tau) + f(s, \theta), \quad on\ Q, \tag{1.1}$$

with initial and boundary conditions,

$$\begin{aligned} w(s, \theta) &= g(s, \theta), \text{for } -\tau \le \theta \le 0, \text{ and } s \in [-1, 1], \\ w(-1, \theta) &= w_l(\theta), \text{for } \theta \ge 0, \\ w(1, \theta) &= w_r(\theta), \text{for } \theta \ge 0, \end{aligned}$$

where $L_\varepsilon^s = -\varepsilon \frac{\partial^2 w}{\partial s^2} - a(s, \theta) \frac{\partial w}{\partial s} + d(s, \theta) \frac{\partial w}{\partial \theta} + b(s, \theta)$ is differential operator and $\tau > 0$ represents the time delay, $\varepsilon \in (0, 1]$ is a small parameter, are given constants. Let us denote $\overline{Q} = [-1, 1] \times [0, T]$, $\Upsilon = \Upsilon_l \cup \Upsilon_b \cup \Upsilon_r$, $w_l(\theta)$, $w_r(\theta)$, $a(s, \theta) \sim a_0(s, \theta)s$, $b(s, \theta) \ge \alpha > 0$, $d(s, \theta) \ge \beta > 0$, $c(s, \theta) \ge \delta > 0$, and $g(s, \theta)$ are specified functions.

Here, Q defines a temporal-spatial domain given by the product of the spatial interval $\omega = (-1, 1)$ and the time interval $(0, T]$. The boundary Υ of this domain is decomposed into and $\Upsilon_l = \{(-1, \theta) : 0 \le \theta \le T\}$ and $\Upsilon_r = \{(1, \theta) : 0 \le \theta \le T\}$, which correspond to the left and right ends of the rectangular domain(Q), and Υ_b, which is specified as the interval $[-1, 1] \times [-\tau, 0]$ representing a part of the boundary involving a delay effect with the delay parameter $\tau \ge 0$. This parameter is connected to the temporal domain through the relationship $T = q\tau$, where q is a positive integer, indicating a proportional relation between the domain

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time level and the delay parameter. Furthermore, the singular perturbation parameter ε is considered within the range $(0, 1]$, suggesting the analysis could be relevant with small perturbations. The functions $b(s, \theta), c(s, \theta), d(s, \theta), f(s, \theta)$, and the boundary functions Υ_b, Υ_r, Υ_l are assumed to be sufficiently smooth and bounded, ensuring that they behave well within the defined domain and under the given conditions.

The reduced problem corresponding to (1.1) is

$$\begin{cases} -a(s,\theta)\dfrac{\partial w(s,\theta)}{\partial s} + b(s,\theta)w(s,\theta) + d(s,\theta)\dfrac{\partial w(s,\theta)}{\partial \theta} = \\ c(s,\theta)w(s,\theta-\tau) + f(s,\theta), \quad \forall (s,\theta) \in \mathrm{Q}, \\ w(s,\theta) = g(s,\theta), \quad (s,\theta) \in \Upsilon_b = [-1,1] \times [-\tau, 0]. \end{cases} \tag{1.2}$$

The boundary layer under discussion exhibits characteristics of a parabolic type, a conclusion derived from the analysis where the solution to the reduced problem is lines that are parallel to the domains boundary and is achieved by setting the parameter s to a constant. Furthermore, the initial functions $g(s, \theta)$ are essential to meet compatibility conditions at corner points of the domain, namely at $(0, 0)$, $(1, 0)$, $(0, -\tau)$, and $(1, -\tau)$. Satisfying compatibility conditions at these points ensures a consistency where the boundary or initial conditions may change sharply. The required compatibility conditions for the function $F(s, \theta) = cw(s, \theta - \tau) + f(s, \theta)$ at the corners are stated below,

$$\frac{\partial^i w_l(\theta)}{\partial \theta^j} = 0, \frac{\partial^i w_r(\theta)}{\partial \theta^j} = 0, \frac{\partial^i g(s,\theta)}{\partial s^j} = 0,\ 0 \le i + 2j \le 6,$$
$$\frac{\partial^{i+j} F(s,\theta)}{\partial s^i \partial \theta^j} = 0,\ 0 \le i + 2j \le 4, \quad (s,\theta) \in \Upsilon_b.$$

Under the above compatibility condition, (1.1) possess unique solution [1], possessing layer at both the end points $s = -1$ and $s = 1$. As (1.1) is characterized by boundary layers, uniform meshes do not work properly [2], which results to the construction of non-uniform and layer adapted meshes.

Delay partial differential equations (DPDEs) and, more specifically, singularly perturbed delay differential equations (SPDPDEs) with turning point is quite insightful. These equations indeed possesses challenge due to the incorporation of delay terms and the singularity developed due to convection coefficient. The small parameter ε introduces complexity by causing rapid changes in the solution, necessitating efficient numerical methods for accurate solutions. The challenge of solving SPPDEs analytically, as highlighted, is largely due to the boundary and interior layers introduced by the small parameter ε. This difficulty is more in classical numerical methods by the need for an impractically large number of mesh points to accurately capture these layers, rendering such approaches ineffective for these problems. So, to overcome the layers, construction of graded meshes is highlighted as a significant advancement in enhancing the robustness of numerical methods, facilitating the handling of problems with boundary and interior layers. Recent advancements have shown that various numerical methods, including finite difference and finite element methods on layer-adapted meshes, offer effective solutions to these complex problems. Reaction-diffusion problems involving boundary layers with Shishkin, Bakhvalov, and adaptive meshes are discussed in a number of works [2–4]. Moreover, turning point problems are discussed in [5–8]. In order to develop the robustness of the method, the construction of the meshes plays crucial role in handling the boundary and interior layers. In recent days, problem possessing boundary and interior layer are better handled by various numerical methods such as finite difference and finite element on layer adapted meshes [9, 10]. But, in all the above discussed articles they either used Shishkin or Bakhvalov mesh to over come the boundary or interior layer. Moreover, in this article we have used modified graded mesh to overcome the boundary or interior layer, which is occurred due to singular perturbation parameter or the coefficient of convection term.

Notation: Throughout the manuscript, we denote by C as a generic constant that does not vary with respect to the perturbation parameter ε or the number of mesh points N. To analyze the convergence of the method, we employ the discrete maximum norm, which is characterized as follows:

$$\|w\|_Q = \max_{s\in Q} |w(s)|.$$

The paper is presented as follows. Starting with the examination of the analytical properties of the continuous problem in Section 2, followed by the introduction of modified graded meshes in Section 3 to refine the computational grid. Section 4 deals with the discretization techniques. Subsequent sections 5 assess the convergence characteristics of the method. Practical implementations presented in Section 6, validating the theoretical outcomes. The paper ends up with Section 7, highlighting the findings and contributions made to the field of numerical analysis.

2. Analytical behavior of the continuous problem

This section explores the bounds of the solution and its partial derivatives within the given domain. Additionally, to demonstrate ε-uniform convergence in a later section, we break down the analytical solution w of (1.1). The operator L^s_ε adheres to the following maximum principle.

Lemma 2.1. *[11] (Maximum principle) Let $\psi(s,\theta)$ be a function such that $\psi(s,\theta) \geq 0$ for all $(s,\theta) \in \Upsilon$. If for all $(s,\theta) \in Q$, we have $L^s_\varepsilon\psi(s,\theta) \geq 0$, then $\psi(s,\theta)$ is non-negative for all $(s,\theta) \in \overline{Q}$.*

Proof. Assume that $\psi(s,\theta) \geq 0$, for all $(s,\theta) \in \Upsilon$ and $L^s_\varepsilon\psi(s,\theta) \geq 0$ for all $(s,\theta) \in Q$. We aim to show that $\psi(s,\theta) \geq 0$, for all $(s,\theta) \in \bar{Q}$.
By hypothesis, $\psi(s,\theta) \geq 0$, for all $(s,\theta) \in \Upsilon$. This means that on the boundary ∂Q, $\psi(s,\theta)$ is non-negative. Suppose, for contradiction, that there exists a point $(s_0,\theta_0) \in \overline{Q}$ such that $\psi(s_0,\theta_0) < 0$. Let (s_0,θ_0) be a point in $\bar{Q}$ where $\psi(s,\theta)$ attains its minimum value. Since $\psi(s,\theta)$ is continuous and $\bar{Q}$ is compact, such a minimum exists. Let us consider two cases:
Case 1: Let $(s_0,\theta_0) \in \partial Q$

If (s_0,θ_0) is on the boundary ∂Q, then by the boundary condition, $\psi(s_0,\theta_0) \geq 0$. This contradicts our assumption that $\psi(s_0,\theta_0) < 0$.
Case 2: Let $(s_0,\theta_0) \in Q$

If (s_0,θ_0) is in the interior Q, then at this minimum point, the operator L^s_ε applied to $\psi(s,\theta)$ must satisfy $L^s_\varepsilon\psi(s_0,\theta_0) \leq 0$. However, by hypothesis, $L^s_\varepsilon\psi(s_0,\theta_0) \geq 0$. Therefore, we must have $L^s_\varepsilon\psi(s_0,\theta_0) = 0$. Since (s_0,θ_0) is a minimum point, the first and second derivatives of $\psi(s,\theta)$ at (s_0,θ_0) must satisfy:

$$\frac{\partial\psi}{\partial\theta}(s_0,\theta_0) \leq 0, \frac{\partial^2\psi}{\partial s^2}(s_0,\theta_0) \geq 0.$$

Combining these with the expression for L^s_ε, we get:

$$L^s_\varepsilon\psi(s_0,\theta_0) = -d\frac{\partial\psi}{\partial\theta}(s_0,\theta_0) + \varepsilon\frac{\partial^2\psi}{\partial s^2}(s_0,\theta_0) + b\psi(s_0,\theta_0) \leq 0.$$

This implies that:

$$0 \leq L^s_\varepsilon\psi(s_0,\theta_0) \leq 0.$$

Therefore, $L^s_\varepsilon\psi(s_0,\theta_0) = 0$. Given that $\psi(s,\theta)$ attains its minimum at (s_0,θ_0) with $\psi(s_0,\theta_0) < 0$, the only way for $L^s_\varepsilon\psi(s_0,\theta_0) = 0$ to hold is if $\psi(s,\theta)$ is a constant function in Q. But since $\psi(s,\theta) \geq 0$ on ∂Q, this constant must be non-negative, contradicting $\psi(s_0,\theta_0) < 0$.

Therefore, our initial assumption that there exists a point $(s_0, \theta_0) \in \overline{Q}$ where $\psi(s_0, \theta_0) < 0$ must be false. Hence, $\psi(s, \theta) \geq 0$ for all $(s, \theta) \in \overline{Q}$.

Lemma 2.2. *[12] (Uniform stability bounds) For the function η defined within the domain specified by the differential operator L_ε^s,*

$$\|\eta\| \leq (1 + \alpha T) \max\{\|L_s^\varepsilon \eta\|, \|\eta_l\|, \|\eta_r\|, \|\eta_b\|\},$$

and for any solution $w(s, \theta)$ corresponding to equation (1.1), it follows that

$$\|w(s, \theta)\| \leq (1 + \alpha T) \max\{\|f\|, \|w_l\|, \|w_r\|, \|g\|\},$$

where $\alpha = \max\{0, 1 - b\}$.

Proof. To prove the uniform stability bounds, we start by considering the function η within the specified domain of the differential operator L_ε^s. Let us denote by Q the domain where these functions are defined. First, we use the definition of the operator norm and the properties of the differential operator L_ε^s. By the given conditions, we have:

$$\|L_\varepsilon^s \eta\| \leq \|\eta\| \leq (1 + \alpha T)\|L_\varepsilon^s \eta\|.$$

Now, considering the boundary conditions η_l, η_r, and η_b, we can write:

$$\|\eta\| \leq (1 + \alpha T) \max\{\|L_\varepsilon^s \eta\|, \|\eta_l\|, \|\eta_r\|, \|\eta_b\|\}.$$

To establish the stability bounds for the solution $w(s, \theta)$, we proceed similarly. Let $w(s, \theta)$ be a solution to equation (1.1). By analogous reasoning and utilizing the properties of the differential operator and boundary conditions w_l, w_r, and $g(s, \theta)$, we have:

$$\|L_\varepsilon^s w\| \leq \|w\| \leq (1 + \alpha T)\|L_\varepsilon^s w\|.$$

Thus, the solution $w(s, \theta)$ satisfies the uniform stability bound:

$$\|w(s, \theta)\| \leq (1 + \alpha T) \max\{\|f\|, \|w_l\|, \|w_r\|, \|g\|\}.$$

Hence, the lemma is proven.

Theorem 2.3. *Let the data a, b, c, d, f in $C^{(4+\gamma,2+\gamma/2)}$ $(\overline{Q})$, $w_l(\theta)$, $w_r(\theta)$ in $C^{2+\gamma/2}([0, T])$, $g(s, \theta) \in C^{(6+\gamma,3+\gamma/2)}(\Upsilon_b) \cap (\overline{Q})$, where $\gamma \in (0, 1)$, satisfy the necessary compatibility requirement on the corner. As a result, the equation (1.1) possesses a unique solution u, with $w \in C^{(6+\gamma,3+\gamma/2)}$ $(\overline{Q})$. Furthermore, for all non-negative integers i, j such that $0 \leq i + 2j \leq 6$, the derivatives of the solution $w(s, \theta)$ adhere to prescribed bounds,*

$$\left\| \frac{\partial^{i+j} w}{\partial s^i \partial \theta^j} \right\| \leq C\varepsilon^{-i}.$$

Proof. The proof of existence and uniqueness is detailed in [13]. Their comprehensive analysis provides the theoretical foundation for the discussions presented here. Moreover, for the proof of bounds on derivatives, the variable s is substituted to $\varepsilon\vartheta = s$, adheres the methodology of [2].

In Theorem (2.3), we establish a bound for the exact solution $w(s, \theta)$. However, this bound alone does not suffice to confirm the ε-uniform convergence of the proposed method. To obtain more precise estimates for the derivatives of the exact solution, we employ a decomposition strategy, dividing the solution into its smooth components and layer components. This approach allows us to separately analyze the behavior of in the smooth regions and near the layers.

Let $w(s, \theta)$ be a function defined as the sum of $v(s, \theta)$ and $\hat{w}(s, \theta)$ for all (s, θ) within the domain $\overline{Q}$,

$$w(s, \theta) = v(s, \theta) + \hat{w}(s, \theta),$$

where the smooth component $v(s,\theta)$ satisfy the differential equations

$$(L_\varepsilon^s)\, v(s,\theta) = c(s,\theta)v(s,\theta-\tau) + f(s,\theta), \quad (s,\theta) \in Q,$$

with initial condition for $v(s,\theta)$ is

$$v(s,\theta) = w(s,\theta), \quad (s,\theta) \in \Upsilon_b,$$

and the boundary condition is specified as

$$v(-1,\theta) = w(-1,\theta),\ v(1,\theta) = w(1,\theta), \quad 0 \le \theta \le T.$$

We further decompose smooth component $v(s,\theta)$ as

$$v(s,\theta) = v_0(s,\theta) + \varepsilon v_1(s,\theta), \quad (s,\theta) \in \overline{Q},$$

where $v_0(s,\theta)$ is the solution of reduced problem

$$\begin{aligned} \left(d\frac{\partial}{\partial\theta} - a\frac{\partial}{\partial s} + b(s,\theta)\right) v_0(s,\theta) = &\quad -c(s,\theta)v_0(s,\theta-\tau) + f(s,\theta), \quad \forall\, (s,\theta) \in Q \\ v_0(s,\theta) = &\quad \varphi_b(s,\theta), \quad (s,\theta) \in \Upsilon_b, \end{aligned}$$

and $v_1(s,\theta)$ satisfies the following problem

$$\begin{aligned} \left(d\frac{\partial}{\partial\theta} - a\frac{\partial}{\partial s} + b(s,\theta)\right) v_1(s,\theta) = &\quad c(s,\theta)v_1(s,\theta-\tau) + \frac{\partial^2 v_0(s,\theta)}{\partial s^2}, \quad (s,\theta) \in Q \\ v_1(s,\theta) = &\quad 0, \quad (s,\theta) \in \Upsilon_b. \end{aligned}$$

Further v satisfies

$$\begin{aligned} (L_\varepsilon^s)\, v(s,\theta) = &\quad c(s,\theta)v(s,\theta-\tau) + f(s,\theta), \quad (s,\theta) \in Q, \\ v(s,\theta) = &\quad w(s,\theta), \quad (s,\theta) \in \Upsilon_b, \\ v(-1,\theta) = &\quad v_0(-1,\theta) \quad (s,\theta) \in \Upsilon_l, \quad v(1,\theta) = v_0(1,\theta), \quad (s,\theta) \in \Upsilon_r, \end{aligned}$$

and the singular components $\hat{w}$ satisfies

$$\begin{aligned} (L_\varepsilon^s)\, \hat{w}(s,\theta) = &\quad c(s,\theta)\hat{w}(s,\theta-\tau), \quad (s,\theta) \in Q, \\ \hat{w}(s,\theta) = &\quad 0, \quad (s,\theta) \in \Upsilon_b, \\ \hat{w}(-1,\theta) = &\quad w_l(\theta) - v_0(0,\theta), \quad (s,\theta) \in \Upsilon_r, \quad \hat{w}(1,\theta) = w_r(\theta) - v_1(1,\theta), \quad (s,\theta) \in \Upsilon_r. \end{aligned}$$

Furthermore, the decomposition $\hat{w}(s,\theta) = \hat{w}_l(s,\theta) + \hat{w}_r(s,\theta)$ is defined by

$$\begin{aligned} (L_\varepsilon^s)\, \hat{w}_l(s,\theta) = &\quad c(s,\theta)\hat{w}_l(s,\theta-\tau), \quad (s,\theta) \in Q, \\ \hat{w}_l(s,\theta) = &\quad 0, \quad (s,\theta) \in \Upsilon_b \cup \Upsilon_r, \\ \hat{w}_l(0,\theta) = &\quad w_l(\theta) - v_0(0,\theta) \quad (s,\theta) \in \Upsilon_l, \end{aligned}$$

and

$$\begin{aligned} (L_\varepsilon^s)\, \hat{w}_r(s,\theta) = &\quad c\hat{w}_r(s,\theta-\tau), \quad (s,\theta) \in Q, \\ \hat{w}_r(s,\theta) = &\quad 0, \quad (s,\theta) \in \Upsilon_b \cup \Upsilon_l, \\ \hat{w}_r(0,\theta) = &\quad w_r(\theta) - v_1(1,\theta) \quad (s,\theta) \in \Upsilon_r. \end{aligned}$$

The Theorem (2.4) presented below gives the estimation of both the singular and the regular components.

Theorem 2.4. *Let the data a, b, c, d, f in $C^{(4+\gamma,2+\gamma/2)}$ $(\overline{Q})$, $w_l(t)$, $w_r(t)$ in $C^{2+\gamma/2}([0,T])$, $g(s,\theta) \in C^{(6+\gamma,3+\gamma/2)}(\Upsilon_b) \cap (\overline{Q})$, where $\gamma \in (0,1)$, fulfills the necessary compatibility condition on the corner. Then, for every pair of non-negative integers i and j satisfying $0 \le i+2j \le 6$, the following inequalities are established for the smooth component v and the boundary layer components $\hat{w}_r$ and $\hat{w}_l$ in the decomposition of the solution w,*

$$\left\| \frac{\partial^{i+j} v}{\partial s^i \partial \theta^j} \right\| \le C(1+\varepsilon^{1-i}),$$

$$\left\| \frac{\partial^{i+j} \hat{w}_r}{\partial s^i \partial \theta^j} \right\| \le C\varepsilon^{-i} e^{-(1-s)/\sqrt{\varepsilon}},$$

and

$$\left\| \frac{\partial^{i+j} \hat{w}_l}{\partial s^i \partial \theta^j} \right\| \le C\varepsilon^{-i} e^{-(-1-s/)\sqrt{\varepsilon}}.$$

Proof. For the proof of theorem can be found in [14].

3. The modified graded mesh and mesh discretization

Modified graded mesh is a technique used in mesh generation for numerical simulations, specially in the finite difference method. In this, we divide the prescribed domain into elements of different sizes, with a gradual increase or decrease in element size. This graded mesh helps in capturing the solution behavior more accurately by placing more elements in areas where the solution varies rapidly or where high gradients are expected. This approach improves the overall efficiency and accuracy of the simulation, as it allows for a more refined discretization of the domain.

3.1. **Spatial discretization of modified graded mesh.** Modified graded mesh is generated by utilizing the following function which is defined piecewise in the prescribed domain $[-1,1]$,

$$\begin{cases} \mu_0 = -1, \\ \mu_j = 4\varepsilon \dfrac{2j}{N} - 1, \quad j = 1,2,3,\cdots,\dfrac{N}{4} \\ \mu_{j+1} = (1+\rho h)(\mu_j + 1) - 1, \quad j = \dfrac{N}{4}+1,\cdots,\dfrac{N}{2}-2 \\ \mu_{\frac{N}{2}} = 0, \\ \mu_j = -\mu_{N-j}, \quad j = \dfrac{N}{2}+1, +\dfrac{N}{2}+2,\cdots,N. \end{cases} \tag{3.1}$$

This set of equations defines a piecewise function for μ_j, where μ_0 is explicitly set to -1, $\mu_{N/2}$ is explicitly set to 0, and the remaining values are determined according to the given conditions. These equations define a sequence of values for μ_j where the pattern changes based on the ranges of j and with help of a non-linear equation in the parameter h,

$$\ln\left(\frac{1}{\varepsilon}\right) = (N/2)\ln(1+\rho h). \tag{3.2}$$

This method ensures a specific distribution of grid points within different sub-intervals of the interval $[-1,1]$. In each of the subinterval $[-1,-1+\varepsilon]$ and $[1-\varepsilon,1]$, $N/4$ points are distributed with a uniform step length of $8\varepsilon/N$. The parameter N represents the total number of grid points, and ε is the small positive value representing the width of the boundary regions. The choice of h within the central subinterval $[1+\varepsilon,1-\varepsilon]$ is determined iteratively using a nonlinear equation (3.2).

Remark 3.1. The mesh size of proposed modified graded satisfies the following bound.

$$h_j = \begin{cases} 8\varepsilon\frac{1}{N}, \quad \text{for } j = 1,\ 2\ ,\ \cdots,\ \frac{N}{4} \\ (1+\rho h)(\mu_{j-1} - \mu_{j-2}), \quad \text{for } j = \frac{N}{4}+1,\ \frac{N}{2}+2,\ \cdots,\ \frac{3N}{4} \\ 8\varepsilon\frac{1}{N}, \quad \text{for } j = \frac{3N}{4},\ \frac{3N}{4}+1,\ \cdots,N. \end{cases}$$

Lemma 3.2. *The mesh defined in (3.1) satisfies the following estimates.*

$$|h_{j+1} - h_j| \leq \begin{cases} Ch \quad for \quad j = \frac{N}{4}+1,\ \frac{N}{4}+2, \cdots,\ \frac{3N}{4}, \\ 0 \quad otherwise. \end{cases}$$

Proof. First, we examine the result for $j = 1,\ 2,\ \cdots,\ N/2$. There is nothing to establish because the mesh is uniform in that part.

We have the following for $j = \frac{N}{4}+1,\ \frac{N}{4}+2,\ \cdots,\ \frac{3N}{4}$,

$$\begin{aligned} |h_{j+1} - h_j| &= |(1+\rho h)\mu_j - (1+\rho h)\mu_{j-1} - (1+\rho h)\mu_{j-1} + (1+\rho h)\mu_{j-2}|, \\ &= (1+\rho h)|\mu_j - 2\mu_{j-1} + \mu_{j-2}|, \\ &= (1+\rho h)|(1+\rho h)(\mu_{j-1}+1) - 1 - 2\mu_{j-1} + \mu_{j-2}|, \\ &= (1+\rho h)|(\rho h - 1)\mu_{j-1} + \rho h + \mu_{j-2}|, \\ &= (1+\rho h)|(\rho h - 1)\{(1+\rho h)(\mu_{j-2}+1) - 1\} + \rho h + \mu_{j-2}|, \\ &= (1+\rho h)(|\rho^2 h^2 \mu_{j-2} + \rho^2 h^2|), \\ &\leq Ch^2, \\ &\leq Ch. \end{aligned}$$

In this case, we have taken $0 < \rho, h < 1$.

Lemma 3.3. *The parameter h for the modified graded mesh defined in* (3.1) *satisfies the bound below,*

$$h \leq CN^{-1} \ln\left(\frac{1}{\varepsilon}\right).$$

Proof. For the detailed proof, one can refer to [15].

3.2. **Temporal Discretization.** We introduce a mesh with uniform step length in the time direction, as there is no effect of layer in the temporal variable (θ). This uniform mesh in the time direction is denoted and defined as follows,

$$Q^M = \{\theta_{\hat{s}} = \hat{s}\Delta\theta, \quad \hat{s} = 0,\ 1,\ \cdots,\ M, \quad \Delta\theta M = T\}.$$

Here, M represents the number of mesh intervals in the temporal direction.

4. Discretization method

In order to discretize the spatial domain, we employ a non-uniform grid Q^N on Q, which includes N number of mesh intervals. Specifically, $N/4$ of these points are distributed uniformly within the layer region $[-1, -1+\varepsilon]$ and another $N/4$ are placed non-uniformly in the interval $[-1+\varepsilon, 0]$. The same distribution pattern applies for the interval $[0, 1-\varepsilon]$ (non-uniform) and $[1-\varepsilon, 1]$ (uniform). Furthermore, we introduce two equidistant grids, Q^M and Q^l, containing M and l grid points spacing uniform $\Delta\theta$ on $[0, T]$ and $[-\tau, 0]$ respectively. Consequently, the discretized domain is represented by

$$Q^N = Q_s^N \times Q^M, \quad \Upsilon_b^N = Q_s^N \times Q^l, \tag{4.1}$$

with boundary points $\Upsilon^N = \overline{Q}^N \cap \Upsilon$ and left/right boundaries defined as $\Upsilon_l^N = Q^N \cap \Upsilon_l$ and $\Upsilon_r^N = Q^N \cap \Upsilon_r$, respectively. Additionally, each discrete interval $Q_i^N = \omega^N \times Q_{i,\theta}^l$ is formed by uniform meshes over $[(i-1)\tau, i\tau]$. To deal with spatial and temporal derivatives, the discretization procedure employs the upwind difference approach and the backward Euler scheme, which are given as follows,

$$
\begin{cases}
\begin{cases}
\left(d(i,j)\mathcal{D}_\theta^- - \varepsilon\dfrac{\mathcal{D}_s^+ - \mathcal{D}_s^-}{\widehat{h_j}} - a(i,j)\mathcal{D}_s^- + b(i,j+1)\right)\mathfrak{U}_{i,j+1} \\
= c(i,j+1)\mathfrak{U}_{i,j-l} + f_{i,j+1}, \qquad \text{for } i = \frac{N}{2}+1,\cdots,N-1 \\
\mathfrak{U}_{0,j+1} = w_l(\theta_{j+1}) \\
\mathfrak{U}_{N,j+1} = w_r(\theta_{j+1}) \\
\mathfrak{U}_{i,-j} = g(s_i,-\theta_p),\ i=1,2,\cdots,N-1,\ p=0,1,\cdots,l
\end{cases} \\
\begin{cases}
\left(d(i,j)\mathcal{D}_\theta^- - \varepsilon\dfrac{\mathcal{D}_s^+ - \mathcal{D}_s^-}{\widehat{h_j}} - a(i,j)\mathcal{D}_s^+ + b(i,j+1)\right)\mathfrak{U}_{i,j+1} \\
= c(i,j+1)\mathfrak{U}_{i,j-l} + f_{i,j+1}, \qquad \text{for } i = 1,2,\cdots,\frac{N}{2} \\
\mathfrak{U}_{0,j+1} = w_l(\theta_{j+1}) \\
\mathfrak{U}_{N,j+1} = w_r(\theta_{j+1}) \\
\mathfrak{U}_{i,-j} = g(s_i,-\theta_p),\ i=1,2,\cdots,N-1,\ p=0,1,\cdots,l
\end{cases}
\end{cases}
\tag{4.2}
$$

The tri-diagonal system of linear equations shown below can be obtained by rearranging the equation (4.2) above,

$$(\alpha_{i,j+1})\mathfrak{U}_{i-1,j+1} + (\beta_{i,j+1})\mathfrak{U}_{i,j+1} + (\gamma_{i,j+1})\mathfrak{U}_{i+1,j+1} = h_{i,j}, \tag{4.3}$$

where

$$\alpha_{i,j+1} = \frac{-\varepsilon\Delta\theta}{h_i\hat{h}_j} + \frac{a_{i,j+1}\Delta\theta}{h_i}, \quad \beta_{i,j+1} = d_{i,j+1} + \frac{\varepsilon\Delta\theta}{h_{i+1}\hat{h}_i} + \frac{\varepsilon\Delta\theta}{h_i\hat{h}_i} - a(i,j+1)\frac{\Delta\theta}{h_i} + b_{i,j+1}\Delta\theta,$$

$$\gamma_{i,j+1} = \frac{-\varepsilon\Delta\theta}{h_{j+1}\hat{h}_j}, \quad h_{i,j} = -c(i,j+1)\mathfrak{U}_{i,j-l}\Delta\theta + f_{i,j+1}\Delta\theta + \mathfrak{U}_{i,j},$$

Also, by rearranging the equation (4.2), we get the tri-diagonal system which is stated below,

$$(\bar{\alpha}_{i,j+1})\mathfrak{U}_{i-1,j+1} + (\bar{\beta}_{i,j+1})\mathfrak{U}_{i,j+1} + (\bar{\gamma}_{i,j+1})\mathfrak{U}_{i+1,j+1} = \bar{h}_{i,j}, \tag{4.4}$$

where

$$\bar{\alpha}_{i,j+1} = \frac{-\varepsilon\Delta\theta}{h_i\hat{h}_j} + \frac{a_{i,j+1}\Delta\theta}{h_i}, \quad \bar{\beta}_{i,j+1} = d_{i,j+1} + \frac{\varepsilon\Delta\theta}{h_{i+1}\hat{h}_i} + \frac{\varepsilon\Delta\theta}{h_i\hat{h}_i} + a(i,j+1)\frac{\Delta\theta}{h_{i+1}} + b_{i,j+1}\Delta\theta,$$

$$\bar{\gamma}_{i,j+1} = \frac{-\varepsilon\Delta\theta}{h_{j+1}\hat{h}_j} - a(i,j+1)\frac{\Delta\theta}{h_{i+1}}, \quad \bar{h}_{i,j} = -c(i,j+1)\mathfrak{z}_{i,j-l}\Delta\theta + f_{i,j+1}\Delta\theta + \mathfrak{z}_{i,j},$$

$$\hat{h}_j = \frac{h_{j+1}+h_j}{2},$$

and the step length $\Delta\theta$ is chosen such that the product $l\Delta\theta = \tau$, and $\theta_j = j\Delta\theta$ for all $j \geq -l$, with l being a positive integer. Furthermore, the mesh function is denoted as $\widetilde{v}(s_i,\theta_j) = \widetilde{v}_{i,j}$, and is defined by

$$\mathcal{D}_s^+\widetilde{v}_{i,j} = \frac{\widetilde{v}_{i+1,j} - \widetilde{v}_{i,j}}{h_{i+1}}, \quad \mathcal{D}_s^-\widetilde{v}_{i,j} = \frac{\widetilde{v}_{i,j} - \widetilde{v}_{i-1,j}}{h_i},$$

$$\mathcal{D}_\theta^-\widetilde{v}_{i,j} = \frac{\widetilde{v}_{i,j} - \widetilde{v}_{i,j-1}}{h_{i+1}},$$

and

$$\delta_s^2\widetilde{v}_{i,j} = \frac{(\mathcal{D}_s^+ - \mathcal{D}_s^-)\widetilde{v}_{i,j}}{\hat{h}_i}.$$

5. Error convergence

Lemma 5.1. *[16] (Discrete minimum principle) Suppose that $\psi(s_i,\theta_j)$, the mesh function satisfies $\psi(s_i,\theta_j) \geq 0$ on Υ^N. If $L_\varepsilon^{s,N}\psi(s_i,\theta_j) \geq 0$ on $(s_i,\theta_j) \in Q^N$, then $\psi(s_i,\theta_j) \geq 0$ on $\overline{Q}^N$.*

Proof. Assume, for contradiction, that there exists a point $(s_0,\theta_0) \in \mathbb{N}$ such that $\psi(s_0,\theta_0) < 0$, and that $\psi(s_0,\theta_0)$ is the minimum value of ψ in ∂Q.

Since $\psi(s_0,\theta_0)$ is a minimum, then for the differential operator $L_\varepsilon^{s,N}$, the first-order partial derivatives of ψ vanish at the point $\psi(s_0,\theta_0)$, and also the second-order partial derivatives (involved in $L_\varepsilon^{r,N}$) at (s_0,θ_0) should indicate upward concavity for ψ.

Given that $L_\varepsilon^{s,N}\psi(s_0,\theta_0) \geq 0$ by assumption, and $\psi(s_0,\theta_0)$ is a minimum, ψ must have upward concavity at this point. However, this contradicts our assumption that $\psi(s_0,\theta_0) < 0$, as upward concavity would imply that the minimum value cannot be negative if all other values are non-negative.

Furthermore, if ψ attains its minimum on $\overline{\partial Q}$, then by the assumption that $\psi(s_0,\theta_0) \geq 0$ for all $(s_0,\theta_0) \in \Upsilon^N$ (and hence on $\overline{\partial Q}$), ψ must also be non-negative. Thus, we conclude that $\psi(s_0,\theta_0) \geq 0$ on $\overline{Q}$.

Lemma 5.2. *Let $\mathfrak{U}(s_i,\theta_j)$ be the mesh function which satisfies the finite difference scheme (4.2), then we have*

$$\|\mathfrak{U}((s_i,\theta_j))\| \leq (1+\alpha T)\max(\|L_\varepsilon^{\theta,N}\||, \|w_l\|_{\Upsilon^N}, \|w_r\|_{\Upsilon^N}, \|g\|_{\Upsilon^N}) \tag{5.1}$$

Proof. By formulating the barrier function

$$\hat{B}^{\pm}(s_i,\theta_j) = (1+\alpha T)\max(\|L_\varepsilon^{\theta,N}\||, \|w_l\|_{\Upsilon^N}, \|w_r\|_{\Upsilon^N}, \|g\|_{\Upsilon^N}) \pm \mathfrak{U}(s_i,\theta_j) \tag{5.2}$$

and invoking (5.1) on (5.2) provides us with (5.1).

Theorem 5.3. *Let w and $\mathfrak{U}$ denote the solutions of the continuous problem (1.1) and its discretized counterpart (4.2) respectively, with the additional condition that both the solutions satisfy the compatibility criteria at the corners. Under these conditions, the error estimation between the continuous and discretized solution is given by,*

$$\max|(w-\mathfrak{U})(s_i,\theta_j)| \leq C[\Delta\theta + N^{-1}\ln\left(\frac{1}{\varepsilon}\right)], \quad (s_i,\theta_j) \in Q^N. \tag{5.3}$$

In this context, the value of C does not depend on N, $\Delta\theta$ and perturbation parameter ε.

Proof. Drawing upon the techniques presented in [3] and [2], we were able to corroborate the stated result considering the mesh briefed in Section 3.1, Section 3.2 and Lemma (3.3). Our confirmation process involved meticulously following the steps formulated in the above references.

To establish the results stated above for various time level, we first split our domain Q into $Q = Q_1 \cup Q_2$, where $Q_1 = (-1,1) \times [0,\tau]$ and $Q_2 = (-1,1) \times [\tau, 2\tau]$. And the discretized domain are defined as $Q^N = Q_1^N \times Q_2^N$, where $Q_1^N = \omega_s^N \times Q^l$ and $Q_2^N = \omega_s^N \times Q^M$.

Initially, for θ within the interval $[0,\tau]$, the right-hand side of equation (1.1) becomes $c(s,\theta)w(s,\theta-\tau) + f(s,\theta)$, which is free from the parameter ε. This allows for the direct application of the results presented in [3], leading to the equation (5.4),

$$\max|(w-\mathfrak{U})(s_i,\theta_j)| \leq C[\Delta\theta + N^{-1}\ln\left(\frac{1}{\varepsilon}\right)]. \tag{5.4}$$

Conversely, for $\theta \geq \tau$, it is considerably complex to make bound on the absolute maximum difference between numerical and analytical solution, as the term $w(s,\theta-\tau)$ is not free from ε. This requires a thorough examination of the relationship between the analytical solution w and the numerical solution $\mathfrak{U}$ within the interval $[\tau, 2\tau]$. Consider the following singularly perturbed PDE

$$\begin{aligned} L^s_\varepsilon w(s,\theta) &= c(s,\theta)w(s,\theta-\tau)+f(s,\theta), \quad (s,\theta)\in Q_2^N \\ w(s,\theta) &= w(s,\tau_l), \quad s\in(0,1) \\ w(-1,\theta) &= w_l(\theta), \quad w(1,\theta)=w_r(\theta), \quad \theta\in[\tau,2\tau] \end{aligned} \tag{5.5}$$

To compute the numerical solution of the specified SPPDE, we discretize the equation (5.5) by applying the backward Euler method for the temporal derivatives and the upwind finite difference scheme for the spatial derivatives.

$$L^\theta_\varepsilon \mathfrak{U}(s_i,\theta_j) \equiv \begin{cases} D^-_\theta \mathfrak{U}_{i,j} - \varepsilon\delta^2_s \mathfrak{U}_{i,j} - a_{i,j}D^-_s\mathfrak{U}_{i,j} + b_{i,j}\mathfrak{U}_{i,j} \\ = c_{i,j}\mathfrak{U}_{i,j-l} + f(s_i,\theta_j), \quad if a(s_i,\theta_j)\geq 0, \\ D^-_\theta \mathfrak{U}_{i,j} - \varepsilon\delta^2_s \mathfrak{U}_{i,j} - a_{i,j}D^+_s\mathfrak{U}_{i,j} + b_{i,j}\mathfrak{U}_{i,j} \\ = c_{i,j}\mathfrak{U}_{i,j-l} + f(s_i,\theta_j), \quad if a(s_i,\theta_j)\leq 0, \end{cases} \tag{5.6}$$

with initial and boundary condition

$$\begin{aligned} \mathfrak{U}(s_i,\theta_j) &= \mathfrak{U}_1(s_i,\theta_j), \quad (s_i,\theta_j)\in Q^N, \\ \mathfrak{U}(0,\theta_j) &= w_l(\theta_j), \\ \mathfrak{U}(1,\theta_j) &= w_r(\theta_j), \quad \theta_j\in Q^l_{2,\theta}, \end{aligned}$$

where the approximate solution over the interval Q_1^N is denoted by $\mathfrak{U}_1(s_i,\theta_j)$.
We now decompose the analytical solution w of (1.1) into its regular component r and layer component s, such that $w=r+s$. Furthermore, the regular component r can be expressed as $r=r_0+\varepsilon r_1$, where r_0 is the solution to the reduced problem,

$$\begin{aligned} \left(d\frac{\partial}{\partial\theta} - a\frac{\partial}{\partial s} + b(s,\theta)\right) r_0(s,\theta) &= cr_0(s,\theta-\tau)+f(s,\theta), \quad (s,\theta)\in(0,1)\times(\tau,2\tau) \\ r_0(s,\theta) &= r(s,\theta), \quad (s,\theta)\in(-1,1)\times[0,\tau] \\ r_0(0,\theta) &= r(0,\theta), \quad \theta\in[\tau,2\tau], \end{aligned}$$

and

$$\begin{aligned} \left(d\frac{\partial}{\partial\theta} - a\frac{\partial}{\partial s} + b(s,\theta)\right) r_1(s,\theta) &= c(s,\theta)r_1(s,\theta-\tau)+\frac{\partial^2 r_0(s,\theta)}{\partial s^2} \\ r_1(s,\theta) &= 0, \quad (s,\theta)\in(-1,1)\times[0,\tau] \\ r_1(-1,\theta) &= r_1(1,\theta)=0, \quad \theta\in[\tau,2\tau]. \end{aligned}$$

Further r satisfies

$$\begin{aligned} L^s_\varepsilon r(s,\theta) &= c(s,\theta)r(s,\theta-\tau)+f(s,\theta), \quad (s,\theta)\in(0,1)\times[\tau,2\tau] \\ r(s,\theta) &= w(s,\theta), \quad (s,\theta)\in(-1,1)\times[0,\tau] \\ r(-1,\theta) &= r_0(-1,\theta), \quad r(1,\theta)=r_1(1,\theta), \quad \theta\in[\tau,2\tau]. \end{aligned}$$

and the layer part $\hat{s}$ satisfies

$$\begin{aligned} L^s_\varepsilon \hat{s}(s,\theta-\tau) &= c\hat{s}(s,\theta-\tau), \quad (s,\theta)\in(-1,1)\times(\tau,2\tau) \\ \hat{s}(s,\theta) &= 0, \quad (s,\theta)\in(-1,1)\times[0,\tau] \\ \hat{s}(1,\theta) &= 0, \quad \hat{s}(0,\theta)=w_l(\theta)-r_0(0,\theta), \quad \theta\in[\tau,2\tau]. \end{aligned}$$

For further investigation of equation (5.6), we decompose the numerical solution, denoted by $\mathfrak{U}$, into two distinct components: the regular part, $\mathcal{R}$, and the layer part, $\mathcal{S}$. Here, $\mathcal{S}$ is identified as the component for the layer phenomena within the solution structure, and $\mathcal{R}$ signifies the regular portion that solves the associated non-homogeneous problem,

$$\begin{aligned} L_{\varepsilon}^{s,N}\mathcal{R} &= c\mathcal{R}(s_i,\theta_{j-l}) + f(s_i,\theta_j), \quad (s_i,\theta_j) \in Q_2^N \\ \mathcal{R}(s_i,\theta_j) &= \mathfrak{U}_1(s_i,\theta_j), \quad (s_i,\theta_j) \in (-1,1)\times(0,\tau) \\ \mathcal{R}(0,\theta_j) &= r(0,\theta_j), \quad \mathcal{R}(1,\theta_j) = r(1,\theta_j), \quad \theta_j \in Q_{2,\theta}^l, \end{aligned}$$

and the singular component $\mathcal{S}$ bound to satisfy,

$$\begin{aligned} L_{\varepsilon}^{s,N}\mathcal{S} &= c\mathcal{S}(s_i,\theta_{j-l}), \quad (s_i,\theta_j) \in Q_2^N \\ \mathcal{S}(s_i,\theta_j) &= 0, \quad (s_i,\theta_j) \in Q_1^N \\ \mathcal{S}(1,\theta_j) &= 0, \quad \mathcal{S}(0,\theta_j) = w_l(\theta_j) - r(0,\theta_j), \quad \theta_j \in Q_{2,\theta}^l. \end{aligned}$$

Therefore, the error can be written in the form,

$$\mathfrak{U} - w = (\mathcal{R} - r) + (\mathcal{S} - \hat{s})$$

We will now determine the bounds for both the smooth and layer components. To establish the bounds of the smooth component, we will employ a conventional method. The smooth error component can be expressed as follows,

$$\begin{aligned} L_{\varepsilon}^{s,N}(\mathcal{R} - r) &= c\mathcal{R}(s_i,\theta_{j-l}) + f(s_i,\theta_j) - L_{\varepsilon}^{s,N} r \\ &= c\mathcal{R}(s_i,\theta_{j-l}) + L_{\varepsilon}^{s,N} w + cw(s_i,\theta_{j-l}) - L_{\varepsilon}^{s,N} r \\ &= c(r(s_i,\theta_{j-l}) - \mathcal{R}(s_i,\theta_{j-l})) + L_{\varepsilon}^{s,N} r \end{aligned}$$

thus we obtain

$$L_{\varepsilon}^{s,N}(\mathcal{R} - r) = c(r(s_i,\theta_{j-l}) - \mathcal{R}(s_i,\theta_{j-l})) - \varepsilon(\frac{\partial^2}{\partial s^2} - \delta_s^2)r + d(s_i,\theta_j)(\frac{\partial}{\partial \theta} - \delta_\theta)r - a(s_i,\theta_j)(\frac{\partial}{\partial s} - \delta_s)r.$$

Using (5.3) with absolute values on both sides yields the following inequality:

$$|L_{\varepsilon}^{s,N}(\mathcal{R} - r)| \le C(\Delta\theta + N^{-1}\ln\left(\frac{1}{\varepsilon}\right)) + \varepsilon|(\frac{\partial^2}{\partial s^2} - \delta_s^2)r + d(s_i,\theta_j)(\frac{\partial}{\partial \theta} - \delta_\theta)r - a(s_i,\theta_j)(\frac{\partial}{\partial s} - \delta_s)r|.$$

Expanding through Taylor series expansion and appropriately arranging the reminder term, we get

$$\le C(\Delta\theta + N^{-1}\ln\left(\frac{1}{\varepsilon}\right)) + \frac{\varepsilon}{12}(s_{i+1} - s_{i-1})^2\left\|\frac{\partial^4 r}{\partial s^4}\right\| + d(s_i,\theta_j)\frac{(\theta_j - \theta_{j-1})}{2}\left\|\frac{\partial^2 r}{\partial \theta^2}\right\| + a(s_i,\theta_j)\frac{(s_i - s_{i-1})}{2}\left\|\frac{\partial^2 r}{\partial s^2}\right\|.$$

Utilizing the discrete minimum principle, bounds of mesh lengths, and derivative estimations, we have

$$|(\mathcal{R} - r)(s_i,\theta_j)| \le C[\Delta\theta + N^{-1}\ln\left(\frac{1}{\varepsilon}\right)]. \tag{5.7}$$

Like continuous solution s, the mesh function $\mathcal{S}$ is discretized for estimating the layer component.

$$\begin{aligned} L_{\varepsilon}^{s,N}\mathcal{S} &= c\mathcal{S}(s_i,\theta_{j-l}), \quad (s_i,\theta_j) \in Q_2^N \\ \mathcal{S}(s_i,\theta_j) &= 0, \quad (s_i,\theta_j) \in Q_2^N \\ \mathcal{S}(1,\theta_j) &= 0, \quad \mathcal{S}(0,\theta_j) = w_l(\theta_j) - r(0,\theta_k), \quad \theta_j \in Q_{2,\theta}^l. \end{aligned}$$

The singular component error can be expressed as,

$$\begin{aligned} L_\varepsilon^{s,N}(\mathcal{S}-\hat{s}) &= L_\varepsilon^{s,N}\mathcal{S} - L_\varepsilon^{s,N}\hat{s}, \\ &= c\mathcal{S}(s_i,\theta_{j-l}) - L_\varepsilon^{s,N}\hat{s}, \\ &= -\varepsilon(\frac{\partial^2}{\partial s^2} - \delta_s^2)s + (\frac{\partial}{\partial\theta} - \delta_\theta)\hat{s}, \end{aligned}$$

then the classical estimates gives

$$|L_\varepsilon^{s,N}(\mathcal{S}-\hat{s})(s_i,\theta_j)| \le C[\Delta\theta + N^{-1}\ln\left(\frac{1}{\varepsilon}\right)].$$

The operator $L_\varepsilon^{s,N}$ adheres to the discrete minimum principle, and furthermore, owing to the property that the inverse operator is uniformly bounded, the aforementioned inequality reduces to,

$$|(\mathcal{S}-\hat{s})(s_i,\theta_j)| \le C[\Delta\theta + N^{-1}\ln\left(\frac{1}{\varepsilon}\right)]. \tag{5.8}$$

Equation (5.7) and (5.8) combinedly finalize the proof for the interval $[\tau, 2\tau]$. We proceed to evaluate the error in subsequent time intervals utilizing th method of induction.

6. Numerical findings

This section presents numerical results to show the efficiency of the method and the validation of the theoretical findings.

Example 6.1. Consider the following problem

$$\begin{cases} \frac{\partial w(s,\theta)}{\partial\theta} + \varepsilon\frac{\partial^2 w(s,\theta)}{\partial s^2} + 2(2s-1)\frac{\partial w(s,\theta)}{\partial s} + 4w(s,\theta) = w(s,\theta-1), \quad (s,\theta)\in(0,1)\times(0,2], \\ w(s,\theta) = 1, \quad (s,\theta)\in(0,1)\times[-1,0], \\ w(0,\theta)=1, \quad w(1,\theta)=1, \quad \theta\in[0,2]. \end{cases}$$

As the exact solution to the problem is not known, so we employ the double mesh principle to get the numerical solution. Additionally, the maximum pointwise error is calculated for every value of the perturbation parameter ε

$$e_\varepsilon^{N,\Delta\theta} = \max|(w-\mathfrak{U})(s_j,\theta_k)| \quad (s_j,\theta_k)\in Q^N,$$

where ω is the exact solution and $\mathfrak{U}$ is approximate solution obtained through the scheme proposed. The order of convergence is given by,

$$p_\varepsilon^{N,\Delta\theta} = \frac{log(e^{N,\Delta\theta}/e^{2N,\Delta\theta/2})}{log2}.$$

The findings of numerical solutions for Example 6.1, presented through various tables and figures. Table 1 presents the computed maximum pointwise error and their related order of convergence. The development of these solutions and their error is visually represented in Figures 1. All calculated at $N=128$, $\varepsilon=10^{-4}$.

Table 1. *Maximum point-wise error and order of convergence on a modified graded mesh for Example 6.1*

	Number of Intervals $N/\Delta\theta$			
ε	$128/\frac{1}{10}$	$256/\frac{1}{20}$	$512/\frac{1}{40}$	$1024/\frac{1}{80}$
10^{-4}	1.2568E-02	6.7256E-03	3.5443E-03	1.8155E-03
	0.9021	0.9241	0.9651	
10^{-6}	1.6804E-02	9.4107E-03	4.9999E-03	2.5697E-03
	0.9338	0.8365	0.9124	
10^{-8}	2.0207E-02	1.1812E-02	6.3899E-03	3.3235E-03
	0.7746	0.8864	0.9740	
10^{-10}	2.3184E-02	1.4025E-02	7.7176E-03	4.0488E-03
	0.8359	0.7251	0.8618	

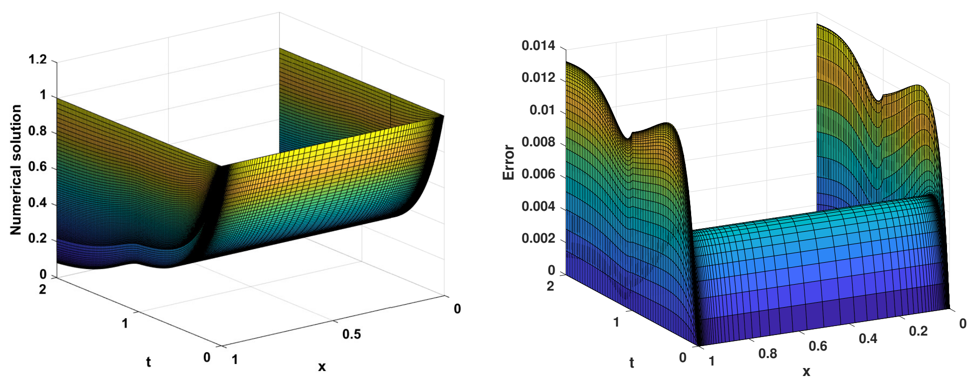

Figure 1. Solution and error profile for Example 6.1

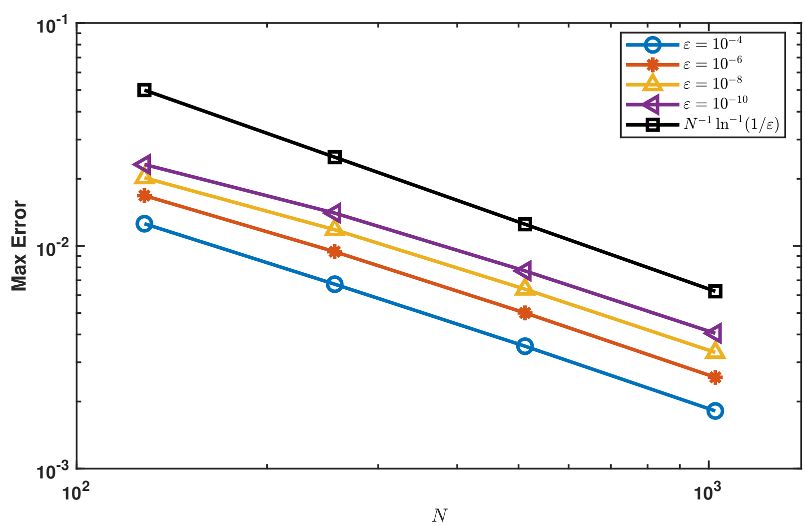

Figure 2. Log-log plot for Example 6.1

7. Conclusions

Our study's results affirm the efficiency of the finite difference method, improved with the upwind scheme and utilized on a modified graded mesh, for addressing singularly perturbed time-delayed one-dimensional turning point problems. Evaluating the error bounds in the discrete maximum norm shows that using a modified graded mesh achieves a higher order of convergence in numerical solutions. Ultimately, this research not only enhances the understanding of singularly perturbed problems but also highlights the benefits of incorporating modified graded meshes into numerical simulation frameworks.

Competing interests. The authors declare no competing interests.

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