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BIVARIATE POLY-FUBINI POLYNOMIALS AND NUMBERS OF PARAMETERS a, b, c

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ABSTRACT. In this paper, a new class of poly-Fubini polynomials and numbers with a, b, c parameters that generalizes classical Fubini polynomials and numbers is introduced. Fundamental properties of these bivariate poly-Fubini polynomials such as addition formulas, explicit formulas, derivative and integral formulas and other summation identities are established. Moreover, we also obtain relations of these generalized bivariate poly-Fubini polynomials with the classical Fubini polynomials and numbers, bivariate Fubini polynomials, Stirling numbers of the second kind and other special polynomials and numbers.

1. INTRODUCTION

The classical Fubini polynomials or geometric polynomials $F_n(y)$ are defined in [20] by

$$F_n(y) = \sum_{k=0}^{n} S_2(n,k)k!y^k, \tag{1.1}$$

where $S_2(n,k)$ is the Stirling numbers of the second kind [8,11]. Setting $y = 1$ in (1.1), we obtain the n^{th} Fubini number (sometimes called ordered Bell number) F_n, defined by

$$F_n(1) := F_n = \sum_{k=0}^{n} S_2(n,k)k!. \tag{1.2}$$

The number F_n combinatorially counts all the possible set partitions of an n-element set such that the order of the blocks matters.

The polynomials $F_n(y)$ satisfy the generating function:

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}, \tag{1.3}$$

and the recurrence relation:

$$F_{n+1}(y) = y\frac{d}{dy}\left[F_n(y) + yF_n(y)\right] \quad (see\ \ [10]). \tag{1.4}$$

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As a generalization of the classical Fubini polynomials, the Fubini polynomials of two variables or bivariate Fubini polynomials $F_n(x, y)$ were introduced by Kargin [16] given by

$$\frac{e^{xt}}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!} \quad \text{(see also [2,17,18])}. \tag{1.5}$$

Setting $x = 0$, (1.5) reduces to the classical Fubini polynomials.

For an integer k, the polylogarithm function $\mathrm{Li}_k(x)$ is defined via the formal power series

$$\mathrm{Li}_k(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^k}, \quad x \in \mathbb{C}, |x| < 1. \tag{1.6}$$

If $k \leq 0$, say $k = -s$, then it converges for $|x| < 1$ and is given by

$$\mathrm{Li}_{-s}(x) = \frac{\sum_{j=0}^{s} \left\langle {s \atop j} \right\rangle x^{s-j}}{(1-x)^{s+1}},$$

where $\left\langle {s \atop j} \right\rangle$ are the Eulerian numbers. The number $\left\langle {s \atop j} \right\rangle$ is the number of permutations of $\{1, 2, \cdots, s\}$ with j permutation ascents. Moreover,

$$\left\langle {s \atop j} \right\rangle = \sum_{l=0}^{j+1} \binom{s+1}{l} (j-l+1)^s.$$

In the case when $k = 1$ in (1.6),

$$\mathrm{Li}_1(x) = -\ln(1-x),$$

where $\ln(z)$ is the principal branch of the complex logarithm ln(z) with the imaginary part restricted by $-\pi <$ Im$(\ln(z)) \leq \pi$.

In [5], Acala and Macababat defined poly-Fubini polynomials of two variables $F_n^{(k)}(x, y)$ using polylogarithm via the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))} e^{xt} = \sum_{n=0}^{\infty} F_n^{(k)}(x,y)\frac{t^n}{n!}. \tag{1.7}$$

These polynomials satisfy the explicit formula

$$F_n^{(k)}(x,y) = \frac{1}{n+1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} F_{n+1}(x-j,y). \tag{1.8}$$

Setting $k = 1$, (1.7) reduces to (1.5).

In this paper, we introduce a generalization of the Fubini polynomials of two variables using polylogarithm and introducing parameters a, b, c. We then establish its fundamental properties and obtain its relations with other related special polynomials and numbers.

2. Poly-Fubini Polynomials of Parameters a, b, c

We now define poly-Fubini polynomials of parameters a, b, c.

Definition 2.1. For $k \in \mathbb{Z}$, we define the *bivariate poly-Fubini polynomials of parameters* a, b, c denoted by $F_n^{(k)}(x, y; a, b, c)$ through the generating function:

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))} c^{xt} = \sum_{n=0}^{\infty} F_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}. \tag{2.1}$$

The polynomials $F_n^{(k)}(y;a,b,c) := F_n^{(k)}(0,y;a,b,c)$ are called *univariate poly-Fubini polynomials* of parameters a,b,c. When $c=e$, the polynomials $F_n^{(k)}(y;a,b,e) := F_n^{(k)}(x,y;a,b)$ are called the *bivariate poly-Fubini polynomials of parameters* a,b and the polynomials $F_n^{k}(0,y;a,b) := F_n^{(k)}(y;a,b)$ are called *univariate poly-Fubini polynomials of parameters* a,b.

Setting $a=1, b=c=e$, and $k=1$, the bivariate poly-Fubini polynomials of parameters a,b,c, $F_n^{(k)}(x,y;a,b,c)$ reduce to the bivariate Fubini polynomials introduced by Kargin [16].

The polynomials $F_n^{(k)}(x,y;a,b,c)$ satisfy the following addition formula.

Theorem 2.2 (Addition Formulas). *For $k\in\mathbb{Z}$ and $n\geq 0$,*

$$F_n^{(k)}(x+z,y;a,b,c) = \sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}F_m^{(k)}(x,y;a,b,c)z^{n-m} \tag{2.2}$$

$$= \sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}F_m^{(k)}(z,y;a,b,c)x^{n-m}. \tag{2.3}$$

Proof. Using (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}F_n^{(k)}(x+z,y;a,b,c)\frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))}c^{xt}c^{zt}\\ &= \sum_{n=0}^{\infty}F_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}\sum_{n=0}^{\infty}(\ln c)^n z^n\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}F_m^{(k)}(x,y;a,b,c)z^{n-m}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (2.2), and interchanging the roles of x and z in (2.2) gives (2.3). □

Taking $z=0$ in (2.3), we obtain a relationship between the two-variable poly-Fubini polynomials of parameters a,b,c, $F_n^{(k)}(x,y;a,b,c)$ and the univariate poly-Fubini polynomials of parameters a,b,c, $F_n^{(k)}(y;a,b,c)$ in the next corollary.

Corollary 2.3. For $k\in\mathbb{Z}$ and $n\geq 0$,

$$F_n^{(k)}(x,y;a,b,c) = \sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}F_m^{(k)}(y;a,b,c)x^{n-m}. \tag{2.4}$$

The poly-Fubini polynomials of two variables satisfy the following derivative and integral properties:

Theorem 2.4 (Derivative and Integral Formulas). *For $k\in\mathbb{Z}$ and $n\geq 0$,*

$$\frac{\partial}{\partial x}F_{n+1}^{(k)}(x,y;a,b,c) = (n+1)(\ln c)F_n^{(k)}(x,y;a,b,c) \tag{2.5}$$

$$\int F_n^{(k)}(x,y;a,b,c)dx = \frac{1}{(n+1)\ln c}F_{n+1}^{(k)}(x,y;a,b,c)+C, \tag{2.6}$$

for any constant C.

Proof. Using Corollary 2.3, we have

$$F_{n+1}^{(k)}(x,y;a,b,c) = \sum_{m=0}^{n+1}\binom{n+1}{m}(\ln c)^{n+1-m}F_m^{(k)}(y;a,b,c)x^{n+1-m}. \tag{2.7}$$

Differentiating both sides of (2.7) with respect to x gives

$$\begin{aligned}\frac{\partial}{\partial x}F_{n+1}^{(k)}(x,y;a,b,c) &= \sum_{m=0}^{n+1}\binom{n+1}{m}(\ln c)^{n+1-m}(n+1-m)F_m^{(k)}(y;a,b,c)x^{(n+1-m)-1}\\ &= (n+1)(\ln c)\sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}F_m^{(k)}(y;a,b,c)x^{n-m}\\ &= (n+1)(\ln c)F_n^{(k)}(x,y;a,b,c).\end{aligned}$$

Equation (2.6) follows directly from (2.5). □

The next identity gives the relation between $F_n^{(k)}(x,y;a,b,c)$ and $F_n^{(k)}(x,y)$.

Theorem 2.5. *For $k\in\mathbb{Z}$ and $n\geq 0$,*

$$F_n^{(k)}(x,y;a,b,c) = (\ln a+\ln b)^{n+1}F_n^{(k)}\left(\frac{x\ln c+\ln a}{\ln a+\ln b},y\right).$$

Proof. From (2.1) and (1.7), we get

$$\begin{aligned}\sum_{n=0}^{\infty}F_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))}c^{xt}\\ &= \frac{\mathrm{Li}_k(1-e^{-t\ln ab})}{ta^{-t}(1-y[(ab)^t-1])}e^{tx\ln c}\\ &= \frac{\ln ab\cdot\mathrm{Li}_k(1-e^{-t\ln ab})}{t\ln ab\cdot[1-y(e^{t\ln ab}-1)]}e^{tx\ln c}\cdot e^{t\ln a}\\ &= \frac{\ln ab\cdot\mathrm{Li}_k(1-e^{-t\ln ab})}{t\ln ab\cdot(1-y(e^{t\ln ab}-1))}e^{t\ln ab\left(\frac{x\ln c+\ln a}{\ln ab}\right)}\\ &= \sum_{n=0}^{\infty}(\ln a+\ln b)^{n+1}F_n^{(k)}\left(\frac{x\ln c+\ln a}{\ln a+\ln b},y\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we get the desired result. □

Using Theorem 2.5 and (1.8), we obtain the explicit formula of $F_n^{(k)}(x,y;a,b,c)$ in terms of the two-variable Fubini polynomials.

Theorem 2.6. *For $k\in\mathbb{Z}$ and $n\geq 0$,*

$$F_n^{(k)}(x,y;a,b,c) = \frac{(\ln a+\ln b)^{n+1}}{n+1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}F_{n+1}\left(\frac{x\ln c+\ln a}{\ln a+\ln b}-j,y\right).$$

Theorem 2.7. *For $k\in\mathbb{Z}$ and $n\geq 0$,*

$$F_n^{(k)}(x,y;a,b,c) = \sum_{j=0}^{n}\binom{n}{j}(\ln a+\ln b)^{n+1}F_{n-j}\left(\frac{x\ln c+\ln a}{\ln a+\ln b},y\right)c_j,$$

where $c_j = \displaystyle\sum_{m=0}^{j}\frac{(-1)^{m+j}m!}{(m+1)^{k-1}(j+1)}S_2(j+1,m+1)$.

Proof. Using (2.1), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} F_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!} &= \frac{c^{xt}}{t(a^{-t}-y(b^t-a^{-t}))}\sum_{m=0}^{\infty}\frac{(1-e^{-t\ln(ab)})^{m+1}}{(m+1)^k}\\
&= \frac{c^{xt}}{t(a^{-t}-y(b^t-a^{-t}))}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}m!}{(m+1)^{k-1}}\left\{\frac{(e^{-t\ln(ab)}-1)^{m+1}}{(m+1)!}\right\}\\
&= \frac{c^{xt}}{t(a^{-t}-y(b^t-a^{-t}))}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}m!}{(m+1)^{k-1}}\sum_{j=m+1}^{\infty}S_2(j,m+1)\frac{[-t\ln(ab)^j}{j!}\\
&= \frac{c^{xt}}{a^{-t}-y(b^t-a^{-t})}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}m!}{(m+1)^{k-1}}\sum_{j=m+1}^{\infty}(-1)^j[\ln(ab)]^jS_2(j,m+1)\frac{t^{j-1}}{j!}\\
&= \frac{c^{xt}}{a^{-t}-y(b^t-a^{-t})}\sum_{m=0}^{\infty}\sum_{j=m}^{\infty}\frac{(-1)^{m+j}m![\ln(ab)]^{j+1}}{(m+1)^{k-1}(j+1)}S_2(j+1,m+1)\frac{t^j}{j!}\\
&= \frac{e^{\left(\frac{x\ln c+\ln a}{\ln(ab)}\right)t\ln(ab)}}{(1-y(e^{t\ln(ab)}-1))}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{m+j}m![\ln(ab)]^{j+1}}{(m+1)^{k-1}(j+1)}S_2(j+1,m+1)\frac{t^j}{j!}\\
&= \left(\sum_{n=0}^{\infty}F_n\left(\frac{x\ln c+\ln a}{\ln a+\ln b},y\right)[\ln(ab)]^n\frac{t^n}{n!}\right)\\
&\quad\times\left(\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{m+j}m![\ln(ab)]^{j+1}}{(m+1)^{k-1}(j+1)}S_2(j+1,m+1)\frac{t^j}{j!}\right).
\end{aligned}
$$

Hence,

$$
\sum_{n=0}^{\infty} F_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(\ln a+\ln b)^{n+1}F_{n-j}\left(\frac{x\ln c+\ln a}{\ln a+\ln b},y\right)c_j\frac{t^n}{n!},
$$

where

$$
c_j = \sum_{m=0}^{j}\frac{(-1)^{m+j}m!}{(m+1)^{k-1}(j+1)}S_2(j+1,m+1).
$$

□

3. Relations with other Polytype Polynomials of Parameters a,b,c, Stirling Numbers of the Second Kind and other Special Polynomials and Numbers

In [13, 14], Jolany et.al. defined a new generalization of Bernoulli numbers and polynomials. They introduced the generalization of poly-Bernoulli polynomials with a,b parameters denoted by $B_n^{(k)}(x;a,b)$ via the the generating function

$$
\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}e^{xt} = \sum_{n=0}^{\infty}B_n^{(k)}(x;a,b)\frac{t^n}{n!}.
$$

Moreover, they extended the definition of generalized poly-Bernoulli polynomials with three parameters denoted by $B_n^{(k)}(x;a,b,c)$ as follows:

$$
\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt} = \sum_{n=0}^{\infty}B_n^{(k)}(x;a,b,c)\frac{t^n}{n!}.
$$

These generalizations will reduce to the poly-Bernoulli polynomials $B_n^{(k)}(x)$ introduced by Bayad and Hamahata [7] by means of the following exponential generating function:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}. \tag{3.1}$$

The numbers $B_n^{(k)}:=B_n^{(k)}(0)$ are called the poly-Bernoulli numbers which were introduced by Kaneko [15] as generalizations of the classical Bernoulli numbers B_n.

It can be seen from the generating function (3.1) that, for any $n\geq 0$,

$$(-1)^n B_n^{(1)}(-x)=B_n(x),$$

where $B_n(x)$ are the classical Bernoulli polynomials given by the generating function:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

The poly-Bernoulli numbers $B_n^{(k)}$ satisfy the relation (see [6]):

$$B_n^{(-k)}=\sum_{m\geq 0}m!S_2(n+1,m+1)m!S_2(k+1,n+1),$$

On the other hand, in [4], Acala and Corcino defined generalized poly-Euler polynomials with three parameters a,b,c denoted by $E_n^{(k)}(x;a,b,c)$ as follows:

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^t+a^{-t})}c^{xt}=\sum_{n=0}^{\infty}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}.$$

These are generalizations of the poly-Euler polynomials defined by Hamahata [12] via the generating function:

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{t(1+e^t)}e^{xt}=\sum_{n=0}^{\infty}E_n^{(k)}(x)\frac{t^n}{n!}.$$

These polynomials satisfy the explicit formula:

$$E_n^{(k)}(x)=\frac{1}{n+1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}E_n(x-j). \tag{3.2}$$

where $E_n(x):=E_n^{(1)}(x)$ are the classical Euler polynomials given by the generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

When $x=0$, $E_n^{(k)}:=E_n^{(k)}(0)$ are called the poly Euler numbers which reduce to classical Euler numbers when $k=1$.

Theorem 3.1. *For $k\in\mathbb{Z}$ and $n>0$,*

$$F_{n-1}^{(k)}(x,y;a,b,c)=\frac{1}{n}\sum_{s=0}^{n}\binom{n}{s}(\ln ab)^s\left[B_{n-s}^{(k)}\left(x+\log_c(ab);a,b,c\right)-B_{n-s}^{(k)}\left(x;a,b,c\right)\right]\alpha_s,$$

where $\alpha_s=\displaystyle\sum_{j=0}^{\infty}\frac{y^j}{(y+1)^{j+1}}j^s.$

Proof. It follows from the generating function (2.1) that

$$\sum_{n=1}^{\infty} nF_{n-1}^{(k)}(x,y;a,b,c)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-(ab)^{-t})}{a^{-t}-y(b^t-a^{-t})}c^{xt} \tag{3.3}$$

Expanding the right-hand side of (3.3), we obtain

$$\begin{aligned}
\frac{\mathrm{Li}_k(1-(ab)^{-t})}{a^{-t}-y(b^t-a^{-t})}c^{xt} &= \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}\cdot\frac{(b^t-a^{-t})}{(y+1)a^{-t}-yb^t}c^{xt}\\
&= \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}\cdot\frac{(ab)^t-1}{(y+1)-y(ab)^t}c^{xt}\\
&= \frac{1}{y+1}\left[\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{[x+\log_c(ab)]t} - \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}\right]\\
&\qquad\times\left(1-\frac{y}{y+1}e^{t\ln(ab)}\right)^{-1}\\
&= \frac{1}{y+1}\left[\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{[x+\log_c(ab)]t} - \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}\right]\\
&\qquad\times\left(\sum_{j=0}^{\infty}\left(\frac{y}{y+1}\right)^j e^{tj\ln ab}\right)\\
&= \frac{1}{y+1}\left(\sum_{n=0}^{\infty}\left[B_n^{(k)}\left(x+\log_c(ab);a,b,c\right) - B_n^{(k)}\left(x;a,b,c\right)\right]\frac{t^n}{n!}\right)\\
&\qquad\times\left(\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{y}{y+1}\right)^j (j\ln ab)^n\frac{t^n}{n!}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{\mathrm{Li}_k(1-(ab)^{-t})}{a^{-t}-y(b^t-a^{-t})}c^{xt} = \sum_{n=0}^{\infty}&\left[\sum_{s=0}^{n}\binom{n}{s}\left(\sum_{j=0}^{\infty}\frac{y^j}{(y+1)^{j+1}}(j\ln ab)^s\right)\right.\\
&\left.\times\left(B_{n-s}^{(k)}\left(x+\log_c(ab);a,b,c\right) - B_{n-s}^{(k)}\left(x;a,b,c\right)\right)\right]\frac{t^n}{n!}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty} nF_{n-1}^{(k)}(x,y;a,b,c)\frac{t^n}{n!}\\
&\quad=\sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s}(\ln ab)^s\left[B_{n-s}^{(k)}\left(x+\log_c(ab);a,b,c\right) - B_{n-s}^{(k)}\left(x;a,b,c\right)\right]\alpha_s\frac{t^n}{n!},
\end{aligned} \tag{3.4}$$

where $\alpha_s = \sum_{j=0}^{\infty}\frac{y^j}{(y+1)^{j+1}}j^s$.

Differentiating both sides of (3.4) with respect to t, gives

$$\begin{aligned}
&\sum_{n=1}^{\infty} nF_{n-1}^{(k)}(x,y;a,b,c)\frac{t^{n-1}}{(n-1)!}\\
&\quad=\sum_{n=1}^{\infty}\sum_{s=0}^{n}\binom{n}{s}(\ln ab)^s\left[B_{n-s}^{(k)}\left(x+\log_c(ab);a,b,c\right) - B_{n-s}^{(k)}\left(x;a,b,c\right)\right]\alpha_s\frac{t^{n-1}}{(n-1)!}.
\end{aligned} \tag{3.5}$$

Comparing the coefficients of $\frac{t^{n-1}}{(n-1)!}$ in (3.5), gives the desired result. □

Expressing $B_n^{(k)}(x,a,b,c)$ in terms of $B_n^k(x)$ in Theorem 3.1 using the relation,

$$B_n^{(k)}(x,a,b,c) = (\ln ab)^n B_n^{(k)}\left(\frac{x\ln c - \ln b}{\ln ab}\right) \quad \text{(see Theorem 3.5 [9])}$$

we obtain the following expression of $F_n^{(k)}(x,y;a,b,c)$ in terms of the poly-Bernoulli polynomials $B_n^{(k)}(x)$.

Corollary 3.2. For $k \in \mathbb{Z}$ and $n > 0$,

$$F_{n-1}^{(k)}(x,y;a,b,c) = \frac{(\ln ab)^n}{n}\sum_{s=0}^{n}\binom{n}{s}\left[B_{n-s}^{(k)}\left(\frac{x\ln c + \ln a}{\ln ab}\right) - B_{n-s}^{(k)}\left(\frac{x\ln c - \ln b}{\ln ab}\right)\right]\alpha_s.$$

Theorem 3.3. *For $k \in \mathbb{Z}$ and $n > 0$,*

$$\begin{aligned}(y+1)nF_{n-1}^{(k)}(x-\log_c a, y;a,b,c) &- ynF_{n-1}^{(k)}(x+\log_c b, y;a,b,c)\\ &= B_n^{(k)}(x+\log_c b;a,b,c) - B_n^{(k)}(x-\log_c a;a,b,c).\end{aligned}$$

Proof. Consider the equation

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))}[(y+1)a^{-t}-yb^t]c^{xt}t = \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}(b^t-a^{-t})c^{xt}. \tag{3.6}$$

Expanding the left-hand side of (3.6), we obtain

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))}[(y+1)a^{-t}-yb^t]c^{xt}t \tag{3.7}$$

$$= \sum_{n=0}^{\infty}\left[(y+1)F_n^{(k)}(x-\log_c a;a,b,c) - yF_n^{(k)}(x+\log_c b;a,b,c)\right]\frac{t^{n+1}}{n!}$$

$$= \sum_{n=1}^{\infty}\left[(y+1)nF_{n-1}^{(k)}(x-\log_c a,y;a,b,c) - ynF_{n-1}^{(k)}(x+\log_c b,y;a,b,c)\right]\frac{t^n}{n!}. \tag{3.8}$$

Similarly, expanding into series, the right-hand side of (3.6) is equal to

$$\sum_{n=0}^{\infty}\left[B_n^{(k)}(x+\log_c b;a,b,c) - B_n^{(k)}(x-\log_c a;a,b,c)\right]\frac{t^n}{n!}. \tag{3.9}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (3.8) and (3.9) completes the proof. $\square$

Theorem 3.4. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$\begin{aligned}(y+1)F_n^{(k)}(x-\log_c a, y;a,b,c) &- yF_n^{(k)}(x+\log_c b, y;a,b,c)\\ &= \frac{1}{2}\left[E_n^{(k)}(x+\log_c b;a,b,c) + E_n^{(k)}(x-\log_c a;a,b,c)\right].\end{aligned}$$

Proof. Consider the equation

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))}[(y+1)a^{-t}-yb^t]c^{xt} = \frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^t+a^{-t})}(b^t+a^{-t})c^{xt}. \tag{3.10}$$

Expanding the left-hand side of (3.10), we obtain

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))}[(y+1)a^{-t}-yb^t]c^{xt} \tag{3.11}$$

$$= 2\sum_{n=0}^{\infty}\left[(y+1)F_n^{(k)}(x-\log_c a,y;a,b,c) - yF_n^{(k)}(x+\log_c b,y;a,b,c)\right]\frac{t^n}{n!}. \tag{3.12}$$

Similarly, expanding into series, the right-hand side of (3.10) is equal to

$$\sum_{n=0}^{\infty}\left[E_n^{(k)}(x+\log_c b;a,b,c) + E_n^{(k)}(x-\log_c a;a,b,c)\right]\frac{t^n}{n!}. \tag{3.13}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (3.12) and (3.13) completes the proof. □

Theorem 3.5. *For $k \in \mathbb{Z}$ and $n \geq 0$, the generalized poly-Fubini polynomials $F_n^{(k)}(x, y; a, b, c)$ satisfy the following relations:*

$$F_n^{(k)}(x, y; a, b, c) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} (\ln c)^l S_2(l, m) F_{n-l}^{(k)}(-m \ln c, y; a, b) x^{(m)}, \tag{3.14}$$

$$F_n^{(k)}(x, y; a, b, c) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} (\ln c)^l S_2(l, m) \binom{n}{m} F_{n-l}^{(k)}(y; a, b)(x)_m, \tag{3.15}$$

$$F_n^{(k)}(x, y; a, b, c) = \sum_{m=0}^{\infty} \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{l}} S_2(l+s, s) F_{n-m-l}^{(k)}(y; a, b) B_m^{(s)}(x \ln c), \tag{3.16}$$

$$F_n^{(k)}(x, y; a, b, c) = \sum_{m=0}^{n} \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} F_{n-m}^{(k)}(j, y; a, b) H_m^{(s)}(x \ln c; \lambda), \tag{3.17}$$

where

$$\left(\frac{t}{e^t - 1}\right)^s e^{xt} = \sum_{n=0}^{\infty} B_n^{(s)}(x) \frac{t^n}{n!} \quad \text{and} \quad \left(\frac{1-\lambda}{e^t - \lambda}\right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x; \lambda) \frac{t^n}{n!}$$

Here, $(x)_m$ and $x^{(m)}$ are the falling and rising factorials respectively, defined as

$$(x)_m = x(x-1)\cdots(x-m+1) \text{ and } x^{(m)} = x(x+1)\cdots(x+m-1) \text{ for } m \geq 1, \text{ and } (x)_0 = x^{(0)} = 1.$$

Proof. For relation (3.14), we note that (2.1) can be written as

$$\sum_{n=0}^{\infty} F_n^{(k)}(x, y; a, b, c) \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t} - y(b^t - a^{-t}))} (1 - (1 - e^{-t \ln c}))^{-x}.$$

Applying Newton's binomial theorem

$$(A + w)^{-x} = \sum_{m=0}^{\infty} \binom{x+m-1}{m} A^{-x-m} (-w)^m, \quad (|w| < |A|).$$

and

$$(e^t - 1)^m = m! \sum_{n=0}^{\infty} S_2(n, m) \frac{t^n}{n!}, \tag{3.18}$$

where $S_2(n, m)$ are the Stirling numbers of the second kind, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n^{(k)}(x, y; a, b, c) \frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t} - y(b^t - a^{-t})} \sum_{m=0}^{\infty} \binom{x+m-1}{m} (1 - e^{-t \ln c})^m. \\
&= \sum_{m=0}^{\infty} x^{(m)} \frac{(e^{t \ln c} - 1)^m}{m!} \frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t} - y(b^t - a^{-t})} e^{-mt \ln c} \\
&= \sum_{m=0}^{\infty} x^{(m)} \left(\sum_{n=0}^{\infty} S_2(n, m) \frac{(t \ln c)^n}{n!} \right) \left(\sum_{n=0}^{\infty} F_n^{(k)}(-m \ln c, y; a, b) \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \sum_{l=m}^{n} (\ln c)^l S_2(l, m) \binom{n}{l} F_{n-l}^{(k)}(-m \ln c, y; a, b) x^{(m)} \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives relation (3.14).

For relation (3.15), we can express (2.1) as

$$\sum_{n=0}^{\infty} F_n^{(k)}(x; a, b, c) \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t} - y(b^t - a^{-t}))} ((e^{t \ln c} - 1) + 1)^x.$$

Again, using binomial theorem and (3.18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))}\sum_{m=0}^{\infty}\binom{x}{m}(e^{t\ln c}-1)^m \\ &= \sum_{m=0}^{\infty}(x)_m\frac{(e^{t\ln c}-1)^m}{m!}\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))} \\ &= \sum_{m=0}^{\infty}(x)_m\left(\sum_{n=0}^{\infty}S_2(n,m)\frac{(t\ln c)^n}{n!}\right)\left(\sum_{n=0}^{\infty}F_n^{(k)}(y;a,b)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{l=m}^{n}(\ln c)^l S_2(l,m)\binom{n}{l}F_{n-l}^{(k)}(y;a,b)(x)_m\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients completes the proof of (3.15).

For relation (3.16), we express (2.1) as

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!} &= \frac{(e^t-1)^s}{s!}\cdot\frac{t^s e^{xt\ln c}}{(e^t-1)^s}\cdot\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))}\cdot\frac{s!}{t^s}. \\ &= \left(\sum_{n=0}^{\infty}S_2(n+s,s)\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}B_m^{(s)}(x\ln c)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}F_n^{(k)}(y;a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s} \\ &= \left(\sum_{n=0}^{\infty}S_2(n+s,s)\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}B_m^{(s)}(x\ln c)\frac{t^m}{m!}F_{n-m}^{(k)}(y;a,b)\frac{t^{n-m}}{(n-m)!}\right)\frac{s!}{t^s} \\ &= \sum_{m=0}^{\infty}\left\{\sum_{n=m}^{\infty}\sum_{l=0}^{n-m}S_2(l+s,s)\frac{t^{l+s}}{(l+s)!}B_m^{(s)}(x\ln c)F_{n-m-l}^{(k)}(y;a,b)\frac{t^{n-m-l}}{(n-m-l)!}\frac{t^m}{m!}\frac{s!}{t^s}\right\} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{s!}}S_2(l+s,s)F_{n-m-l}^{(k)}(y;a,b)B_m^{(s)}(x\ln c)\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients completes the proof of (3.16).

For relation (3.17), we express (2.1) as

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!} &= \frac{(1-\lambda)^s}{(e^t-\lambda)^s}e^{xt\ln c}\cdot\frac{(e^t-\lambda)^s}{(1-\lambda)^s}\cdot\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))} \\ &= \frac{1}{(1-\lambda)^s}\left(\sum_{n=0}^{\infty}H_n^{(s)}(x\ln c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^{-t}-y(b^t-a^{-t}))}e^{jt}\right) \\ &= \frac{1}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}\left(\sum_{n=0}^{\infty}H_m^{(s)}(x\ln c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}F_n^{(k)}(j,y;a,b)\frac{t^n}{n!}\right) \\ &= \frac{1}{(1-\lambda)^s}\left(\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}H_m^{(s)}(x\ln c;\lambda)F_{n-m}^{(k)}(j,y;a,b)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}F_{n-m}^{(k)}(j,y;a,b)H_m^{(s)}(x\ln c;\lambda)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients completes the proof of (3.17). □

In particular, when $c = e$ in Theorem 3.5, we have the following identities for the bivariate poly-Fubini polynomials of parameters a, b.

Corollary 3.6. For $k \in \mathbb{Z}$ and $n \geq 0$, the generalized poly-Fubini polynomials $F_n^{(k)}(x, y; a, b)$ satisfy the following relations:

$$F_n^{(k)}(x,y;a,b) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} S_2(l,m) F_{n-l}^{(k)}(-m,y;a,b) x^{(m)},$$

$$F_n^{(k)}(x,y;a,b) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} S_2(l,m) \binom{n}{m} F_{n-l}^{(k)}(y;a,b)(x)_m,$$

$$F_n^{(k)}(x,y;a,b) = \sum_{m=0}^{\infty} \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{l}} S_2(l+s,s) F_{n-m-l}^{(k)}(y;a,b) B_m^{(s)}(x),$$

$$F_n^{(k)}(x,y;a,b) = \sum_{m=0}^{n} \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} F_{n-m}^{(k)}(j,y;a,b) H_m^{(s)}(x;\lambda).$$

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