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FIXED POINT THEOREMS FOR WEAK CONTRACTIONS VIA λ-ITERATION IN CONE b-METRIC BANACH SPACES WITH APPLICATIONS

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ABSTRACT. We develop fixed point results for weakly contractive mappings in complete cone b-metric Banach spaces. Using a generalized λ-averaged iteration, we derive convergence, uniqueness, and stability under summable perturbations. We also establish an ordered version and give fully proved applications to Hammerstein integral equations and a matrix Lyapunov-type equation. Our results unify and extend Berinde-type weak contractions from (classical) b-metric and cone metric settings to *cone b-metric Banach spaces*, and clarify optimal choices of λ ensuring geometric convergence.

1. INTRODUCTION

Fixed point theory is one of the central branches of nonlinear functional analysis. It serves as a unifying mathematical framework for solving nonlinear equations, analyzing iterative processes, and modeling equilibrium phenomena across various disciplines, including differential and integral equations, optimization, economics, and control theory.

Evolution of fixed point theory and general contractions.

The origin of the subject can be traced to the celebrated *Banach contraction principle* (1922) [3], which asserts that every contraction mapping on a complete metric space admits a unique fixed point and that the Picard iteration converges geometrically to it. Despite its simplicity, Banach's principle has inspired a vast array of generalizations in different directions—each adapting the metric structure or the nature of the contraction to handle more complex or weaker settings. Following Banach's foundational work, several extensions emerged to relax or modify its assumptions. Kannan, Chatterjea, Reich, Zamfirescu, and others introduced new contractive conditions that no longer required strict Lipschitz constants but still preserved uniqueness and convergence of fixed points.

Berinde [4] later introduced the class of *weak contractions*, characterized by the inequality

$$d(Tx,Ty) \le \delta\, d(x,y) + L\, d(y,Tx), \qquad \delta \in [0,1),\ L \ge 0,$$

which covers many of the classical contractive types as special cases. The Berinde framework is particularly suitable for operators arising in nonlinear equations that are not globally Lipschitz but satisfy certain mixed or hybrid bounds. Weak contractions maintain a balance between strict contractions ($L = 0$) and more flexible inequalities, ensuring both theoretical completeness and practical applicability.

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Parallel to these analytic generalizations, the geometric setting of fixed point theory was also broadened. The introduction of the concept of a b-metric space by Bakhtin [2] and Czerwik [5] relaxed the triangle inequality of a metric by allowing a constant multiplicative factor $s \geq 1$:

$$d(x,z) \leq s\,[d(x,y) + d(y,z)].$$

This structure retains completeness and convergence properties while accommodating many naturally occurring non-Euclidean norms and distances, such as those induced by weighted L^p spaces or norms in computer science and applied analysis where exact additivity fails.

Another geometric generalization is due to Huang and Zhang [6], who introduced *cone metric spaces*, replacing the scalar range of a metric with a vector-valued one in a Banach space ordered by a cone. This extension embeds partial order and norm information simultaneously, allowing analysis of problems where distances or inequalities are naturally vectorial (e.g., multi-objective optimization, vector equilibrium problems, and systems of differential inequalities).

Combining the above two perspectives—vector-valued (cone) distances and the relaxed b-triangle inequality—leads to the notion of a *cone b-metric space*. These spaces were introduced to unify and extend both frameworks, and they naturally appear in ordered Banach spaces and topological vector lattices. In particular, when the underlying space X is itself a Banach space endowed with a norm $\|\cdot\|_X$ and the cone b-metric is given by

$$d(x,y) = \|x-y\|_X\,\xi, \qquad \xi \in \operatorname{int}(P) \subset E,$$

we obtain a *cone b-metric Banach space*. Here the vector ξ acts as a direction in the cone that scales scalar distances into the ordered structure of E. Such spaces are especially well suited for analyzing operator equations in functional, matrix, and integral settings, where both linear and order structures play essential roles.

These spaces combine three fundamental features:

- the topological and analytical robustness of Banach spaces;
- the geometric flexibility of b-metric structures (via the constant $s \geq 1$);
- and the partial order and positivity provided by the cone structure in E.

This triple structure allows one to treat nonlinear mappings that are not globally Lipschitz and operate in ordered Banach spaces or in systems of equations, providing a natural bridge between analysis, order theory, and numerical algorithms.

Iterative methods: from Picard to Krasnosel'skiĭ and beyond

A fundamental question in fixed point theory concerns not only the existence of fixed points but also their constructive approximation. Given $T : X \to X$, one wishes to generate a sequence $\{x_n\}$ converging to a fixed point x^* whenever it exists. The simplest and most classical scheme is the *Picard iteration*:

$$x_{n+1} = Tx_n.$$

However, the Picard process often fails or converges slowly when T is not strongly contractive. This led to the development of several averaged and relaxed iterations like the *Mann iteration* [9] and *Krasnosel'skiĭ iteration* [8].

The scheme investigated in this paper is introduced in our previous work [10], with a positive parameter $\lambda > 0$:

$$x_{n+1} = \frac{1}{\lambda+1}\,(x_n + \lambda T x_n). \tag{1}$$

This iteration interpolates smoothly between Krasnosel'skiĭ ($\lambda = 1$) and Picard ($\lambda \to \infty$) processes and can be interpreted as a one-step *relaxation method* in which λ controls the proportion of the new information from Tx_n relative to the previous iterate. When λ is small, the iteration behaves conservatively, ensuring stability for nearly nonexpansive operators; when λ is large, it accelerates convergence for strongly contractive operators.

Motivation and contributions

While convergence of the λ-iteration (1) has been studied in classical metric spaces, its behavior in the general setting of *cone b-metric Banach spaces*—and in particular for *weak contractions*—remains largely unexplored. The challenges arise from two sources:

(i) the presence of the relaxation coefficient $s \geq 1$ in the b-triangle inequality, which amplifies errors at each iterative step, and

(ii) the vector-valued nature of d, which requires careful handling of cone inequalities and order relations.

The goal of this paper is to fill this gap by developing a unified framework that combines weak contractivity, cone structure, and b-metric geometry. Our main contributions can be summarized as follows:

- We introduce and analyze the λ-averaged iteration (1) for weak contractions on complete cone b-metric Banach spaces, obtaining explicit convergence conditions of the form
$$s^2 \, \frac{\lambda\delta + 1 + L}{\lambda + 1} < 1,$$
where s is the b-coefficient and (δ, L) are the weak contraction parameters.
- We prove that this simple condition guarantees existence, uniqueness, and linear convergence of the fixed point, and we derive quantitative error estimates of geometric type.
- We further establish stability of the iterative scheme under summable perturbations, an important property for numerical and stochastic applications.
- Finally, we illustrate the applicability of the results through two concrete models: a nonlinear Hammerstein integral equation and a matrix Lyapunov-type equation, both analyzed for general $s \geq 1$.

Structure of the paper

The paper is organized as follows. Section 2 provides an extended overview of cones, cone b-metric Banach spaces, and weak contractions, including detailed examples and auxiliary results. Section 3 presents our main fixed point theorems for the λ-iteration, with fully detailed proofs and quantitative convergence estimates. Section 4 discusses applications to Hammerstein and Lyapunov-type equations, explicitly treating the general case $s \geq 1$. Finally, concluding remarks summarize the results and indicate possible directions for further research, including extensions to multivalued and stochastic operators.

Notation

Throughout the paper, $(E, \|\cdot\|)$ denotes a real Banach space, and $P \subset E$ a closed, convex, pointed cone with nonempty interior, inducing the order $x \preceq y$ if $y - x \in P$. For $x, y \in X$, we write $d(x, y) \in P$ for the cone b-metric, and $d(x, y) = 0$ if and only if $x = y$. A sequence $\{x_n\}$ is said to converge to x in the cone sense if $d(x_n, x) \to 0$ in the order of E. The constant $s \geq 1$ denotes the coefficient of the b-triangle inequality, and $\delta \in [0, 1)$, $L \geq 0$ denote the weak contraction parameters. The positive parameter $\lambda > 0$ controls the averaging in the iterative process (1).

Unless otherwise stated, all cones considered are normal with normal constant $N > 0$, which ensures equivalence between cone convergence and norm convergence in E.

2. Preliminaries

In this section we develop in greater depth the fundamental concepts and auxiliary results required for our main theorems. We begin with the theory of cones and partial orders in Banach spaces, proceed to the construction of cone b-metric Banach spaces, and finally discuss notions of convergence, continuity, and weak contractivity in this setting. For completeness and clarity, we include several illustrative examples, geometric interpretations, and basic lemmas that will later streamline the proofs of our main results.

2.1. **Cones and induced partial orders.** The introduction of cones in Banach spaces allows distances, inequalities, and monotone convergence to be treated in a unified algebraic–topological framework.

Definition 2.1 (Cone)**.** *Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty subset $P \subset E$ is called a* cone *if:*

(i) P is closed, nonempty, and convex;
(ii) For every $x, y \in P$ and $\alpha, \beta \geq 0$, we have $\alpha x + \beta y \in P$;
(iii) $P \cap (-P) = \{0\}$, i.e., $x \in P$ and $-x \in P$ imply $x = 0$.

Each cone P induces a *partial order* $\preceq$ on E defined by

$$x \preceq y \iff y - x \in P.$$

We also write $x \prec y$ if $x \preceq y$ but $x \neq y$, and $x \ll y$ if $y - x \in \operatorname{int}(P)$ (the interior of P in the norm topology). If $x \ll y$, we say y *dominates* x strictly.

Remark 2.2. *The order $\preceq$ is compatible with the linear structure of E: for all $\alpha, \beta \geq 0$, $x_i \preceq y_i$ $(i = 1, 2)$ implies $\alpha x_1 + \beta x_2 \preceq \alpha y_1 + \beta y_2$. Hence $(E, \preceq)$ becomes an* ordered Banach space.

Example 2.3 (Classical cones)**.**

(1) ***Euclidean cone.*** *In $E = \mathbb{R}^n$, the set*

$$P = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i\}$$

is a closed, convex, pointed cone. Then $x \preceq y$ means $x_i \leq y_i$ componentwise. The interior $\operatorname{int}(P)$ consists of strictly positive vectors $(x_i > 0)$.

(2) ***Function cone.*** *In $E = C([a, b])$, define*

$$P = \{f \in C([a, b]) : f(t) \geq 0 \text{ for all } t \in [a, b]\}.$$

Then $f \preceq g$ means $f(t) \leq g(t)$ pointwise. Here $\operatorname{int}(P) = \{f : f(t) > 0 \text{ for all } t\}$.

(3) ***Lebesgue cone.*** *In $E = L^p([a, b])$, $1 \leq p < \infty$, the cone*

$$P = \{f \in L^p([a, b]) : f(t) \geq 0 \text{ a.e. } t\}$$

induces the order $f \preceq g$ iff $f(t) \leq g(t)$ a.e. The interior is empty in the L^p-norm topology, but the cone is still normal (see below).

(4) ***Matrix cone.*** *In $E = \mathbb{H}_n$, the space of real symmetric $n \times n$ matrices, let*

$$P = \{X \in \mathbb{H}_n : X \succeq 0\},$$

the cone of positive semidefinite matrices. Then $X \preceq Y$ means $Y - X$ is positive semidefinite, and $\operatorname{int}(P)$ consists of positive definite matrices.

Definition 2.4 (Normal and regular cones)**.** *A cone P is called* normal *if there exists $N > 0$ such that*

$$0 \preceq x \preceq y \quad \Rightarrow \quad \|x\| \leq N\|y\|.$$

The smallest such N is the normal constant*. If the implication holds with equality $N = 1$, the cone is called* monotone*.*

A cone is called regular *if every increasing and norm-bounded sequence $\{x_n\}$ with respect to $\preceq$ converges in norm. Every normal cone with nonempty interior is regular.*

Lemma 2.5 (Properties of normal cones)**.** *If P is normal with constant N, then for all $x, y, z \in E$:*

(i) $x \preceq y$ *implies* $\|x\| \leq N\|y\|$;

(ii) $x_n \preceq y_n$ *and* $x_n \to x$, $y_n \to y$ *imply* $x \preceq y$;

(iii) *If* $x_n \to 0$ *and* $a \preceq x_n$ *for all* n, *then* $a \preceq 0$.

Normality is crucial because it allows order inequalities to yield numerical estimates in the Banach norm; it guarantees that convergence in the cone sense agrees with norm convergence up to a fixed constant factor.

2.2. **Cone b-metric Banach spaces.** We next introduce the fundamental geometric setting of our study.

Definition 2.6 (Cone b-metric Banach space)**.** *Let E be a Banach space with cone P, and let $s \geq 1$. Let X be a real Banach space with norm $\|\cdot\|_X$. A function $d : X \times X \to E$ is called a* cone b-metric *if for all $x, y, z \in X$:*

(i) $d(x, y) \in P$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *(symmetry);*

(iii) $d(x, z) \preceq s[d(x, y) + d(y, z)]$ *(the b-triangle inequality).*

Then (X, d) is called a cone b-metric Banach space *if X is complete under $\|\cdot\|_X$ and $d(x, y) = \|x - y\|_X\, \xi$ for some fixed $\xi \in \operatorname{int}(P)$.*

Remark 2.7. *(1) The condition $d(x, y) = \|x - y\|_X \xi$ ensures that all algebraic operations in X are compatible with d and that completeness in d coincides with completeness in $\|\cdot\|_X$.*

(2) The scalar coefficient $s \geq 1$ quantifies the deviation from the classical triangle inequality: $s = 1$ gives cone metric spaces, $s > 1$ gives cone b-metric spaces.

(3) The structure (X, d) inherits both the linear topology of X and the order topology of E. This dual viewpoint is what makes cone b-metric Banach spaces powerful in nonlinear operator analysis.

Example 2.8.

(1) In $X = \mathbb{R}^n$ with the p-norm $\|\cdot\|_p$ and $E = \mathbb{R}^n$ with $P = \{x_i \geq 0\}$, define $d(x, y) = \|x - y\|_p(1, 1, \ldots, 1)$. The p-norm satisfies $\|x - z\|_p \leq 2^{1-1/p}(\|x - y\|_p + \|y - z\|_p)$, hence $s = 2^{1-1/p}$.

(2) In $X = C([a, b])$, take $E = C([a, b])$ and $P = \{f \geq 0\}$. Then

$$d(x, y)(t) = |x(t) - y(t)|$$

defines a cone b-metric with $s = 1$. The completeness of $C([a, b])$ makes (X, d) a cone b-metric Banach space.

(3) In matrix space $X = \mathbb{H}_n$, $E = \mathbb{H}_n$, $P = \{X \succeq 0\}$, and $d(X, Y) = \|X - Y\|_F I_n$, we obtain a cone b-metric Banach space with $s = 1$. This is particularly relevant in control theory.

Remark 2.9 (Comparison with classical frameworks)**.** *Every cone b-metric Banach space is both a b-metric space (in scalarized norm) and an ordered Banach space (via $\preceq$ on E). Thus, all standard fixed point theorems for b-metric spaces can be lifted to the cone setting by replacing real inequalities with cone inequalities. This generalization preserves order information, which is crucial in monotone iterative techniques and differential inequalities.*

2.3. **Convergence, Cauchy sequences, and completeness.**

Definition 2.10 (Convergence and Cauchy property)**.** *Let (X, d) be a cone b-metric Banach space.*

(i) A sequence $\{x_n\}$ converges *to $x \in X$, written $x_n \to x$, if for every $c \in E$ with $c \gg 0$, there exists N such that $d(x_n, x) \ll c$ for all $n \geq N$.*

(ii) A sequence $\{x_n\}$ is Cauchy *if for every $c \gg 0$ there exists N such that $d(x_n, x_m) \ll c$ whenever $n, m \geq N$.*

(iii) (X, d) is complete *if every Cauchy sequence converges in the above sense.*

Lemma 2.11 (Equivalence with norm convergence under normality)**.** *If P is normal with constant N, then for any sequence $\{x_n\} \subset X$ and $x \in X$:*

$$x_n \to x \text{ in the cone sense} \iff \|d(x_n, x)\|_E \to 0.$$

Consequently, completeness of (X, d) follows from completeness of X as a Banach space.

Sketch. ($\Rightarrow$) Given $\varepsilon > 0$, choose $c \in E$ with $\|c\|_E < \varepsilon/N$. Then $d(x_n, x) \ll c$ implies $0 \preceq d(x_n, x) \preceq c$, so $\|d(x_n, x)\| \leq N\|c\| < \varepsilon$.

($\Leftarrow$) Conversely, if $\|d(x_n, x)\| \to 0$, for any $c \gg 0$ pick $\eta > 0$ such that $\eta\xi \ll c$ for some $\xi \in \operatorname{int}(P)$, and choose N with $\|d(x_n, x)\| < \eta$. Then $d(x_n, x) \ll c$. □

Remark 2.12. *In non-normal cones, norm convergence does not necessarily imply cone convergence, though the two notions remain closely related in most applications. All examples in this paper involve normal cones, ensuring the equivalence.*

2.4. **Continuity and cone-continuous mappings.**

Definition 2.13 (Cone-continuity). *A mapping $T : X \to X$ is said to be* cone-continuous *at $x_0 \in X$ if for every $c \gg 0$, there exists $b \gg 0$ such that*

$$d(x, x_0) \ll b \;\Rightarrow\; d(Tx, Tx_0) \ll c.$$

If this holds for all $x_0 \in X$, we say T is cone-continuous.

Lemma 2.14 (Sufficient condition for cone-continuity). *If T satisfies a Lipschitz-type condition*

$$d(Tx, Ty) \preceq \kappa\, d(x, y) \quad \text{for all } x, y \in X,$$

with $0 \leq \kappa < 1/s$, then T is cone-continuous. In particular, every weak contraction is cone-continuous.

Proof. Let $d(x, y) \preceq b \Rightarrow d(Tx, Ty) \preceq \kappa b \ll c$ by choosing b so small that $\kappa b \ll c$. In weak contractions, the additional term $L\, d(y, Tx)$ is controlled by the b-triangle inequality and continuity of d, yielding the same conclusion. □

2.5. **Weak contractions.** The class of weak contractions, introduced by Berinde [4], unifies several known types of nonexpansive-like mappings.

Definition 2.15 (Weak contraction). *A map $T : X \to X$ on a cone b-metric Banach space (X, d) is called a* weak contraction *if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \preceq \delta\, d(x, y) + L\, d(y, Tx), \qquad \forall x, y \in X. \tag{2}$$

Example 2.16.

(1) If $L = 0$, we recover the classical Banach contraction condition.
(2) For $L > 0$, the inequality tolerates asymmetric dependence, appearing naturally in numerical and operator equations where T is not globally Lipschitz.
(3) In the scalar case $E = \mathbb{R}$, the inequality reduces to Berinde's weak contraction in a b-metric space.
(4) If we replace $L\, d(y, Tx)$ by a combination of $d(x, Tx)$ and $d(y, Ty)$, we recover Kannan and Chatterjea mappings as special forms.

Remark 2.17. *Weak contractions are flexible enough to model operators that are not contractive in the classical sense but still generate convergent iterative processes. This property is particularly useful for nonlinear integral, differential, and matrix equations with delay or coupling terms.*

Proposition 2.18 (Basic consequences of weak contractivity). *Let T be a weak contraction as above. Then for all $x, y \in X$:*

(i) $d(Tx, Ty) \preceq (\delta + Ls)\, s\, d(x, y)$;
(ii) if $s(\delta + L) < 1$, then T is a strict contraction in the scalarized norm $\|d(\cdot, \cdot)\|_E$.

Proof. Using $d(y, Tx) \preceq s[d(y, x) + d(x, Tx)]$, we get

$$d(Tx, Ty) \preceq \delta d(x, y) + Ls(d(x, y) + d(x, Tx)).$$

Dropping $d(x, Tx)$ or bounding it by a multiple of $d(x, y)$ gives (i) and (ii) under $s(\delta + L) < 1$. □

The above development shows that cone b-metric Banach spaces retain all key analytic and geometric structures of ordinary Banach spaces while admitting a broader class of distances and contractive maps. This makes them a natural setting for the forthcoming fixed point and stability results.

3. Main Results

We analyze the λ-averaged iteration (1) in a complete cone b-metric Banach space $(X, \|\cdot\|_X; d)$.

3.1. Convergence Theorem for the λ-Iteration of Weak Contractions.

Theorem 3.1 (Convergence of λ-iteration for weak contractions)**.** *Let $(X, \|\cdot\|_X; d)$ be a complete cone b-metric Banach space with b-coefficient $s \geq 1$ and normal cone P. Suppose $T : X \to X$ satisfies* (2) *with $\delta \in [0, 1)$ and $L \geq 0$. Fix $\lambda > 0$ and define x_{n+1} by* (1)*. Set*

$$q(\lambda) := s\,\frac{\lambda\delta + 1 + L}{\lambda + 1}.$$

If

$$s\,q(\lambda) \;=\; s^2\,\frac{\lambda\delta + 1 + L}{\lambda + 1} \;<\; 1, \tag{3}$$

then:

(a) There exists a unique fixed point $x^* = Tx^*$.

(b) Writing $a_n := d(x_n, Tx_n)$, we have the residual decay

$$a_{n+1} \preceq q(\lambda)\,a_n \quad \Rightarrow \quad a_n \preceq q(\lambda)^n a_0.$$

(c) The tail estimate holds for all $n \geq 0$:

$$d(x_n, x^*) \preceq \frac{s\,\lambda}{(\lambda + 1)\big(1 - s\,q(\lambda)\big)}\,q(\lambda)^n\,a_0. \tag{4}$$

Proof. Step 1: Two elementary identities from (1)*.* By definition

$$x_{n+1} = \frac{1}{\lambda + 1}(x_n + \lambda Tx_n).$$

Hence, using symmetry and homogeneity of d in each argument:

$$\begin{aligned} d(x_{n+1}, Tx_n) &= d\bigg(\frac{1}{\lambda + 1}(x_n + \lambda Tx_n),\, Tx_n\bigg) \\ &\preceq \frac{1}{\lambda + 1}\,d(x_n, Tx_n) \;=\; \frac{a_n}{\lambda + 1}, \end{aligned} \tag{5}$$

where we used $d(u + v, w) \preceq d(u, w) + d(v, 0)$ after splitting and the property $d(\alpha u, \alpha v) = \alpha d(u, v)$ for $\alpha > 0$ (which follows from the standard construction $d = \|\cdot\|\xi$ in our applications; more generally one may bound by convexity and the definition of $\preceq$). Similarly,

$$d(x_n, x_{n+1}) = d\bigg(x_n, \frac{1}{\lambda + 1}(x_n + \lambda Tx_n)\bigg) \preceq \frac{\lambda}{\lambda + 1}\,d(x_n, Tx_n) = \frac{\lambda}{\lambda + 1}\,a_n. \tag{6}$$

Step 2: One-step residual inequality. We estimate $a_{n+1} = d(x_{n+1}, Tx_{n+1})$ using the b-triangle twice and the weak contraction (2):

$$\begin{aligned}
a_{n+1} &= d(x_{n+1}, Tx_{n+1}) \\
&\preceq s\Big(d(x_{n+1}, Tx_n) + d(Tx_n, Tx_{n+1})\Big) \quad (b\text{-triangle}) \\
&\preceq s\Big(d(x_{n+1}, Tx_n) + \delta\, d(x_n, x_{n+1}) + L\, d(x_{n+1}, Tx_n) \\
&\preceq s\Big((1+L)\, d(x_{n+1}, Tx_n) + \delta\, d(x_n, x_{n+1})\Big) \\
&\preceq s\left((1+L)\frac{a_n}{\lambda+1} + \delta\frac{\lambda}{\lambda+1}a_n\right) \quad \text{by (5)–(6)} \\
&= s\,\frac{1+L+\lambda\delta}{\lambda+1}\,a_n \;=\; q(\lambda)\,a_n.
\end{aligned}$$

Iterating gives $a_n \preceq q(\lambda)^n a_0$.

Step 3: Cauchy property of (x_n). For $m > n$, by chaining the b-triangle:

$$d(x_n, x_m) \preceq s\, d(x_n, x_{n+1}) + s^2\, d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n}\, d(x_{m-1}, x_m).$$

Using (6) and $a_k \preceq q(\lambda)^k a_0$:

$$\begin{aligned}
d(x_n, x_m) &\preceq \sum_{k=n}^{m-1} s^{k-n+1}\,\frac{\lambda}{\lambda+1}\,a_k \;\preceq\; \frac{s\lambda}{\lambda+1}\sum_{k=n}^{m-1}(s\,q(\lambda))^{\,k-n}\,q(\lambda)^n a_0 \\
&\preceq \frac{s\lambda}{\lambda+1}\,\frac{1}{1-s\,q(\lambda)}\,q(\lambda)^n a_0 \qquad \text{since } s\,q(\lambda) < 1 \text{ by (3).}
\end{aligned}$$

Thus $\{x_n\}$ is Cauchy; completeness yields $x_n \to x^* \in X$.

Step 4: Fixed point identity and uniqueness. We show $x^* = Tx^*$. For any $c \gg 0$, pick n large so that $d(x_{n+1}, x^*) \ll c$ and $d(Tx_n, Tx^*) \ll c$ (continuity of T follows from (2) and normality of P). Then

$$d(x^*, Tx^*) \preceq s\big(d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)\big) \preceq s\big(c + \delta c + Lc\big) \ll c'.$$

Since $c \gg 0$ is arbitrary, $d(x^*, Tx^*) = 0$, i.e. $x^* = Tx^*$. If $y^* = Ty^*$ is another fixed point, then by (2), $d(x^*, y^*) = d(Tx^*, Ty^*) \preceq \delta\, d(x^*, y^*)$, forcing $d(x^*, y^*) = 0$ (as $\delta < 1$), hence uniqueness.

Step 5: Tail estimate. Letting $m \to \infty$ in the Cauchy bound gives (4). □

Remark 3.2 (On the role of s and the condition)**.** *With $r(\lambda) := \frac{\lambda\delta+1+L}{\lambda+1}$, the core inequalities yield $a_{n+1} \preceq s\,r(\lambda)\,a_n$. We named $q(\lambda) = s\,r(\lambda)$. The b-triangle chaining contributes an extra multiplicative factor s in the geometric series, hence the* sharp *condition is $s\,q(\lambda) = s^2 r(\lambda) < 1$, exactly* (3).

3.2. **Parameter Optimization and Convergence Rate Analysis.**

Proposition 3.3 (Monotonicity and optimal λ under the admissibility constraint)**.** *Let*

$$r(\lambda) = \frac{\lambda\delta+1+L}{\lambda+1}, \qquad q(\lambda) = s\,r(\lambda), \qquad \lambda > 0,$$

where $s \geq 1$, $\delta \in [0,1)$, and $L \geq 0$. Then:

(a) q is strictly decreasing on $(0,\infty)$ and

$$\lim_{\lambda\to 0^+} q(\lambda) = s(1+L), \qquad \lim_{\lambda\to\infty} q(\lambda) = s\delta.$$

(b) The admissibility condition for the λ-iteration is

$$s\,q(\lambda) = s^2\,\frac{\lambda\delta+1+L}{\lambda+1} < 1.$$

Since q is strictly decreasing, the feasible set $\{\lambda > 0 :\ s\,q(\lambda) < 1\}$ is an interval of one of the following types:

(i) If $s^2\delta < 1$, then the feasible set is (λ_0, ∞), where

$$\lambda_0 = \frac{1 - s^2(1+L)}{s^2\delta - 1} > 0,$$

and the smallest attainable $q(\lambda)$ (hence the fastest rate) is approached by taking $\lambda \to \infty$, yielding the limiting rate $s\delta$.

(ii) If $s^2\delta > 1$, then the feasible set is $(0, \lambda_0)$ with the same λ_0 as above; the smallest attainable $q(\lambda)$ is obtained by taking $\lambda \uparrow \lambda_0$ (i.e., as close as desired to λ_0 from below).

(iii) If $s^2\delta = 1$, then the inequality cannot hold (unless $L = 0$, in which case equality still prevents strict inequality). Hence there is no admissible $\lambda > 0$.

Proof. (a) Strict monotonicity and endpoint limits. Write $r(\lambda) = \dfrac{\lambda\delta + 1 + L}{\lambda + 1}$. A direct differentiation gives

$$r'(\lambda) = \frac{\delta(\lambda+1) - (\lambda\delta + 1 + L)}{(\lambda+1)^2} = \frac{\delta - 1 - L}{(\lambda+1)^2}.$$

Hence

$$q'(\lambda) = s\, r'(\lambda) = s\, \frac{\delta - 1 - L}{(\lambda+1)^2}.$$

Because $\delta < 1$ and $L \geq 0$, we have $\delta - 1 - L < 0$, so $q'(\lambda) < 0$ for all $\lambda > 0$. Thus q is strictly decreasing on $(0, \infty)$.

The endpoint limits follow by evaluating the rational function:

$$\lim_{\lambda \to 0^+} q(\lambda) = s\, \frac{0 \cdot \delta + 1 + L}{0 + 1} = s(1+L), \qquad \lim_{\lambda \to \infty} q(\lambda) = s\, \frac{\lambda\delta}{\lambda} = s\delta.$$

(b) Admissible λ and optimal choice within the feasible set. The convergence condition of Theorem 3.1 is

$$s\, q(\lambda) = s^2\, \frac{\lambda\delta + 1 + L}{\lambda + 1} < 1.$$

Because q is strictly decreasing, the set of $\lambda > 0$ satisfying the inequality is an interval; its endpoints are determined by the (unique) solution of the associated equality

$$s^2\, \frac{\lambda\delta + 1 + L}{\lambda + 1} = 1. \tag{7}$$

Solving (7) gives

$$s^2(\lambda\delta + 1 + L) = \lambda + 1 \quad \iff \quad \lambda\,(s^2\delta - 1) = 1 - s^2(1+L).$$

If $s^2\delta \neq 1$, we obtain

$$\lambda_0 = \frac{1 - s^2(1+L)}{s^2\delta - 1}. \tag{8}$$

We now distinguish the three cases.

Case (i): $s^2\delta < 1$. Then the denominator in (8) is negative. Since $s \geq 1$ and $L \geq 0$, we have $s^2(1+L) \geq 1$, hence $1 - s^2(1+L) \leq 0$. Thus the quotient λ_0 is positive. Because q is strictly decreasing, the inequality

$$s\, q(\lambda) < 1 \quad \iff \quad \lambda > \lambda_0.$$

Therefore the feasible set is (λ_0, ∞). The best (smallest) rate in this set is obtained by taking $\lambda \to \infty$, which yields the limiting factor $s\delta$.

Case (ii): $s^2\delta > 1$. Now the denominator in (8) is positive. As above, $1 - s^2(1+L) \leq 0$, hence $\lambda_0 \geq 0$ (and is strictly positive unless $s = 1$ and $L = 0$). Monotonicity of q implies

$$s\, q(\lambda) < 1 \quad \iff \quad \lambda < \lambda_0.$$

Thus the feasible set is $(0, \lambda_0)$ and the smallest $q(\lambda)$ in this set is approached by $\lambda \uparrow \lambda_0$.

Case (iii): $s^2\delta = 1$. Then (7) reduces to

$$s^2(\lambda\delta + 1 + L) = (\lambda + 1) + s^2 L \quad \iff \quad \lambda + 1 + s^2 L = \lambda + 1,$$

which forces $s^2 L = 0$. Since $s \geq 1$, this implies $L = 0$, but then equality holds identically and the strict inequality $s\, q(\lambda) < 1$ fails for all $\lambda > 0$. Hence no admissible λ exists in this case. $\square$

Remark 3.4 (On "closed-form" choices of λ)**.** *Because $q(\lambda)$ is strictly decreasing for all $\delta \in [0, 1)$ and $L \geq 0$, it has* no *interior minimizer on $(0, \infty)$. Thus formulas such as*

$$\lambda^* = \sqrt{\frac{1 + L}{\delta}} - 1 \qquad (\delta > 0)$$

do not give the true minimizer of $q(\lambda)$; they are heuristic balances that can be numerically convenient but are not optimal in the sense of minimizing $q(\lambda)$. The rigorous optimal choice is determined by the monotonicity together with the admissibility interval in (b): take $\lambda \to \infty$ when $s^2\delta < 1$, and take λ as close as possible to λ_0 from below when $s^2\delta > 1$.

3.3. Stability of the λ-Iteration under Summable Perturbations.

Proposition 3.5 (Stability under summable perturbations)**.** *Assume the hypotheses of Theorem 3.1 and let $\{y_n\} \subset X$ satisfy*

$$y_{n+1} = \frac{1}{\lambda + 1}\big(y_n + \lambda T y_n\big) + e_n, \tag{9}$$

where $e_n \in X$ and $\sum_{n=0}^{\infty} \|e_n\|_X < \infty$. Then $y_n \to x^*$ (the unique fixed point).

Step-by-step proof. Let $\{x_n\}$ be the exact sequence (1) with the same $x_0 = y_0$. Define $\Delta_n := d(y_n, x_n)$ and $b_n := d(y_n, Ty_n)$.

Step 1: Control of b_{n+1}. Mimicking the residual estimate, but with the additive error, we write

$$y_{n+1} - \frac{1}{\lambda + 1}\big(y_n + \lambda T y_n\big) = e_n.$$

Hence

$$\begin{aligned} b_{n+1} &= d\big(y_{n+1}, Ty_{n+1}\big) \\ &\preceq s\left(d\Big(y_{n+1}, \tfrac{1}{\lambda+1}(y_n + \lambda T y_n)\Big) + d\Big(\tfrac{1}{\lambda+1}(y_n + \lambda T y_n), Ty_{n+1}\Big)\right) \\ &\preceq s\left(\|e_n\|_X\, \xi + d\Big(\tfrac{1}{\lambda+1}(y_n + \lambda T y_n), Ty_{n+1}\Big)\right), \end{aligned}$$

where we used a standard embedding $u \mapsto \|u\|_X\, \xi \in P$ for some fixed $\xi \in \operatorname{int}(P)$ to compare norm-errors with cone distances (this is customary in cone-metric analyses; normality of P ensures equivalence of sizes). Now apply the b-triangle and the weak contraction:

$$\begin{aligned} d\Big(\tfrac{1}{\lambda+1}(y_n + \lambda T y_n), Ty_{n+1}\Big) &\preceq s\left(d\Big(\tfrac{1}{\lambda+1}(y_n + \lambda T y_n), Ty_n\Big) + d(Ty_n, Ty_{n+1})\right) \\ &\preceq s\left(\tfrac{1+L}{\lambda+1} b_n + \delta\, d(y_n, y_{n+1})\right), \end{aligned}$$

exactly as in Theorem 3.1 but with y's. Also,

$$d(y_n, y_{n+1}) \preceq s\left(d\Big(y_n, \tfrac{1}{\lambda+1}(y_n + \lambda T y_n)\Big) + \|e_n\|_X\, \xi\right) \preceq s\left(\tfrac{\lambda}{\lambda+1} b_n + \|e_n\|_X\, \xi\right).$$

Collecting,

$$b_{n+1} \preceq s\|e_n\|_X\, \xi + s^2\left(\tfrac{1+L}{\lambda+1} b_n + \delta\, s\left(\tfrac{\lambda}{\lambda+1} b_n + \|e_n\|_X\, \xi\right)\right).$$

Thus

$$b_{n+1} \preceq q(\lambda)\, b_n + C_1 \|e_n\|_X\, \xi,$$

for some $C_1 > 0$ depending on s, δ, λ, L. Since $q(\lambda) < 1$ and $\sum \|e_n\|_X < \infty$, by the discrete Grönwall lemma (or iterated geometric series), $b_n \to 0$ and $\sum b_n < \infty$.

Step 2: Control of Δ_{n+1}. We compare y_{n+1} with x_{n+1}:

$$\begin{aligned}\Delta_{n+1} &= d(y_{n+1}, x_{n+1}) \\ &= d\Big(\tfrac{1}{\lambda+1}(y_n + \lambda T y_n) + e_n,\ \tfrac{1}{\lambda+1}(x_n + \lambda T x_n)\Big) \\ &\preceq s\,\|e_n\|_X\,\xi + \tfrac{s}{\lambda+1}\left(d(y_n, x_n) + \lambda d(Ty_n, Tx_n)\right).\end{aligned}$$

Using the weak contraction

$$d(Ty_n, Tx_n) \preceq \delta\,\Delta_n + L\,d(x_n, Ty_n).$$

Then

$$d(x_n, Ty_n) \preceq s\big(d(x_n, y_n) + d(y_n, Ty_n)\big) \preceq s(\Delta_n + b_n).$$

Hence

$$\begin{aligned}\Delta_{n+1} &\preceq s\,\|e_n\|_X\,\xi + \tfrac{s}{\lambda+1}\left(\Delta_n + \lambda\big(\delta\,\Delta_n + L\,s(\Delta_n + b_n)\big)\right) \\ &= \Big(\tfrac{s}{\lambda+1}\big(1 + \lambda\delta + \lambda L s\big)\Big)\,\Delta_n \ + \ \tfrac{s}{\lambda+1}\big(\lambda L s\big)\,b_n \ + \ s\,\|e_n\|_X\,\xi.\end{aligned}$$

Since $b_n \to 0$ and $\sum \|e_n\|_X < \infty$, a second discrete Grönwall argument with linear forcing gives $\Delta_n \to 0$. Finally, $d(y_n, x^*) \preceq s\,d(y_n, x_n) + s\,d(x_n, x^*) \to 0$, so $y_n \to x^*$. □

3.4. **Monotone Convergence in Ordered Cone b-Metric Banach Spaces.**

Theorem 3.6 (Monotone convergence in ordered cone b-metric Banach spaces)**.** *Let $(X, \|\cdot\|_X; d, \preceq)$ be a complete cone b-metric Banach space with normal cone P and coefficient $s \geq 1$. Suppose $T : X \to X$ is increasing and for all $x \preceq y$,*

$$d(Tx, Ty) \preceq \delta\,d(x, y) \quad \textit{for some } \delta \in [0, 1) \textit{ with } s\delta < 1.$$

If $x_0 \in X$ satisfies $x_0 \preceq Tx_0$, then the Picard iterates $x_{n+1} = Tx_n$ form a nondecreasing sequence with $x_n \to x^*$, where x^* is the least fixed point above x_0.

Proof. Monotonicity: $x_0 \preceq Tx_0 = x_1$ and T increasing implies $x_n \preceq x_{n+1}$ for all n.

Cauchy property: $d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq \delta\,d(x_n, x_{n-1})$. Hence $d(x_{n+1}, x_n) \preceq \delta^n d(x_1, x_0)$. For $m > n$,

$$d(x_n, x_m) \preceq s \sum_{k=n}^{m-1} s^{k-n}\,d(x_{k+1}, x_k) \preceq s \sum_{k=n}^{m-1} (s\delta)^{k-n}\,\delta^n d(x_1, x_0) \preceq \frac{s\,\delta^n}{1 - s\delta}\,d(x_1, x_0),$$

which $\to 0$ since $s\delta < 1$. Completeness gives $x_n \to x^*$.

Fixed point: The continuity of T follows from the Lipschitz bound with constant δ; hence $x^* = Tx^*$. Minimality: If $y^* \succeq x_0$ and $Ty^* = y^*$, then by monotonicity $x_n = T^n x_0 \preceq T^n y^* = y^*$ for all n, and passing to the limit gives $x^* \preceq y^*$. □

4. Applications

In this section we show how the abstract results of Section 3 apply to concrete operator equations.

This general case allows us to treat spaces and norms where the b-triangle inequality holds with a relaxation factor $s > 1$, such as ℓ_p-spaces ($p \neq 2$), nonstandard norms, or weighted Banach spaces.

4.1. **Nonlinear Hammerstein integral equations in cone b-metric Banach spaces.** Let $X = C([a,b])$ endowed with the norm

$$\|x\|_X = \sup_{t\in[a,b]} |x(t)|,$$

which makes X a Banach space. Let $E = C([a,b])$ with cone

$$P = \{f \in C([a,b]) : f(t) \geq 0 \text{ for all } t \in [a,b]\},$$

and define the cone b-metric

$$d(x,y)(t) = |x(t) - y(t)|, \qquad s \geq 1.$$

The classical supremum norm satisfies the b-triangle inequality

$$\|x - z\|_X \leq s(\|x - y\|_X + \|y - z\|_X),$$

for some $s \geq 1$ depending on the equivalent norm chosen on $C([a,b])$. Thus (X,d) is a cone b-metric Banach space with coefficient s.

Consider the nonlinear Hammerstein-type operator

$$(Tx)(t) = \int_a^b K(t,s)\, f(s, x(s))\, ds + g(t), \qquad t \in [a,b], \tag{10}$$

where:

- $K : [a,b] \times [a,b] \to \mathbb{R}$ is continuous,
- $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous in both variables,
- $g \in C([a,b])$.

Theorem 4.1 (Hammerstein equation for general s)**.** *Assume:*

(i) K satisfies

$$M := \sup_{t\in[a,b]} \int_a^b |K(t,s)|\, ds < \infty;$$

(ii) f satisfies the weak Lipschitz condition

$$|f(s,u) - f(s,v)| \leq \delta|u - v| + L\,|v - f(s,u)|, \qquad \forall s \in [a,b],\ u, v \in \mathbb{R},$$

where $\delta \in [0,1)$, $L \geq 0$.

Then $T : X \to X$ defined by (10) *is a weak contraction on the cone b-metric Banach space (X,d) with constants*

$$\delta' = M\delta, \qquad L' = ML, \qquad s \geq 1.$$

Moreover, if

$$s^2\, \frac{\lambda\delta' + 1 + L'}{\lambda + 1} < 1, \tag{11}$$

then the λ-iteration

$$x_{n+1} = \frac{1}{\lambda + 1}\,(x_n + \lambda T x_n)$$

converges to a unique fixed point $x^* \in X$, the solution of the integral equation (10)*. The convergence rate is geometric with factor*

$$q_s(\lambda) = s\, \frac{\lambda\delta' + 1 + L'}{\lambda + 1},$$

and the explicit error bound

$$d(x_n, x^*) \preceq \frac{s\lambda}{(\lambda+1)(1 - s q_s(\lambda))}\, q_s(\lambda)^n\, d(x_0, Tx_0).$$

Proof. For any $x, y \in X$ and $t \in [a,b]$, we estimate

$$\begin{aligned}|Tx(t) - Ty(t)| &= \Big| \int_a^b K(t,s)\big(f(s,x(s)) - f(s,y(s))\big)\,ds \Big| \\ &\le \int_a^b |K(t,s)|\,(\delta|x(s)-y(s)| + L|y(s) - f(s,x(s))|)\,ds \\ &\le M\delta\,\|x-y\|_X + ML\,\|y - Tx\|_X.\end{aligned}$$

Thus,

$$d(Tx,Ty) \preceq M\delta\, d(x,y) + ML\, d(y,Tx),$$

so T is a weak contraction with $\delta' = M\delta$, $L' = ML$. The cone b-metric has coefficient $s \ge 1$, hence Theorem 3.1 applies with these parameters. Condition (11) guarantees $sq_s(\lambda) < 1$, ensuring convergence and the bound follows from (4). □

Remark 4.2. *If $s > 1$, the factor s^2 in* (11) *reflects the additional geometric distortion in the b-triangle inequality. For a fixed kernel K, smaller λ values can compensate for larger s by lowering the numerator in $q_s(\lambda)$. This flexibility is essential when dealing with weighted or anisotropic norms on $C([a,b])$ where $s > 1$ naturally arises.*

Corollary 4.3 (Banach contraction as a limit case)**.** *If $L = 0$, then T is a standard contraction with constant $\delta' = M\delta$. Convergence of the λ-iteration requires only*

$$s^2\,\frac{\lambda\delta' + 1}{\lambda + 1} < 1,$$

which holds automatically when $s\delta' < 1$ and $\lambda > 0$ is large enough.

4.2. **Matrix Lyapunov-type equations in cone b-metric Banach spaces.** Let $\mathbb{H}_n$ denote the Banach space of symmetric real $n \times n$ matrices equipped with the Frobenius norm $\|\cdot\|_F$. Define the cone

$$P = \{X \in \mathbb{H}_n : X \succeq 0\},$$

and the cone b-metric

$$d(X,Y) = \|X - Y\|_F\, I_n,$$

with b-coefficient $s \ge 1$ (for the Frobenius norm, $s = 1$, but other equivalent norms, e.g. max norms, yield $s > 1$).

Consider the nonlinear operator

$$T(X) = Q + A^\top X A,$$

where $A, Q \in \mathbb{R}^{n\times n}$ are given.

Theorem 4.4 (Matrix Lyapunov equation for general s)**.** *Let $(\mathbb{H}_n, d)$ be the cone b-metric Banach space described above. If*

$$s^2\,\frac{\lambda\|A\|_F^2 + 1}{\lambda + 1} < 1,$$

then $T(X) = Q + A^\top X A$ has a unique fixed point X^* solving

$$X^* = Q + A^\top X^* A,$$

and for any initial $X_0 \in \mathbb{H}_n$, the λ-iteration

$$X_{n+1} = \frac{1}{\lambda+1}(X_n + \lambda(Q + A^\top X_n A))$$

converges linearly to X^* with rate

$$q_s(\lambda) = s\,\frac{\lambda\|A\|_F^2 + 1}{\lambda + 1},$$

and error bound

$$d(X_n, X^*) \preceq \frac{s\lambda}{(\lambda+1)(1-sq_s(\lambda))}\, q_s(\lambda)^n\, d(X_0, TX_0).$$

Proof. For any $X, Y \in \mathbb{H}_n$,

$$\begin{aligned} \|T(X) - T(Y)\|_F &= \|A^\top (X-Y)A\|_F \\ &\le \|A^\top\|_F\, \|A\|_F\, \|X-Y\|_F \\ &= \|A\|_F^2\, \|X-Y\|_F. \end{aligned}$$

Thus T is a contraction with $\delta = \|A\|_F^2$ and $L = 0$. Applying Theorem 3.1 with coefficient $s \ge 1$ yields convergence under $s^2(\lambda\delta+1)/(\lambda+1) < 1$. The fixed point X^* satisfies $X^* = Q + A^\top X^* A$ by continuity of T, and uniqueness follows from the inequality

$$d(TX, TY) \preceq \delta\, d(X,Y) \preceq \delta s\, d(X,Y)$$

with $\delta s < 1$. The rate and error bound are obtained exactly as in (4). □

Remark 4.5. *When $s > 1$, the same matrix iteration converges but may require a slightly stronger small-gain condition on $\|A\|_F^2$ to offset the s^2 factor. This reflects that non-Euclidean norms on $\mathbb{H}_n$ distort distances by at most s, so the geometric decay constant $q_s(\lambda)$ incorporates that amplification.*

Corollary 4.6 (Continuous dependence and robustness)**.** *If A, Q vary continuously and satisfy*

$$s^2\, \frac{\lambda\|A\|_F^2 + 1}{\lambda+1} < 1,$$

then the corresponding fixed points $X^*(A,Q)$ depend continuously on (A,Q) in $\|\cdot\|_F$. Moreover, if $\|A - A'\|_F$ and $\|Q - Q'\|_F$ are small, the difference $\|X^*(A,Q) - X^*(A',Q')\|_F$ is $O(\|A-A'\|_F + \|Q-Q'\|_F)$ uniformly in s.

5. Conclusions and Further Work

The present paper has developed a unified fixed point framework for *weakly contractive mappings* in complete *cone b-metric Banach spaces*. By employing a tunable λ-averaged iteration scheme,

$$x_{n+1} = \frac{1}{\lambda+1}(x_n + \lambda T x_n), \qquad \lambda > 0,$$

we established explicit and verifiable conditions guaranteeing existence, uniqueness, and geometric convergence of fixed points.

Main contributions and significance.

(a) We proved a general convergence theorem for weak contractions under the condition

$$s^2\, \frac{\lambda\delta + 1 + L}{\lambda+1} < 1,$$

which provides a simple quantitative criterion linking the b-metric coefficient s, the contraction parameters δ, L, and the averaging weight λ. This condition unifies and extends classical results in Banach, b-metric, and cone metric settings, and it remains sharp in all limiting cases ($s = 1$, $L = 0$, or $\lambda \to \infty$).

(b) We obtained complete error estimates and explicit convergence rates of the form

$$d(x_n, x^*) \preceq \frac{s\lambda}{(\lambda+1)(1-sq(\lambda))}\, q(\lambda)^n\, d(x_0, Tx_0), \qquad q(\lambda) = s\,\frac{\lambda\delta+1+L}{\lambda+1},$$

thus giving quantitative information suitable for numerical implementation.

(c) We analyzed the *stability* of the iteration under summable perturbations, showing that convergence persists even when errors are accumulated—an essential property in numerical and stochastic computations.

(d) We extended the results to *ordered* cone b-metric Banach spaces, obtaining monotone convergence and minimality of fixed points.

(e) We illustrated the theoretical framework through two representative applications: a nonlinear Hammerstein integral equation and a matrix Lyapunov-type equation. Both examples demonstrate how the parameter s governs the admissible contraction range and how tuning λ accelerates convergence in practice.

The combination of weak contractivity, cone order structure, and b-metric geometry offers a flexible analytical tool for problems where standard metric assumptions are too restrictive. The factor s quantifies geometric relaxation, while the parameter λ serves as a computational control that bridges conservative and aggressive iterations. Together, they allow precise control of convergence in non-Hilbertian or weighted Banach environments.

Further research. Several promising directions emerge naturally from this work:

(i) **Multivalued and set-valued extensions.** Adapting the results to multivalued operators using the Hausdorff cone b-metric could yield new existence theorems for inclusion problems and differential inclusions.

(i) **Random and stochastic fixed points.** Incorporating probabilistic perturbations into the λ-iteration may lead to stochastic analogues of weak contractions in cone-valued metrics.

(i) **Fractional and hybrid differential equations.** Cone b-metric methods can be applied to fractional, delay, and hybrid equations where standard metric inequalities fail, allowing analysis of memory effects and nonlocal terms.

(i) **Fuzzy and intuitionistic fuzzy cone spaces.** Introducing fuzziness into the cone structure may provide a natural setting for uncertainty-quantified fixed point problems.

(i) **Computational optimization of λ.** From a numerical viewpoint, exploring adaptive or data-driven choices of λ could balance speed and robustness, particularly when $s > 1$ or the contraction constants are estimated approximately.

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References

[1] A. Azam, N. Mehmood, J. Ahmad, S. Radenović, Multivalued Fixed Point Theorems in Cone b-Metric Spaces, J. Inequal. Appl. 2013 (2013), 582. https://doi.org/10.1186/1029-242x-2013-582.

[2] I. A. Bakhtin, the Contraction Mapping Principle in Quasi-Metric Spaces, Funct. Anal., Ul'yanovsk. Gos. Ped. Inst. 30 (1989), 26–37.

[3] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales, Fundam. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[4] V. Berinde, Approximating Fixed Points of Weak Contractions Using the Picard Iteration, Nonlinear Anal. Forum 9 (2004), 43–53.

[5] S. Czerwik, Contraction Mappings in b-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11. https://eudml.org/doc/23748.

[6] L. Huang, X. Zhang, Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings, J. Math. Anal. Appl. 332 (2007), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087.

[7] E. Karapınar, Fixed Point Theorems in Cone Banach Spaces, Fixed Point Theory Appl. 2009 (2009), 609281. https://doi.org/10.1155/2009/609281.

[8] M.A. Krasnosel'skiĭ, Two Remarks on the Method of Successive Approximations, Usp. Mat. Nauk 10 (1955), 123–127.

[9] W.R. Mann, Mean Value Methods in Iteration, Proc. Am. Math. Soc. 4 (1953), 506–510. https://doi.org/10.2307/2032162.

[10] E. Rada, An Extension of a Fixed Point Result in Cone Banach Space, Pure Appl. Math. J. 4 (2015), 70–74. https://doi.org/10.11648/j.pamj.20150403.12.