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ON A GENERALIZED QUARTER-SYMMETRIC J-CONNECTION IN AN ALMOST HERMITIAN MANIFOLD

JAEMAN KIM

ABSTRACT. We have introduced a generalized type of quarter-symmetric connection (namely, generalized quarter-symmetric J-connection) in an almost Hermitian manifold which unifies the different types of quarter-symmetric metric and non-metric connections in an almost Hermitian manifold. In this paper, we prove that an almost Hermitian manifold is almost Kähler (resp. Hermitian) if and only if it is almost Kähler with respect to a generalized quarter-symmetric J-connection (resp. Hermitian with respect to a generalized quarter-symmetric J-connection). As a consequence, we obtain that an almost Hermitian manifold is Kähler if and only if it is Kähler with respect to a generalized quarter-symmetric J-connection. Furthermore, some properties of almost analytic vector field with respect to a generalized quarter-symmetric J-connection are also given.

1. INTRODUCTION

Let (M^m, g) be a Riemannian manifold of dimension m with a metric tensor g. A linear connection ∇ on M^m satisfies

$$(i)\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z, (ii)\nabla_X(fY) = (Xf)Y + f\nabla_X Y,$$

where f, g are smooth functions on M^m and X, Y, Z are smooth vector fields on M^m. The torsion tensor T of ∇ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

If the torsion tensor T vanishes, then ∇ is said to be a symmetric connection, otherwise it is non-symmetric. If the metric tensor g of M^m satisfies $\nabla g = 0$, then ∇ is said to be a metric connection, otherwise it is non-metric. It is well known that the Levi-Civita connection is the only linear connection which is both symmetric and metric. In particular, a non-symmetric connection ∇ is called quarter-symmetric if its torsion tensor T is of the form

$$T(X,Y) = u(Y)\phi X - u(X)\phi Y,$$

where u is a 1-form on M^m and ϕ is a tensor of type (1,1). In case of $\phi = Id$, such a connection is said to be a semi-symmetric connection. Studies of various types of semi-symmetric connections and their properties include [1,2,3,4,5,9,12] among others. In [11], Tripathi has introduced a generalized type of connection which unifies various semi-symmetric and quarter-symmetric connections. Recently, Tarafdar and Kundu [10] introduced a type of connection which unifies the concepts of various symmetric and semi-symmetric

DEPARTMENT OF MATHEMATICS EDUCATION, KANGWON NATIONAL UNIVERSITY, CHUNCHON 200-701, KANGWON DO, KOREA

E-mail address: jaeman64@kangwon.ac.kr.
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connections. Motivated by this aspect, we have introduced a generalized type of quarter-symmetric connection which unifies several quarter-symmetric metric and non-metric connections. More precisely, this connection $\bar{\nabla}$ is written as follows:

$$\bar{\nabla}_X Y = \nabla_X Y - u(X)\phi Y - f_1\{u_1(X)Y + u_1(Y)X - g(X,Y)U_1\} - f_2 g(X,Y)U_2,$$

where u, u_1, u_2 are the three 1-forms and their associated vector fields U, U_1, U_2 are given by $u(X) = g(U,X), u_1(X) = g(U_1,X), u_2(X) = g(U_2,X)$ and f_1, f_2 are the smooth functions on M^m. Let (M^{2n}, g, J) be an almost Hermitian manifold of dimension $2n(\geq 4)$ with almost complex structure J and compatible metric g such that $J^2X = -X$ and $g(JX, JY) = g(X,Y)$ for every smooth vector fields X, Y on M^{2n}. On an almost Hermitian manifold (M^{2n}, g, J), it is natural to consider the (1,1)-tensor ϕ to be an almost complex structure J. Therefore from now on, we deal with a quarter-symmetric connection $\bar{\nabla}$ in an almost Hermitian manifold defined as

$$\bar{\nabla}_X Y = \nabla_X Y - u(X)JY - f_1\{u_1(X)Y + u_1(Y)X - g(X,Y)U_1\} - f_2 g(X,Y)U_2 \tag{1.1}$$

and this connection is said to be a generalized quarter-symmetric J-connection. In particular,
(1) if $f_1 = f_2 = 0$, then we obtain a quarter-symmetric metric connection given by Yano and Imai [13]; Mishra and Pandey [6] ; Mukhopadhyay, Roy and Barua [8];
(2) if $f_1 \neq 0$,$f_2 = 0$, then we obtain a quarter-symmetric recurrent-metric connection given by Tripathi [11];
(3) if $f_1 = 1$,$f_2 = 0$,$u_1 = u$, then we obtain a special quarter-symmetric recurrent-metric connection given by Tripathi [11];
(4) if $f_1 = 0$,$f_2 \neq 0$, then we obtain a quarter-symmetric non-metric connection given by Tripathi [11];
The present paper investigates the necessary and sufficient condition for an almost Kähler (resp. Hermitian) manifold with respect to a generalized quarter-symmetric J-connection to be an almost Kähler (resp. Hermitian) manifold. Some properties of almost analytic vector field with respect to a generalized quarter-symmetric J-connection are also given.

2. Main results

Let (M^{2n}, g, J) be an almost Hermitian manifold of dimension $2n(\geq 4)$. The Kähler form Ω of (M^{2n}, g, J) is defined by $\Omega(X,Y) = g(JX,Y)$. Note that (M^{2n}, g, J) is Kähler if and only if $\nabla_X J = 0$; (M^{2n}, g, J) is Hermitian if and only if $N(X,Y) = (\nabla_{JX}J)Y - (\nabla_{JY}J)X - J((\nabla_X J)Y) + J((\nabla_Y J)X) = 0$ (i.e., the Nijenhuis tensor N of the Levi-Civita connection ∇ vanishes) ; (M^{2n}, g, J) is almost Kähler if and only if $(\nabla_X\Omega)(Y,Z) + (\nabla_Y\Omega)(Z,X) + (\nabla_Z\Omega)(X,Y) = 0$ for arbitrary vector fields X, Y, Z (i.e., the Kähler form Ω of (M^{2n}, g, J) is closed). It is well known that (M^{2n}, g, J) is Kähler if and only if (M^{2n}, g, J) is both Hermitian and almost Kähler. An almost Hermitian manifold (M^{2n}, g, J) is said to be almost Kähler with respect to a generalized quarter-symmetric J-connection $\bar{\nabla}$ provided that

$$(\bar{\nabla}_X\Omega)(Y,Z) + (\bar{\nabla}_Y\Omega)(Z,X) + (\bar{\nabla}_Z\Omega)(X,Y) = 0$$

holds for every vector field X, Y, Z on M^{2n}.
Contrary to the semi-symmetric case [10], we have the following:

Theorem 2.1. *Let (M^{2n}, g, J) be an almost Hermitian manifold of dimension $2n(\geq 4)$. Then (M^{2n}, g, J) is almost Kähler if and only if (M^{2n}, g, J) is almost Kähler with respect to a generalized quarter-symmetric J-connection $\bar{\nabla}$.*

Proof. Since the relation

$$\begin{aligned} X(\Omega(Y,Z)) &= (\bar{\nabla}_X\Omega)(Y,Z) + \Omega(\bar{\nabla}_X Y, Z) + \Omega(Y, \bar{\nabla}_X Z) \\ &= (\nabla_X\Omega)(Y,Z) + \Omega(\nabla_X Y, Z) + \Omega(Y, \nabla_X Z) \end{aligned}$$

holds, we have

$$(\bar{\nabla}_X\Omega)(Y,Z) = (\nabla_X\Omega)(Y,Z) - \Omega(\bar{\nabla}_XY - \nabla_XY, Z) - \Omega(Y, \bar{\nabla}_XZ - \nabla_XZ).$$

Taking account of (1.1), we get from the last relation

$$\begin{aligned}(\bar{\nabla}_X\Omega)(Y,Z) &= (\nabla_X\Omega)(Y,Z) - \Omega(-u(X)JY - f_1\{u_1(X)Y \\ &+u_1(Y)X - g(X,Y)U_1\} - f_2g(X,Y)U_2, Z) - \Omega(Y, -u(X)JZ \\ &-f_1\{u_1(X)Z + u_1(Z)X - g(X,Z)U_1\} - f_2g(X,Z)U_2) \\ &= (\nabla_X\Omega)(Y,Z) + f_1u_1(X)\Omega(Y,Z) + f_1u_1(Y)\Omega(X,Z) \\ &-f_1g(X,Y)\Omega(U_1,Z) + f_2g(X,Y)\Omega(U_2,Z) + f_1u_1(X)\Omega(Y,Z) \\ &+f_1u_1(Z)\Omega(Y,X) - f_1g(X,Z)\Omega(Y,U_1) + f_2g(X,Z)\Omega(Y,U_2).\end{aligned}$$

Taking cyclic sum of X, Y, Z in the last relation, we get

$$\begin{aligned}&(\bar{\nabla}_X\Omega)(Y,Z) + (\bar{\nabla}_Y\Omega)(Z,X) + (\bar{\nabla}_Z\Omega)(X,Y) \\ &= (\nabla_X\Omega)(Y,Z) + (\nabla_Y\Omega)(Z,X) + (\nabla_Z\Omega)(X,Y),\end{aligned}$$

which yields the required result. □

Since a Kähler manifold is almost Kähler, we immediately have

Corollary 2.2. *On a Kähler manifold (M^{2n}, g, J) $(2n \geq 4)$ equipped with a generalized quarter-symmetric J-connection $\bar{\nabla}$, the Kähler form Ω of (M^{2n}, g, J) is closed with respect to $\bar{\nabla}$.*

Analogous to the definition of Nijenhuis tensor N with respect to the Levi-Civita connection ∇, we define the Nijenhuis tensor $\bar{N}$ with respect to a generalized quarter-symmetric J-connection $\bar{\nabla}$ by

$$\bar{N}(X,Y) = (\bar{\nabla}_{JX}J)(Y) - (\bar{\nabla}_{JY}J)(X) - J((\bar{\nabla}_XJ)(Y)) + J((\bar{\nabla}_YJ)(X)).$$

An almost Hermitian manifold (M^{2n}, g, J) is said to be Hermitian with respect to a generalized quarter-symmetric J-connection $\bar{\nabla}$ provided that $\bar{N}(X,Y) = 0$ holds for every vector field X, Y on M^{2n}. Then we can state the following:

Theorem 2.3. *Let (M^{2n}, g, J) be an almost Hermitian manifold of dimension $2n(\geq 4)$ equipped with a generalized quarter-symmetric J-connection $\bar{\nabla}$. Then the Nijenhuis tensors of ∇ and $\bar{\nabla}$ coincide.*

Proof. Since the covariant derivative of JY with respect to $\bar{\nabla}$ gives us

$$\bar{\nabla}_X(JY) = (\bar{\nabla}_XJ)Y + J(\bar{\nabla}_XY),$$

we have from (1.1)

$$\begin{aligned}(\bar{\nabla}_XJ)Y &= (\nabla_XJ)Y - f_1u_1(JY)X + f_1g(X,JY)U_1 \\ &-f_2g(X,JY)U_2 + f_1u_1(Y)JX - f_1g(X,Y)JU_1 + f_2g(X,Y)JU_2. \qquad (2.2)\end{aligned}$$

Therefore replacing X by JX in (2.2), we get

$$\begin{aligned}(\bar{\nabla}_{JX}J)Y &= (\nabla_{JX}J)Y - f_1u_1(JY)JX + f_1g(X,Y)U_1 \\ &-f_2g(X,Y)U_2 - f_1u_1(Y)X - f_1g(JX,Y)JU_1 + f_2g(JX,Y)JU_2. \qquad (2.3)\end{aligned}$$

Interchanging X and Y in (2.3), we obtain

$$(\bar{\nabla}_{JY}J)X = (\nabla_{JY}J)X - f_1u_1(JX)JY + f_1g(X,Y)U_1$$

$$-f_2g(X,Y)U_2 - f_1u_1(X)Y - f_1g(X,JY)JU_1 + f_2g(X,JY)JU_2. \tag{2.4}$$

Operating J on the both sides of (2.2), we have

$$J((\bar{\nabla}_X J)Y) = J((\nabla_X J)Y) - f_1u_1(JY)JX + f_1g(X,JY)JU_1$$

$$-f_2g(X,JY)JU_2 - f_1u_1(Y)X + f_1g(X,Y)U_1 - f_2g(X,Y)U_2. \tag{2.5}$$

Interchanging X and Y in (2.5), we get

$$J((\bar{\nabla}_Y J)X) = J((\nabla_Y J)X) - f_1u_1(JX)JY + f_1g(JX,Y)JU_1$$

$$-f_2g(JX,Y)JU_2 - f_1u_1(X)Y + f_1g(X,Y)U_1 - f_2g(X,Y)U_2. \tag{2.6}$$

From (2.3), (2.4), (2.5) and (2.6), it follows that

$$\bar{N}(X,Y) = N(X,Y),$$

which gives the required result. □

Since an almost Hermitian manifold with vanishing Nijenhuis tensor is Hermitian, we have

Corollary 2.4. *Let (M^{2n}, g, J) be an almost Hermitian manifold of dimension $2n(\geq 4)$ equipped with a generalized quarter-symmetric J-connection $\bar{\nabla}$. Then (M^{2n}, g, J) is Hermitian if the Nijenhuis tensor $\bar{N}$ with respect to $\bar{\nabla}$ vanishes.*

Since the Nijenhuis tensor vanishes on the Kähler manifold, we have

Corollary 2.5. *On a Kähler manifold (M^{2n}, g, J) $(2n \geq 4)$ equipped with a generalized quarter-symmetric J-connection $\bar{\nabla}$, the Nijenhuis tensor $\bar{N}$ with respect to $\bar{\nabla}$ vanishes.*

Due to theorem 2.1 and theorem 2.3, we immediately obtain

Corollary 2.6. *Let (M^{2n}, g, J) be an almost Hermitian manifold of dimension $2n(\geq 4)$ equipped with a generalized quarter-symmetric J-connection $\bar{\nabla}$. Then (M^{2n}, g, J) is Kähler if and only if (M^{2n}, g, J) is Kähler with respect to a generalized quarter-symmetric J-connection $\bar{\nabla}$.*

A vector field V is said to be an almost analytic vector field [7] if the Lie-derivative of an almost complex structure J with respect to the vector field V vanishes identically, that is,

$$L_V J = 0. \tag{2.7}$$

It is easy to see that (2.7) is equivalent to $[V, JX] = J[V, X]$, which means

$$(\nabla_V J)X - \nabla_{JX}V + J\nabla_X V = 0. \tag{2.8}$$

In this paper, we call V an almost analytic vector field with respect to a generalized quarter-symmetric J-connection $\bar{\nabla}$ if the identity

$$(\bar{\nabla}_V J)X - \bar{\nabla}_{JX}V + J\bar{\nabla}_X V = 0 \tag{2.9}$$

holds for every vector field X on M^{2n}. On a Kähler manifold, the relation (2.8) reduces to

$$J\nabla_X V - \nabla_{JX}V = 0. \tag{2.10}$$

Now we can state the following:

Theorem 2.7. *Let (M^{2n}, g, J) be an almost Hermitian manifold of dimension $2n(\geq 4)$ equipped with a generalized quarter-symmetric J-connection $\bar{\nabla}$ and V be a nowhere vanishing vector field. A necessary and sufficient condition for an almost analytic vector field V with respect to a generalized quarter-symmetric J-connection $\bar{\nabla}$ to be an almost analytic vector field is that the associated 1-form u of $\bar{\nabla}$ vanishes identically. On the other hand, on a Kähler manifold (M^{2n}, g, J) $(2n \geq 4)$ equipped with a generalized quarter-symmetric J-connection $\bar{\nabla}$, the relation*

$$J\bar{\nabla}_X V - \bar{\nabla}_{JX} V = u(X)V + u(JX)JV + f_1 u_1(JX)V - f_1 u_1(X)JV$$

$$-f_1 g(JX, V)U_1 + f_1 g(X, V)JU_1 + f_2 g(JX, V)U_2 - f_2 g(X, V)JU_2 \tag{2.11}$$

holds if and only if the vector field V is an almost analytic vector field.

Proof. Replacing X by V and Y by X in (2.2), we obtain

$$(\bar{\nabla}_V J)X = (\nabla_V J)X - f_1 u_1(JX)V + f_1 g(V, JX)U_1$$

$$-f_2 g(V, JX)U_2 + f_1 u_1(X)JV - f_1 g(V, X)JU_1 + f_2 g(V, X)JU_2. \tag{2.12}$$

Furthermore, substituting V in place of Y in (1.1), we get

$$\bar{\nabla}_X V = \nabla_X V - u(X)JV - f_1\{u_1(X)V$$

$$+u_1(V)X - g(X, V)U_1\} - f_2 g(X, V)U_2. \tag{2.13}$$

Now taking J on both sides of (2.13), we find

$$J\bar{\nabla}_X V = J\nabla_X V + u(X)V - f_1\{u_1(X)JV$$

$$+u_1(V)JX - g(X, V)JU_1\} - f_2 g(X, V)JU_2. \tag{2.14}$$

Finally, replacing X by JX in (2.13), we have

$$\bar{\nabla}_{JX} V = \nabla_{JX} V - u(JX)JV - f_1\{u_1(JX)V$$

$$+u_1(V)JX - g(JX, V)U_1\} - f_2 g(JX, V)U_2. \tag{2.15}$$

Therefore, from (2.12), (2.14) and (2.15), it follows that

$$(\bar{\nabla}_V J)X - \bar{\nabla}_{JX} V + J\bar{\nabla}_X V = (\nabla_V J)X - \nabla_{JX} V + J\nabla_X V$$

$$+u(X)V + u(JX)JV. \tag{2.16}$$

Therefore, if $u = 0$, then it follows from (2.16) that an almost analytic vector field V with respect to a generalized quarter-symmetric J-connection $\bar{\nabla}$ is also almost analytic, and vice versa. Conversely, if we assume $(\bar{\nabla}_V J)X - \bar{\nabla}_{JX} V + J\bar{\nabla}_X V = 0$ and $(\nabla_V J)X - \nabla_{JX} V + J\nabla_X V = 0$, then (2.16) turns into $u(X)V + u(JX)JV = 0$, which yields $u = 0$. On the other hand, if (M^{2n}, g, J) $(2n \geq 4)$ is Kähler, then it follows from (2.14) and (2.15) that

$$J\bar{\nabla}_X V - \bar{\nabla}_{JX} V = J\nabla_X V - \nabla_{JX} V + u(X)V + u(JX)JV + f_1 u_1(JX)V$$

$$-f_1 u_1(X)JV - f_1 g(JX, V)U_1 + f_1 g(X, V)JU_1 + f_2 g(JX, V)U_2 - f_2 g(X, V)JU_2. \tag{2.17}$$

By virtue of (2.10), it follows from (2.17) that an almost analytic vector field V satisfies the relation (2.11), and vice versa. This completes the proof of theorem 2.7. □

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