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ON TRIDIAGONAL CORRELATION MATRICES

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ABSTRACT. We investigate the conditions under which symmetric tridiagonal matrices represent valid correlation matrices. By exploiting a recursive determinant relationship, we derive explicit sufficient conditions for positive definiteness and highlight connections with several existing criteria. For dimensions up to four, we precisely characterize feasible parameter regions, providing both analytical expressions and intuitive geometric interpretations. We pay special attention to two structured cases: stationary processes, where we establish a sharp necessary and sufficient bound, and alternating period-2 correlation structures, whose spectral properties yield exact semidefiniteness criteria. The derived results furnish practical guidelines for verifying the validity of banded correlation models in various applied contexts.

1. INTRODUCTION

Consider a 1-dependent stochastic process $\{X_t, t = 1, 2, \ldots\}$, meaning that X_s and X_t are independent whenever $|t - s| > 1$. The autocorrelation matrix of n successive observations from this process takes the form

$$\mathbf{A}_n = \begin{bmatrix} 1 & \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & 1 & \alpha_2 & \ddots & \ddots & \vdots \\ 0 & \alpha_2 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \alpha_{n-2} & 0 \\ \vdots & \ddots & \ddots & \alpha_{n-2} & 1 & \alpha_{n-1} \\ 0 & \cdots & 0 & 0 & \alpha_{n-1} & 1 \end{bmatrix}_{n\times n}, \tag{1.1}$$

where $\alpha_i := \text{Corr}(X_i, X_{i+1}) \in [-1, 1]$, for $i = 1, \ldots, n-1$. The matrix $\mathbf{A}_n$ is clearly symmetric and tridiagonal, with all diagonal entries equal to one.

Our goal is to characterize the values of α_i for which $\mathbf{A}_n$ is a valid correlation matrix. Banded correlation structures such as $\mathbf{A}_n$ are commonly encountered in applied settings, particularly in longitudinal studies, panel data models, and spatial processes with local (e.g., nearest-neighbor) dependence. It is well known that a matrix with unit diagonal entries is a correlation matrix if and only if it is symmetric and positive semidefinite; that is, $\mathbf{v}^\top \mathbf{A}_n \mathbf{v} \geq 0$ for any $n \times 1$ vector $\mathbf{v}$. Equivalently, $\mathbf{A}_n$ is positive semidefinite if and only if all of its principal minors are nonnegative (see, e.g., Theorem 7.2.5 in [5]). In particular, if all leading

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principal minors are strictly positive, then $\mathbf{A}_n$ is positive definite. Let $f_i = |\mathbf{A}_i|$, $i = 1, \ldots, n$, denote the determinant of the leading principal minor of $\mathbf{A}_n$. It satisfies the following recurrence relation:

$$f_i = f_{i-1} - \alpha_{i-1}^2 f_{i-2}, \qquad i = 1, \ldots, n, \tag{1.2}$$

with initial conditions $f_{-1} = 0$ and $f_0 = 1$; see [4].

To verify that $\mathbf{A}_n$ is a valid correlation matrix, one must, in principle, ensure that all principal minors are nonnegative, which may require checking up to $2^n - 1$ submatrices. However, by leveraging the recurrence relation in (1.2), we can derive sufficient conditions on the α_i values that guarantee positive semidefiniteness. In Section 2, we present several such sufficient conditions. In Section 3, we characterize necessary and sufficient conditions for $\mathbf{A}_n$ when $n \leq 4$. In Section 4, we examine specific parametric classes of $\mathbf{A}_n$ and establish sharp conditions for their positive semidefiniteness.

2. Sufficient Conditions

A simple yet useful sufficient condition for positive semidefiniteness is that all α_i values lie within $[-0.5, 0.5]$. This is formalized in the following proposition.

Proposition 2.1. Let $\mathbf{A}_n$ be a symmetric tridiagonal matrix with unit diagonal elements defined in (1.1). If

$$\max_{1 \leq i \leq n-1} |\alpha_i| \leq \tfrac{1}{2},$$

then $\mathbf{A}_n$ is a correlation matrix; in particular, it is positive semidefinite.

Proof. Let Z_i, for $i = 1, \ldots, n$, and ε_i, for $i = 1, \ldots, n-1$, be independent standard normal random variables. Define

$$X_i = \sqrt{1 - |\alpha_{i-1}| - |\alpha_i|} \cdot Z_i + \sqrt{|\alpha_{i-1}|} \cdot \varepsilon_{i-1} + \mathrm{sgn}(\alpha_i)\sqrt{|\alpha_i|} \cdot \varepsilon_i, \quad \text{for } i = 1, \ldots, n,$$

with the conventions $\alpha_0 = \alpha_n = 0$, $\varepsilon_0 = \varepsilon_n = 0$, and where sgn denotes the sign function. Since $\max_{1 \leq i \leq n-1} |\alpha_i| \leq \frac{1}{2}$, we have $|\alpha_{i-1}| + |\alpha_i| \leq 1$ for all i, ensuring X_i is well-defined. Note that each X_i has zero mean and unit variance:

$$\mathrm{Var}(X_i) = (1 - |\alpha_{i-1}| - |\alpha_i|) + |\alpha_{i-1}| + |\alpha_i| = 1.$$

The covariance between adjacent variables is $\mathrm{Cov}(X_i, X_{i+1}) = \alpha_i$, and for $|i - j| > 1$, X_i and X_j are uncorrelated due to independence. Thus, the correlation matrix of $\mathbf{X} = (X_1, \ldots, X_n)^\top$ is precisely $\mathbf{A}_n$, establishing its positive semidefiniteness. □

Alternatively, Proposition 2.1 can also be proven using the recurrence relation in (1.2):

Proof. Consider any principal $k \times k$ submatrix $\mathbf{A}_k$ of $\mathbf{A}_n$ of the same tridiagonal form:

$$\mathbf{A}_k = \begin{bmatrix} 1 & \alpha_{i_1} & 0 & 0 & \cdots & 0 \\ \alpha_{i_1} & 1 & \alpha_{i_2} & \ddots & \ddots & \vdots \\ 0 & \alpha_{i_2} & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \alpha_{i_{k-2}} & 0 \\ \vdots & \ddots & \ddots & \alpha_{i_{k-2}} & 1 & \alpha_{i_{k-1}} \\ 0 & \cdots & 0 & 0 & \alpha_{i_{k-1}} & 1 \end{bmatrix}_{k \times k},$$

where $i_j \in \{1, \ldots, n-1\}$, for $j = 1, \ldots, k-1$. Let $f_i = |\mathbf{A}_i|$, $i = 1, \ldots, k$, denote the determinant of the leading principal minor of $\mathbf{A}_k$. We prove by induction that $f_k \geq 0$ for all $k \geq 1$. The base cases are immediate: $f_{-1} = 0$, $f_0 = 1$, and $f_1 = 1$. Assume inductively that $f_{k-2} \geq 0$ and $f_{k-1} \geq 0$. Then, using the recurrence relation (1.2) and noting that $|\alpha_{i_j}| \leq \frac{1}{2}$, for $j = 1, \ldots, k-1$, we obtain

$$f_k = f_{k-1} - \alpha_{i_{k-1}}^2 f_{k-2} \geq f_{k-1} - \frac{1}{4} f_{k-2} = \frac{3}{4} f_{k-2} - \alpha_{i_{k-2}} f_{k-3}$$
$$\geq \frac{3}{4} f_{k-2} - \frac{1}{4} f_{k-3} \geq \cdots \geq \frac{k+1}{2^k} f_0 - \frac{k}{2^{k+1}} f_{-1} = \frac{k+1}{2^k} \geq 0,$$

for all $k \geq 1$. Hence, all principal minors are nonnegative, and $\mathbf{A}_n$ is positive semidefinite. □

Several other sufficient conditions for positive semidefiniteness of symmetric tridiagonal matrices have been proposed. In particular, [1] summarizes a collection of such criteria, generalizing earlier work by [2] and [7]. When applied to matrices with unit diagonal entries, their results yield the following:

Proposition 2.2. Let $\mathbf{A}_n$ be a symmetric tridiagonal matrix with unit diagonal elements of the form (1.1). If

$$\max_i |\alpha_i| < \frac{1}{2\cos\left(\frac{\pi}{n+1}\right)},$$

then $\mathbf{A}_n$ is positive definite.

Note that $\frac{1}{2\cos\left(\frac{\pi}{n+1}\right)} \geq \frac{1}{2}$ for all $n \geq 1$, so Proposition 2.2 is strictly stronger than Proposition 2.1. The bound is visualized in Figure 1. Using a Taylor expansion,

$$\cos\left(\frac{\pi}{n+1}\right) = 1 - \frac{\pi^2}{2(n+1)^2} + O(n^{-4}),$$

it follows that the threshold converges to $\frac{1}{2}$ at a quadratic rate.

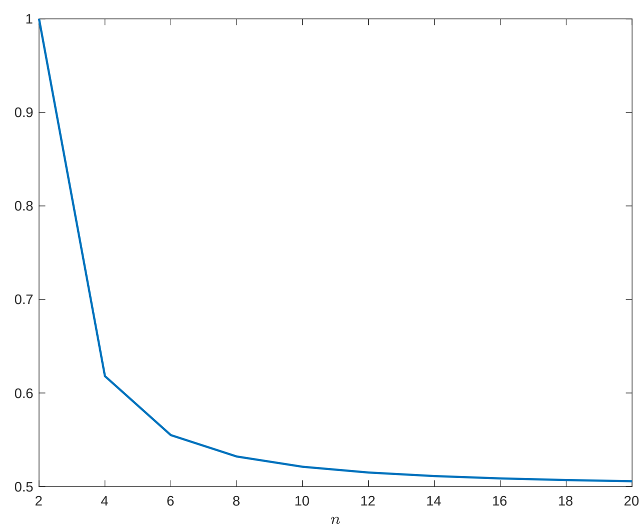

FIGURE 1. Threshold in Proposition 2.2 as a function of n.

An alternative sufficient condition was derived by [3] (Theorem 2.3), using properties of the numerical range. Under the assumption of unit diagonals, their result becomes:

Proposition 2.3. Let $\mathbf{A}_n$ be a symmetric tridiagonal matrix with unit diagonal elements of the form (1.1). If

$$\sum_i^{n-1} \alpha_i^2 < \frac{2}{n(n-1)}, \tag{2.1}$$

then $\mathbf{A}_n$ is positive definite.

Both Propositions 2.2 and 2.3 restrict the magnitude of the α_i values, though the latter is strictly more restrictive, as shown below.

Lemma 2.4. *The condition in Proposition 2.3 is strictly more restrictive than that in Proposition 2.2.*

Proof. When $n = 2$, both conditions yield $|\alpha_1| < 1$. For $n = 3$, Proposition 2.2 implies $\max\{|\alpha_1|, |\alpha_2|\} < \frac{\sqrt{2}}{2}$, while Proposition 2.3 requires $\alpha_1^2 + \alpha_2^2 < \frac{1}{3}$, which implies $\max\{|\alpha_1|, |\alpha_2|\} < \sqrt{1/3} < \frac{\sqrt{2}}{2}$. For $n \geq 4$, the condition in (2.1) implies

$$\max_i |\alpha_i| < \sqrt{\frac{2}{n(n-1)}} < \frac{1}{2} \leq \frac{1}{2\cos\left(\frac{\pi}{n+1}\right)}.$$

Thus, Proposition 2.3 is more stringent. □

Henceforth, we focus primarily on Proposition 2.2, which offers a tractable and sharp condition. The following example compares both conditions.

Example 2.5. Consider the two 4×4 matrices:

$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0.1 & 0 & 0 \\ 0.1 & 1 & 0.2 & 0 \\ 0 & 0.2 & 1 & 0.3 \\ 0 & 0 & 0.3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{C}_2 = \begin{bmatrix} 1 & 0.2 & 0 & 0 \\ 0.2 & 1 & 0.4 & 0 \\ 0 & 0.4 & 1 & 0.6 \\ 0 & 0 & 0.6 & 1 \end{bmatrix}.$$

Matrix $\mathbf{C}_1$ satisfies both Propositions 2.2 and 2.3, since $\max_i |\alpha_i| = 0.3 < 0.6180$ and $\sum \alpha_i^2 = 0.14 < \frac{1}{6}$. Matrix $\mathbf{C}_2$ satisfies Proposition 2.2 but violates Proposition 2.3, as $\sum \alpha_i^2 = 0.56 > \frac{1}{6}$.

Recall that a real matrix $\mathbf{A} = (a_{ij})$ is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \quad \text{for } i = 1, \ldots, n.$$

It is well known that any symmetric diagonally dominant matrix with nonnegative diagonal entries is positive semidefinite (see Theorem 6.1.10 in [5]). For tridiagonal correlation matrices, this gives:

Proposition 2.6. Let $\mathbf{A}_n$ be a symmetric tridiagonal matrix with unit diagonal elements of the form (1.1). If

$$|\alpha_i| + |\alpha_{i+1}| \leq 1, \quad \text{for } i = 1, \ldots, n-2,$$

then $\mathbf{A}_n$ is positive definite.

Although Propositions 2.2 and 2.6 both provide sufficient conditions for positive definiteness, neither implies the other, as illustrated below.

Example 2.7. Consider the following two 4×4 matrices:

$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0.1 & 0 & 0 \\ 0.1 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.6 \\ 0 & 0 & 0.6 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{C}_2 = \begin{bmatrix} 1 & 0.1 & 0 & 0 \\ 0.1 & 1 & 0.1 & 0 \\ 0 & 0.1 & 1 & 0.8 \\ 0 & 0 & 0.8 & 1 \end{bmatrix}.$$

Matrix $\mathbf{C}_1$ satisfies Proposition 2.2 but not Proposition 2.6, while $\mathbf{C}_2$ satisfies Proposition 2.6 but violates Proposition 2.2.

3. Sufficient and Necessary Conditions for $n \leq 4$

In this section, we examine tridiagonal correlation matrices of small dimensions ($n \leq 4$) and derive necessary and sufficient conditions for positive semidefiniteness. For $n = 2$, the matrix

$$\mathbf{A}_2 = \begin{bmatrix} 1 & \alpha_1 \\ \alpha_1 & 1 \end{bmatrix}$$

is positive semidefinite if and only if $|\alpha_1| \leq 1$, which follows directly from the requirement that all principal minors must be nonnegative. For $n = 3$, consider the matrix

$$\mathbf{A}_3 = \begin{bmatrix} 1 & \alpha_1 & 0 \\ \alpha_1 & 1 & \alpha_2 \\ 0 & \alpha_2 & 1 \end{bmatrix}.$$

The seven principal submatrices yield the following nontrivial determinants: $1 - \alpha_1^2$, $1 - \alpha_2^2$, $1 - \alpha_1^2 - \alpha_2^2$ and 1's. Therefore, $\mathbf{A}_3$ is positive semidefinite if and only if

$$\alpha_1^2 + \alpha_2^2 \leq 1. \tag{3.1}$$

It is noteworthy that neither Proposition 2.2 nor Proposition 2.6 is necessary for $n = 3$. Figure 2 illustrates the feasible region defined by (3.1) and compares it with the regions defined by the sufficient conditions in Propositions 2.2 and 2.6. Although Propositions 2.2 and 2.6 cover different regions, each captures approximately 63.7% of the area defined by (3.1).

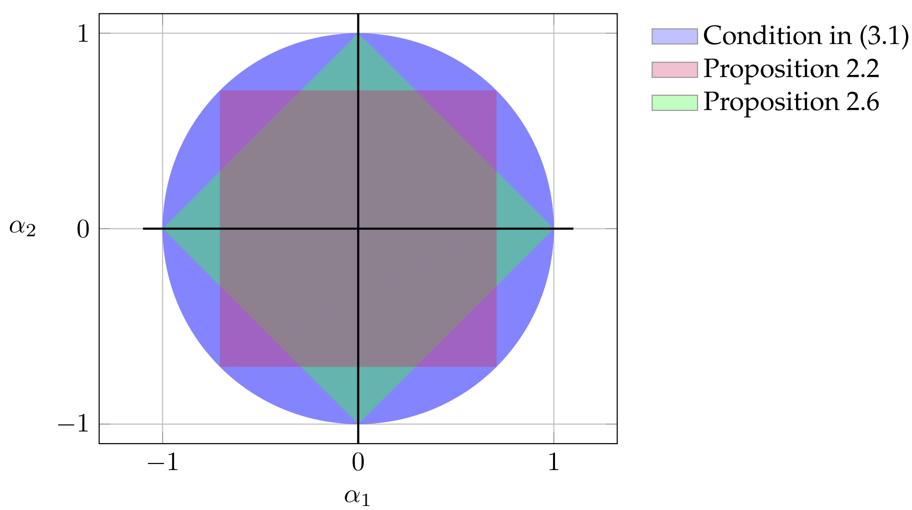

FIGURE 2. Feasible region for $\mathbf{A}_3$ (blue), with sufficient conditions from Propositions 2.2 (purple) and 2.6 (green).

For $n = 4$, consider the matrix

$$\mathbf{A}_4 = \begin{bmatrix} 1 & \alpha_1 & 0 & 0 \\ \alpha_1 & 1 & \alpha_2 & 0 \\ 0 & \alpha_2 & 1 & \alpha_3 \\ 0 & 0 & \alpha_3 & 1 \end{bmatrix}.$$

From (1.2), the determinants of its 15 principal submatrices are $1-\alpha_1^2$, $1-\alpha_2^2$, $1-\alpha_3^2$, $1-\alpha_1^2-\alpha_2^2$, $1-\alpha_1^2-\alpha_3^2$, $1-\alpha_2^2-\alpha_3^2$, $1-\alpha_1^2-\alpha_2^2-\alpha_3^2+\alpha_1^2\alpha_3^2$ and 1's. Thus, $\mathbf{A}_4$ is positive semidefinite if and only if

$$\begin{cases} \alpha_1^2+\alpha_3^2 \le 1; \\ \alpha_1^2+\alpha_2^2+\alpha_3^2-\alpha_1^2\alpha_3^2 \le 1. \end{cases} \tag{3.2}$$

Figure 3 visualizes the feasible region defined by (3.2) in three-dimensional space. We generated random points in the cube $[-1,1]^3$ and checked whether each point satisfied the conditions in (3.2). Approximately 55.84% of the total volume met the criteria, suggesting that the valid parameter space is still substantial, even though the constraints are more complex.

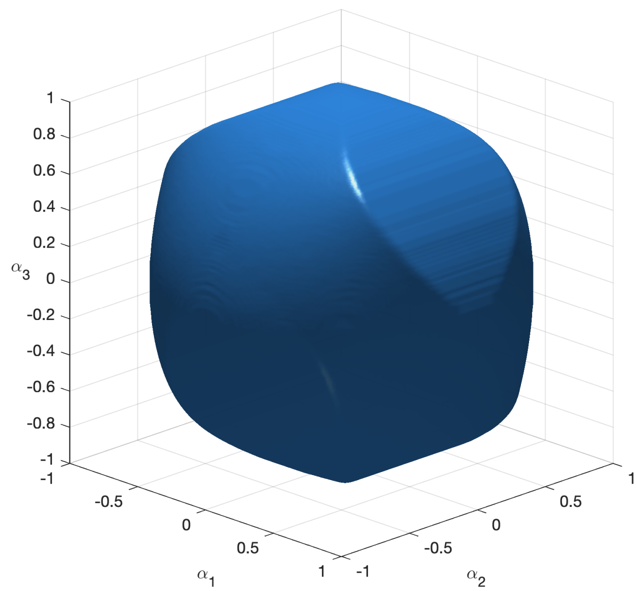

FIGURE 3. Feasible region for $(\alpha_1,\alpha_2,\alpha_3)$ such that $\mathbf{A}_4$ is positive semidefinite.

Table 1 reports the proportion of the volume in $[-1,1]^{n-1}$ for $n=2,3$, and 4 such that the corresponding matrix $\mathbf{A}_n$ is positive semidefinite. The feasible region shrinks noticeably as the dimension increases, with the percentage decreasing approximately linearly. While it would be interesting to study the volume behavior in higher dimensions, such analysis is beyond the scope of this paper.

TABLE 1. Proportion of volume in $[-1,1]^{n-1}$ such that $\mathbf{A}_n$ is positive semidefinite.

Dimension (n)	2	3	4
Percentage (%)	100	78.54	55.84

4. PARAMETRIC CLASSES

4.1. **Stationary 1-dependent process.** Stationary 1-dependent processes are widely used in practice. A canonical example is the moving average process of order 1, or MA(1), which satisfies the 1-dependence condition. In this case, all adjacent autocorrelations are equal, i.e., $\alpha := \alpha_1 = \cdots = \alpha_{n-1}$. The resulting autocorrelation matrix, denoted $\mathbf{A}'_n$, is a symmetric tridiagonal Toeplitz matrix. Remarkably, for this class of matrices, the sufficient condition in Proposition 2.2 is also necessary.

Proposition 4.1. Let $\mathbf{A}'_n$ denote the autocorrelation matrix of a stationary 1-dependent process with constant correlation α. Then $\mathbf{A}'_n$ is positive semidefinite if and only if

$$|\alpha| \leq \frac{1}{2\cos\left(\frac{\pi}{n+1}\right)}. \tag{4.1}$$

Proof. The eigenvalues of $\mathbf{A}'_n$ are given by

$$\lambda_k = 1 + 2|\alpha| \cos\left(\frac{k\pi}{n+1}\right), \quad k = 1, \ldots, n,$$

see, e.g., Problem 1.4.P16 in [5] or Section 4.8.6 in [6]. Since $\cos(\cdot)$ is decreasing on $(0, \pi)$, the smallest eigenvalue occurs at $k = n - 1$. Hence, $\mathbf{A}'_n$ is positive semidefinite if and only if

$$1 - 2|\alpha| \cos\left(\frac{\pi}{n+1}\right) \geq 0.$$

The result follows immediately. □

4.2. **Alternating Period-2 Correlation Structure.** We now consider a generalization of the stationary setting in which the correlation coefficients alternate between two fixed values, α and β. The resulting tridiagonal matrix takes the form

$$\mathbf{A}_n^{\alpha,\beta} = \begin{bmatrix} 1 & \alpha & 0 & \ldots & 0 \\ \alpha & 1 & \beta & \ldots & 0 \\ 0 & \beta & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}_{n\times n}. \tag{4.2}$$

Depending on whether n is even or odd, the last off-diagonal entry is either α or β. This is a special case of a period-2 Jacobi matrix. The eigenvalues of such matrices are well-studied (see Theorems 2.1 and 3.1 in [8]) and can also be derived using discrete Bloch–Floquet theory (see [10]). Specifically, the eigenvalues of $\mathbf{A}_n^{\alpha,\beta}$ are given by:

- For odd n, the eigenvalues are

$$\lambda_k^{\pm} = 1 \pm \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos\left(\frac{2k\pi}{n+1}\right)}, \quad k = 1, \ldots, \frac{n-1}{2}, \tag{4.3}$$

  together with an eigenvalue $\lambda_n = 1$.
- For even n, the eigenvalues are given by

$$\lambda_k^{\pm} = 1 \pm \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos(\theta_k)}, \quad k = 1, \ldots, \frac{n}{2},$$

  where the $\theta_k \in (0, \pi)$ satisfy the transcendental equation

$$\sin\left(\frac{(n+2)\theta_k}{2}\right) + \frac{\beta}{\alpha} \sin\left(\frac{n\theta_k}{2}\right) = 0, \quad k = 1, \ldots, \frac{n}{2}. \tag{4.4}$$

Proposition 4.2. Let $\mathbf{A}_n^{\alpha,\beta}$ be a tridiagonal matrix defined in (4.2). Then it is positive semidefinite if and only if:

- For n odd,

$$\alpha^2 + \beta^2 + 2\alpha\beta \cos\left(\frac{2\pi}{n+1}\right) \leq 1.$$

- For n even,

$$\begin{cases} \alpha^2 + \beta^2 + 2\alpha\beta\cos(\theta_1) \leq 1, & \text{if } \alpha\beta \geq 0; \\ \alpha^2 + \beta^2 + 2\alpha\beta\cos(\theta_2) \leq 1, & \text{if } \alpha\beta < 0, \end{cases}$$

where $\theta_1, \theta_2 \in (0, \pi)$ are the minimum and maximum solution of (4.4), respectively.

Proof. For odd n, the smallest eigenvalue is

$$1 - \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta\cos\left(\frac{2k\pi}{n+1}\right)} \geq 0, \quad k = 1, \ldots, m.$$

which is minimized at either $k = 1$ (when $\alpha\beta \geq 0$) or $k = m$ (when $\alpha\beta < 0$). The matrix is positive semidefinite if and only if this minimum is nonnegative.

For even n, the eigenvalues depend on the solutions θ_k of (4.4), which must be computed numerically. The cosine function is decreasing on $[0, \pi]$, so the minimum eigenvalue corresponds to either θ_1 or θ_2 depending on the sign of $\alpha\beta$. The result follows. □

Note that for $n = 3$, the period-2 Jacobi matrix includes all possible tridiagonal correlation matrices of that size. In this case, Proposition 4.2 reduces to the condition $\alpha^2 + \beta^2 \leq 1$, consistent with the $n = 3$ result in Section 3. This serves as an independent confirmation of that characterization. As previously discussed, Propositions 2.2 and 2.6 are not necessary even in this period-2 case, highlighting the importance of direct spectral analysis for such structured matrices.

Additionally, when $\alpha = \beta$, the condition for odd n in Proposition 4.2 reduces directly to the stationary case condition in (4.1). For even n, the transcendental equation (4.4) becomes

$$2\sin\left(\frac{(n+1)\theta_k}{2}\right)\cos\left(\frac{\theta_k}{2}\right) = 0, \quad k = 1, \ldots, \frac{n}{2}.$$

Since $\theta_k \in (0, \pi)$, we have $\cos\left(\frac{\theta_k}{2}\right) > 0$, and the equation implies

$$\sin\left(\frac{(n+1)\theta_k}{2}\right) = 0,$$

which yields

$$\theta_k = \frac{2k\pi}{n+1}, \quad k = 1, \ldots, \frac{n}{2}.$$

Thus, the result is again consistent with the stationary condition given in (4.1).

For odd n, it is clear that the checking time is very fast. However, for n even, we need to solve the equation (4.4). By using a fine grid (we used 1,000) to find sign-change intervals automatically, then we use the `fzero` function in MATLAB® to find all solutions.

To assess performance, we randomly generated 100 instances for each matrix dimension $n = 50, 100, 150$, . . ., 1000. The average computation times are displayed in Figure 4. As expected, the runtime of the traditional eigenvalue decomposition method increases quadratically with n, consistent with the known computational complexity for tridiagonal matrices. In contrast, Proposition 4.2 yields a nearly constant runtime, with all average values remaining below 10^{-4} seconds. These results confirm that the proposition is not only theoretically sound but also practically efficient.

5. Conclusion and Future Work

We have presented a comprehensive analysis of the conditions under which symmetric tridiagonal matrices represent valid correlation matrices. The results establish explicit bounds on the correlation parameters that ensure positive semidefiniteness. For stationary structures, we show that the condition $|\alpha| \leq \frac{1}{2\cos\left(\frac{\pi}{n+1}\right)}$

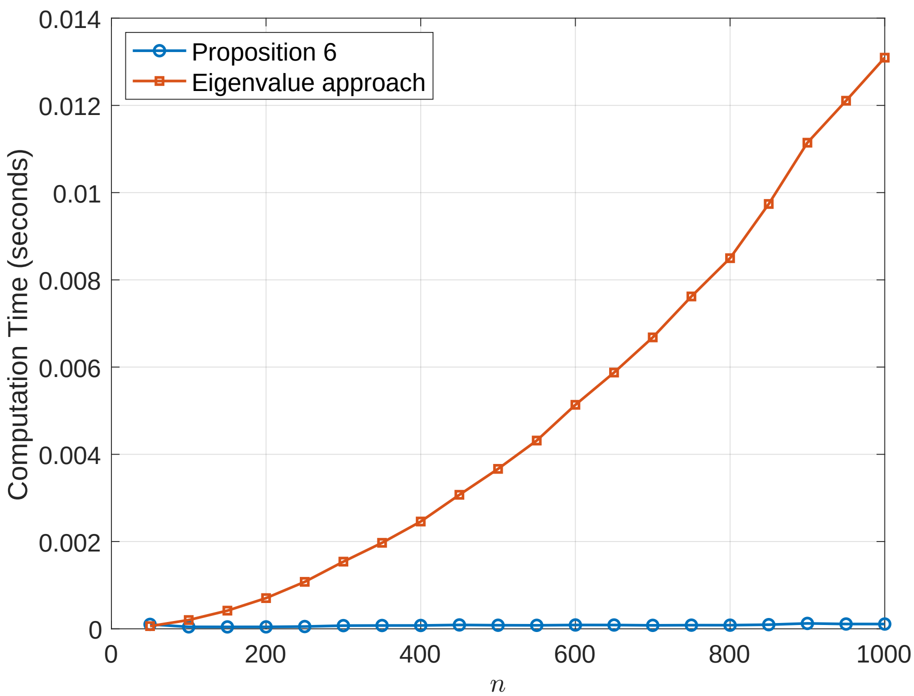

FIGURE 4. Average computation time over 100 trials for each matrix dimension

is both necessary and sufficient, yielding a sharp characterization. Our study of alternating period-2 correlation structures further extends the literature by providing closed-form, eigenvalue-based criteria for positive semidefiniteness. Additionally, numerical investigations for small dimensions offer both analytical insights and geometric interpretations of the feasible parameter space.

Future work may explore the asymptotic behavior of the feasible volume as matrix size grows, or extend these results to broader classes of banded or block tridiagonal correlation structures. Developing efficient algorithms for testing positive semidefiniteness in higher-dimensional structured matrices also remains an important direction.

Competing interests. The author declares no competing interests.

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