DYNAMICS OF A DISCRETE-TIME CHAOTIC LÜ SYSTEM

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ABSTRACT. A discrete-time chaotic Lü system is investigated. Firstly, we give the conditions of local stability of this system around feasible fixed points. Then, we show analytically that discretized Lü system undergoes a flip-Neimark Sacker(NS) bifurcation when one of the system parameter varies near its critical value. We confirm the existence of flip-NS bifurcation via explicit Flip-NS bifurcation criterion and determine the direction of both bifurcations with the help of center manifold theory and bifurcation theory. We carry out numerical simulations to affirm our analytical findings. Furthermore, we present Maximum Lyapunov exponents (MLEs) and Fractal dimension (FD) numerically in order to justify whether chaos exists in the system or not. At the end, we apply hybrid control strategy to eliminate chaotic trajectories of the system.

1. INTRODUCTION

The system changes over time is referred to as dynamical system. Currently, a mathematical model linked to dynamical system is used in ecology, weather forecasting, heartbeat regulation, collapse prevention of power systems, and biomedical applications to human psychiatry, and among other things. The dynamical system can be divided into two parts continuous dynamical system and discrete dynamical system. Numerous academics focused on and conducted in-depth research on systems bifurcation in continuous dynamical systems, but a little works have been studied in systems bifurcations in discrete dynamical system. In continuous dynamical system, three dimensional chaotic systems have been extensively investigated by renowned researchers [17, 18, 23, 24] and the references therein. Lü et al. [18] and Chen et al. [24] constructed new critical chaotic system by anti-control technique in Lorenz system [17, 23]. These systems are known as Lü system and Chen's system respectively. Qualitative analyses of these empirical works found many dynamical properties including local bifurcations, chaotic, periodic, quasi-periodic orbits and route to chaos. They also obtained super-critical and sub-critical bifurcations conditions around positive equilibrium point. However, a lot of exploratory works have been suggested that discrete-time models are more suitable compared to differential equation model as discrete-time model reveal rich chaotic dynamics and give effective computational models for numerical simulations [4, 6, 7, 10, 12, 15, 16, 20–22, 25, 30, 31]. These studies investigated unexpected characteristics, such as the occurrence of (flip-NS) bifurcations and chaotic phenomena, using either numerical methods or center manifold theory applications. These researches focused solely on two-dimensional discrete systems.

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Recently, a very short number of contributions dedicated to study the dynamics of three dimensional discrete systems [1, 5, 8, 9, 11, 13, 19, 27]. For example, discrete-time epidemic models SIR, SEIR and hypertensive or diabetic exposed to COVID-19 discussed in [1, 9, 13] respectively, in [27] the authors investigated discrete financial system and in [19], the authors studied discrete chaotic system. In these works, the researchers concentrated their endeavor to determine the direction and stability of Flip and NS bifurcation by using explicit Flip-NS bifurcation criterion, center manifold theory and bifurcation theory. Neimark-Sacker bifurcation for discrete-time food chain model studied in [8]. The studies in [5, 11] investigated discrete population models. These studies used only explicit (Flip-NS bifurcaton) criterion and numerical simulations for existence of flip and NS bifurcations. Chaos is strongly hooked into the initial conditions for the answer trajectories, or the exponential aberration in solution trajectories for little differences within the initial conditions in discrete system.

In this paper, we consider the following three dimensional chaotic Lü system [18]:

$$
\begin{aligned}
\dot{x} &= a(y - x), \\
\dot{y} &= -xz + cy, \\
\dot{z} &= xy - bz,
\end{aligned}
\tag{1.1}
$$

In system (1.1) $x, y, z \in \mathbb{R}$ are the state variables denoting the rate of convective overtuning, horizontal temperature difference and vertical temperature difference respectively. The parameters $a, b, c \in \mathbb{R}^{+}$ in the system represent the Prandal number, the Rayleigh number, and some physical proportions of the region under study and for more description of these parameters we refer [23] .

Forward Euler scheme is applied in order to get the discrete chaotic Lü system with integral step-size ϱ as follows

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x + \varrho(a(y - x)) \\ y + \varrho(xz + cy) \\ z + \varrho(xy - bz) \end{pmatrix} = \begin{pmatrix} e_1(x, y, z) \\ e_2(x, y, z) \\ e_3(x, y, z) \end{pmatrix}
\tag{1.2}
$$

The Flip and NS bifurcations play an significant role for generation of critical chaotic dynamics in discrete system and trigger a route to chaos. The objective of this work is to analyze systematically the conditions for occurence of flip and NS bifurcations by using an explicit Flip-NS bifurcation criterion and to determine the stability and direction of both bifurcations by the applications of bifurcation theory.

The remaining part of this paper is organised as follows: Sect.2, explores the local stability conditions of feasible fixed points. In Sect.3, we analyze theoritically that under a certain parametric condition, the system (1.2) undergoes a Flip or NS bifurcations. In Sect.4, we present system dynamics numerically including diagrams of bifurcations, phase portraits, MLEs and FD to validate our analytical findings. In Sect.5, we implement a hybrid control strategy to stabilize chaos of the uncontrolled system. In Sect.6, we give a short discussion.

2. LOCAL STABILITY ANALYSIS OF FIXED POINTS

The fixed points of the system (1.2) are the solutions of the following system of non-linear equations:

$$
\begin{aligned}
x &= e_1(x, y, z) \\
y &= e_2(x, y, z) \\
z &= e_3(x, y, z)
\end{aligned}
\tag{2.1}
$$

For all permissible parameter values, system (1.2) has three fixed points, the trivial fixed point is $E_0(0,0,0)$ and the other two non-zero fixed points are $E_{\pm} = (\pm\sqrt{bc}, \pm\sqrt{bc}, c)$.

At any arbitrary fixed point $E(x, y, z)$, the Jacobian matrix of system (1.2) is given by

$$J(E) = \begin{pmatrix} 1 - a\varrho & a\varrho & 0 \\ -z\varrho & 1 + c\varrho & -x\varrho \\ y\varrho & x\varrho & 1 - b\varrho \end{pmatrix} = (j_{kl}), \quad k, l = 1, 2, 3 \tag{2.2}$$

and the eigenvalues of matrix $J(E)$ are the roots of the following characteristic equation

$$P(\mu) := \mu^3 + \vartheta_2\mu^2 + \vartheta_1\mu + \vartheta_0 = 0 \tag{2.3}$$

where,

$$\vartheta_2 = -tr(J),$$
$$\vartheta_1 = \begin{vmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{vmatrix} + \begin{vmatrix} j_{22} & j_{23} \\ j_{32} & j_{33} \end{vmatrix} + \begin{vmatrix} j_{11} & j_{13} \\ j_{31} & j_{33} \end{vmatrix},$$
$$\vartheta_0 = -\left| J \right|.$$

We first introduce the following lemma concerning the necessary and sufficient requirements for stability around a systems fixed point in order to explore the nature of the system (1.2) around $E(x, y, z)$.

Lemma 2.1. *[2] Suppose that $\vartheta_2, \vartheta_1, \vartheta_0 \in \mathbb{R}$. Then, the necessary and sufficient conditions for all roots μ of the equation*

$$\mu^3 + \vartheta_2\mu^2 + \vartheta_1\mu + \vartheta_0 = 0$$

to satisfy $|\mu| < 1$ are
$$|\vartheta_2 + \vartheta_0| < 1 + \vartheta_1, |\vartheta_2 - 3\vartheta_0| < 3 - \vartheta_1, \text{ and } \vartheta_0{}^2 + \vartheta_1 - \vartheta_0\vartheta_2 < 1.$$

Now, the local dynamics of system (1.2) around fixed points E_0 and E_+ are as follows.
At E_0, the Jacobian matrix is

$$J(E_0) = \begin{pmatrix} 1 - a\varrho & a\varrho & 0 \\ 0 & 1 + c\varrho & 0 \\ 0 & 0 & 1 - b\varrho \end{pmatrix}.$$

So, the eigenvalues of $J(E_0)$ are $\mu_1 = 1 - a\varrho, \mu_2 = 1 + c\varrho$ and $\mu_3 = 1 - b\varrho$. We obtain the following Lemma.

Lemma 2.2. *For any parameter values, the fixed point E_0 is a*
$-$ *saddle if $\varrho < \min\left\{\frac{2}{b}, \frac{2}{a}\right\}$,*
$-$ *source if $\varrho > \max\left\{\frac{2}{b}, \frac{2}{a}\right\}$.*

The Jacobian matrix at E_+,

$$J(E_+) = \begin{pmatrix} 1 - a\varrho & a\varrho & 0 \\ -c\varrho & 1 + c\varrho & -\sqrt{bc}\varrho \\ \sqrt{bc}\varrho & \sqrt{bc}\varrho & 1 - b\varrho \end{pmatrix},$$

and the eigenvalues of $J(E_+)$ satisfy the equation

$$P(\mu) := \mu^3 + \kappa_2 \mu^2 + \kappa_1 \mu + \kappa_0 = 0. \tag{2.4}$$

where,

$$\begin{aligned}
\kappa_2 &= -3 + \varrho(a + b - c), \\
\kappa_1 &= 3 - 2\varrho(b - c) + a\varrho(-2 + b\varrho), \\
\kappa_0 &= 1 + \varrho(b - c) + a\varrho(1 + b\varrho(-1 + 2c\varrho))
\end{aligned} \tag{2.5}$$

We give the following Lemma for stability condition of the fixed point E_+.

Lemma 2.3. *The fixed point E_+ of system (1.2) is locally asymptotically stable if and only if the coefficients $\kappa_2, \kappa_1, \kappa_0$ of (2.4) satisfy*

$$|\kappa_2 + \kappa_0| < 1 + \kappa_1, \quad |\kappa_2 - 3\kappa_0| < 3 - \kappa_1, \text{ and } \kappa_0{}^2 + \kappa_1 - \kappa_0\kappa_2 < 1.$$

3. ANALYSIS OF BIFURCATIONS

In this section, we will discuss the existence, direction and stability analysis of flip and NS bifurcations near the fixed point E_+ by using an explicit Flip-NS bifurcation criterion without computing the eigenvalues of the respective system and bifurcation theory [14, 26, 28] . We consider ϱ as the bifurcation parameter, otherwise stated.

3.1. **Flip Bifurcation: Existence, Direction and Stability.**

3.1.1. **Existence of Flip Bifurcation.** To investigate the existence of flip bifurcation, we will use the following Lemma.

Lemma 3.1. *[28] Consider an n-dimensional discrete system as follows:*

$$Y_{k+1} = H_v(Y_k),$$

where $v \in \mathbb{R}$ is being taken as a bifurcation parameter. Furthemore, we write the equation of the jacobian matrix $J(Y^*) = (\theta_{ij})_{n \times n}$ at fixed point $Y^* \in \mathbb{R}^n$ for H_v as follows

$$D_v(\mu) := \mu^n + \iota_1 \mu^{n-1} + ... + \iota_{n-1} \mu + \iota_n = 0 \tag{3.1}$$

where $\iota_i = \iota_i(v, u), i = 1, 2, ..., n$ and u is being taken as the control parameter unless stated which is to be determined. Later, we define a sequence of determinants $(\mathbb{N}_i^\pm(v, u))_{i=0}^n$ with $\mathbb{N}_0^\pm(v, u)) = 1$ which is to be defined as:

$$\mathbb{N}_i^\pm = det(R_1 \pm R_2) \tag{3.2}$$

where,

$$
R_1 = \begin{pmatrix}
1 & \iota_1 & \iota_2 & ... & \iota_{i-1} \\
0 & 1 & \iota_1 & ... & \iota_{i-2} \\
0 & 0 & 1 & ... & \iota_{i-3} \\
... & ... & ... & ... & ... \\
0 & 0 & 0 & ... & 1
\end{pmatrix},
$$

$$
R_2 = \begin{pmatrix}
\iota_{n-i+1} & \iota_{n-i+2} & ... & \iota_{n-1} & \iota_n \\
\iota_{n-i+2} & \iota_{n-i+3} & ... & \iota_n & 0 \\
... & ... & ... & ... & ... \\
\iota_{n-1} & \iota_n & ... & 0 & 0 \\
\iota_n & 0 & ... & 0 & 0
\end{pmatrix}.
$$

(3.3)

Moreover, the followings are satisfied:

(HT1) Eigenvalue condition: $D_{\upsilon_0}(-1) = 0, \mathbb{N}^{\pm}_{n-1}(\upsilon_0, u) > 0, D_{\upsilon_0}(1) > 0, \mathbb{N}^{\pm}_{i}(\upsilon_0, u) > 0, i = n-2, n-4, ..., 1(or \quad 2),$ *when n is even (or odd respectively).*

(HT2) Transversality Condition: $\frac{\sum_{i=1}^{n}(-1)^{n-i}\iota_i^{'}}{\sum_{i=1}^{n}(-1)^{n-i}(n-i+1)\iota_{i-1}} \neq 0,$ *where* $\iota_i^{'}$ *means derivative of* $\iota(\upsilon)$ *at* $\upsilon = \upsilon_0,$ *then the flip bifurcation occurs at critical point* $\upsilon_0.$

If we choose $n = 3$, the subsequent lemma will give the necessary parametric conditions for which the system (1.2) underlie a flip bifurcation.

Lemma 3.2. *The flip bifurcation of system (1.2) takes place around fixed point* E_+ *at* $\varrho = \varrho_F$ *if and only if*

$1 - \kappa_1 + \kappa_0(\kappa_2 - \kappa_0) > 0,$

$1 + \kappa_1 - \kappa_0(\kappa_2 + \kappa_0) > 0,$

$1 + \kappa_2 + \kappa_1 + \kappa_0 > 0,$

$1 - \kappa_2 + \kappa_1 - \kappa_0 = 0,$

$1 + \kappa_0 > 0,$

$1 - \kappa_0 > 0,$

and

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}\iota_i^{'}}{\sum_{i=1}^{n}(-1)^{n-i}(n-i+1)\iota_{i-1}} = \frac{\kappa_2^{'}-\kappa_1^{'}+\kappa_0^{'}}{3-2\kappa_2+\kappa_1} \neq 0,$$

where $\kappa_2, \kappa_1, \kappa_0$ *are given as in (2.5) and* $\kappa_i^{'} = \frac{d\kappa_i}{d\varrho}|_{\varrho=\varrho_F}$ *with*

$$\varrho_F = \frac{1}{3c} + \frac{a(b-6c)+6c(-b+c)}{3c\Gamma} + \frac{\Gamma}{3abc},$$

$$\Gamma = \sqrt[3]{(a^3b^2(b-9c) - 9a^2b^2(b-7c) + 3\sqrt{3}\sqrt{\Lambda})},$$

$$\Lambda = -a^3b^3c^2(a^3(b-8c) - 8(b-c)^3c - 2a^2(b^2-5bc-12c^2) + a(b^3+10b^2c-95bc^2-24c^3)).$$

Proof: We take $n = 3$ and consider ϱ as a bifurcation parameter, then by using Lemma 3.1 and Lemma 2.3, we get:

$\mathbb{N}_2^{-}(\varrho) = 1 - \kappa_1 + \kappa_0(\kappa_2 - \kappa_0) > 0,$

$\mathbb{N}_2^{+}(\varrho) = 1 + \kappa_1 - \kappa_0(\kappa_2 + \kappa_0) > 0,$

$D_\varrho(1) = 1 + \kappa_2 + \kappa_1 + \kappa_0 > 0,$

$D_\varrho(-1) = 1 - \kappa_2 + \kappa_1 - \kappa_0 = 0,$

$\mathbb{N}_1^+(\varrho) = 1 + \kappa_0 > 0,$

$\mathbb{N}_1^-(\varrho) = 1 - \kappa_0 > 0.$

Define the set

$$FB_{E_+} = \{(a, b, c, \varrho) : \varrho = \varrho_F, a, b, c > 0\}.$$

If system parameters value vary in a small neighbourhood of the set FB_{E_+}, one of the eigenvalue of (2.4) is $\mu_3(\varrho_F) = -1$ and other two are $|\mu_{1,2}(\varrho_F)| \neq \pm 1$, and then the system (1.2) underlies a flip bifurcation around the fixed point E_+.

3.1.2. **Direction and Stability of Flip Bifurcation.** Here, the direction of flip bifurcation is to be determined by the applications of center manifold theory and bifurcation theory [14]. We consider the fixed point $E_+(\sqrt{bc}, \sqrt{bc}, c)$ of system (1.2) with arbitrary parameter $(a, b, c, \varrho) \in FB_{E_+}$. Let, $\varrho = \varrho_F$, then the eigenvalues of $J(E_+)$ are:

$$|\mu_i(\varrho_F)| \neq \pm 1, i = 1, 2 \tag{3.4}$$

and $\mu_3(\varrho_F) = -1$

Next, we use the transformation $\hat{x} = x - x^+, \hat{y} = y - y^+, \hat{z} = z - z^+$, where $x^+ = y^+ = \sqrt{bc}, \quad z^+ = c$ and set $A(\varrho) = J(E_+)$. Next, we transfer the fixed point E_+ of system (1.2) to the origin. Applying Taylor expansion, the system (1.2) becomes

$$X \longrightarrow A(\varrho)X + F \tag{3.5}$$

where, $X = (\hat{x}, \hat{y}, \hat{z})^T$ and

$$F(\hat{x}, \hat{y}, \hat{z}, \varrho) = (F_1(\hat{x}, \hat{y}, \hat{z}, \varrho), F_2(\hat{x}, \hat{y}, \hat{z}, \varrho), F_3(\hat{x}, \hat{y}, \hat{z}, \varrho))^T = (0, -xz\varrho, xy\varrho)^T \tag{3.6}$$

Then, we write the system (3.5) as

$$X_{n+1} = AX_n + \tfrac{1}{2}B(X_n, X_n) + \tfrac{1}{6}C(X_n, X_n, X_n) + O(X_n^4)$$

where,

$$B(x, y) = \begin{pmatrix} B_1(x, y) \\ B_2(x, y) \\ B_3(x, y) \end{pmatrix} \text{ and } C(x, y, u) = \begin{pmatrix} C_1(x, y, u) \\ C_2(x, y, u) \\ C_3(x, y, u) \end{pmatrix} \tag{3.7}$$

are the symmetric multi-linear functions of $x, y, z, u \in \mathbb{R}^3$ and these functions can be defined by:

$$B_i(x, y) = \sum_{j,k=1}^3 \left.\frac{\partial^2 F_i(v, \varrho)}{\partial v_j \partial v_k}\right|_{v=0} x_j y_k,$$

$$C_i(x, y, u) = \sum_{j,k,l=1}^3 \left.\frac{\partial^3 F_i(v, \varrho)}{\partial v_j \partial v_k \partial v_l}\right|_{v=0} x_j y_k u_l.$$

In particular,

$$B(x, y) = \begin{pmatrix} 0 \\ -x_3 y_1 \varrho - x_1 y_3 \varrho \\ x_2 y_1 \varrho + x_1 y_2 \varrho \end{pmatrix} \text{ and } C(x, y, u) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.8}$$

Consider two eigenvectors $m_1, m_2 \in \mathbb{R}^3$ of A for eigenvalue $\mu_3(\varrho_F) = -1$ such that

$$A(\varrho_F)m_1 = -m_1 \quad \text{and} \quad A^T(\varrho_F)m_2 = -m_2,$$

where m_1 and m_2 must satisfy the inner product property $\langle m_1, m_2 \rangle = 1$. Therefore, the direction of flip bifurcation can be determined from the sign of $l_1(\varrho_F)$ which is calculated by

$$l_1(\varrho_F) = \frac{1}{6}\langle m_2, C(m_1, m_1, m_1)\rangle - \frac{1}{2}\left\langle m_2, B\left(m_1, (A-I)^{-1}B(m_1, m_1)\right)\right\rangle \tag{3.9}$$

The above discussion can be given concisely in the following theorem.

Theorem 3.3. *Suppose (3.4) holds well and $l_1(\varrho_F) \neq 0$. If ϱ changes its value around the bifurcation point, then flip bifurcation will occur for the system (1.2) at fixed point $E_+(x^+, y^+, z^+)$. Furthermore, if $l_1(\varrho_F) < 0$ (resp. $l_1(\varrho_F) > 0$), then there exists unstable (resp. stable) period-2 points that bifurcate from $E_+(x^+, y^+, z^+)$.*

3.2. NS Bifurcation: Existence, Direction and Stability.

3.2.1. Existence of NS Bifurcation:
We will introduce the following Lemma for the existence of NS bifurcation with the help of explicit Flip-NS bifurcation criterion.

Lemma 3.4. *[26, 28] Let us consider an n-dimensional discrete system as follows: $Y_{k+1} = H_\upsilon(Y_k)$, with identic conditions (3.1), (3.2) and (3.3) stated in Lemma 3.1 . Next, we assume that the following conditions hold well:*

(CT1) Eigenvalue condition: $\mathbb{N}_{n-1}^-(\upsilon_0, u) = 0, \mathbb{N}_{n-1}^+(\upsilon_0, u) > 0, D_{\upsilon_0}(1) > 0, (-1)^n D_{\upsilon_0}(-1) > 0, \mathbb{N}_i^{\pm}(\upsilon_0, u) > 0$ for $i = n-3, n-5, ..., 2(or 1)$ when n is odd (or even respectively)

(CT2) Transversality Condition: $\left(\frac{d}{d\upsilon}\left(\mathbb{N}_{n-1}^-(\upsilon, u)\right)\right)_{\upsilon=\upsilon_0} \neq 0$.

(CT3) Non-resonance condition: $\cos\left(\frac{2\pi}{l}\right) \neq \varphi$,where $l = 3, 4, 5, ...$ and $\varphi = 1 - 0.5 D_{\upsilon_0}(1)\mathbb{N}_{n-3}^-(\upsilon_0, u)/\mathbb{N}_{n-2}^+(\upsilon_0, u)$, then the NS bifurcation will occur at the critical value υ_0.

Furthermore, if we choose $n = 3$, the subsequent lemma will give the necessary and sufficient parametric conditions for which system (1.2) underlies NS bifurcation if bifurcation parameter ϱ passes its critical value.

Lemma 3.5. *The NS bifurcation of system (1.2) occurs around the fixed point E_+ at $\varrho = \varrho_{NS}$ if and only if*
$$1 - \kappa_1 + \kappa_0(\kappa_2 - \kappa_0) = 0,$$
$$1 + \kappa_1 - \kappa_0(\kappa_2 + \kappa_0) > 0,$$
$$1 + \kappa_2 + \kappa_1 + \kappa_0 > 0,$$
$$1 - \kappa_2 + \kappa_1 - \kappa_0 > 0,$$
$$\frac{d}{d\varrho}\left(1 - \kappa_1 + \kappa_0(\kappa_2 - \kappa_0)\right)_{\varrho=\varrho_{NS}} \neq 0,$$

and

$$\cos\left(\frac{2\pi}{l}\right) \neq 1 - \frac{1 + \kappa_2 + \kappa_1 + \kappa_0}{2(1 + \kappa_0)}, l = 3, 4, 5, \ldots$$

where $\kappa_2, \kappa_1, \kappa_0$ are given as in (2.5) with

$$\varrho_{NS} = \frac{1}{48c^2} + \left(16c - \frac{8c^2(a(b-6c) - 6c(-b+c))}{\Psi} - \frac{8\Psi}{ab}\right),$$
$$\Psi = \sqrt{3}(a^3 b^2(b - 9c)c^3 - 9a^2 b^2(b - 7c)c^4 + 3\sqrt{3}\sqrt{\Phi}),$$
$$\Phi = -a^3 b^3 c^8(a^3(b - 8c) - 8(b - c)^3 c - 2a^2(b^2 - 5bc - 12c^2) + a(b^3 + 10b^2 c - 95bc^2 - 24c^3)).$$

Proof: We take $n = 3$ and consider ϱ as a bifurcation parameter, by using Lemma 3.4 and Lemma 2.3, we get:

$$\mathbb{N}_2^-(\varrho) = 1 - \kappa_1 + \kappa_0(\kappa_2 - \kappa_0) = 0,$$
$$\mathbb{N}_2^+(\varrho) = 1 + \kappa_1 - \kappa_0(\kappa_2 + \kappa_0) > 0,$$
$$D_\varrho(1) = 1 + \kappa_2 + \kappa_1 + \kappa_0 > 0,$$
$$(-1)^3 D_\varrho(-1) = 1 - \kappa_2 + \kappa_1 - \kappa_0 > 0,$$
$$\left(\frac{d}{d\varrho}\left(\mathbb{N}_2^-(\varrho)\right)\right)_{\varrho=\varrho_{NS}} = \frac{d}{d\varrho}\left(1 - \kappa_1 + \kappa_0(\kappa_2 - \kappa_0)\right)_{\varrho=\varrho_{NS}} \neq 0$$

and

$$1 - 0.5 D_\varrho(1)\mathbb{N}_0^-(h)/\mathbb{N}_1^+(\varrho) = 1 - \frac{1+\kappa_2+\kappa_1+\kappa_0}{2(1+\kappa_0)} \neq \cos\left(\frac{2\pi}{l}\right), l = 3, 4, 5\ldots$$

Set

$$NSB_{E_+} = \{(a, b, c, \varrho) : \varrho = \varrho_{NS}, a, b, c > 0\},$$

and for parameter perturbation in a small neighbourhood of NSB_{E_+}, two roots (eigenvalues) of (2.4) are complex conjuagte having modulus one and the magnitude of other root is not equal to one, then the system (1.2) experiences NS bifurcation around E_+.

3.2.2. **Direction and Stability of NS Bifurcation.** This section will present the direction of NS bifurcation with the help of center manifold theory and bifurcation theory.

Next, we choose the fixed point $E_+(x^+, y^+, z^+)$ of system (1.2) with arbitrary parameter $(a, b, c, \varrho) \in NSB_{E_+}$. Let, $\varrho = \varrho_{NS}$, then the matrix $J(E_+)$ has the eigenvalues satisfying

$$|\mu_i(\varrho_{NS})| = 1, i = 1, 2 \tag{3.10}$$

and $\mu_3(\varrho_{NS}) \neq 1$.

For eigenvalues $\mu(\varrho_{NS})$ and $\bar{\mu}(\varrho_{NS})$, let $m_1, m_2 \in \mathbb{C}^3$ be two eigenvectors of $A(\varrho_{NS})$ and $A^T(\varrho_{NS})$ respectively such that the following conditions hold:

$$A(\varrho_{NS})m_1 = \mu(\varrho_{NS})m_1, A(\varrho_{NS})\bar{m}_1 = \bar{\mu}(\varrho_{NS})\bar{m}_1,$$
$$A^T(\varrho_{NS})m_2 = \bar{\mu}(\varrho_{NS})m_2, A^T(\varrho_{NS})\bar{m}_2 = \mu(\varrho_{NS})\bar{m}_2, \tag{3.11}$$

and the eigenvector m_1, m_2 must satisfy the inner product property $\langle m_1, m_2 \rangle = 1$ where $\langle m_1, m_2 \rangle = m_{11}\overline{m_{21}} + m_{12}\overline{m_{22}} + m_{13}\overline{m_{23}} = \sum_{i=1}^3 m_{1i}\overline{m_{2i}}$. We decompose $X \in \mathbb{R}^3$ as $X = zm_1 + \bar{z}\bar{m}_1$ by considering ϱ vary near to ϱ_{NS} and for $z \in \mathbb{C}$. The explicit formula of z is $z = \langle m_2, X \rangle$. So, the system (3.5) transformed to the following system for $|\varrho|$ close to ϱ_{NS}:

$$z \longmapsto \mu(\varrho)z + \widehat{g}(z, \bar{z}, \varrho) \tag{3.12}$$

where $\mu(\varrho) = (1 + \widehat{\varphi}(\varrho))e^{i\theta(\varrho)}$ with $\widehat{\varphi}(\varrho_{NS}) = 0$ and $\widehat{g}(z, \bar{z}, \varrho)$ is a smooth complex-valued function. Applying Taylor expansion to the function $\widehat{g}$, we obtain

$$\widehat{g}(z, \bar{z}, \varrho) = \sum_{k+l \geq 2} \frac{1}{k!l!}\widehat{g}_{kl}(\varrho)z^{k-l} \quad \text{with} \quad \widehat{g}_{kl} \in \mathbb{C}, k, l = 0, 1, \ldots.$$

By using symmetric multi-linear vector functions, the Taylor coefficients can be defined

$$\widehat{g}_{20}(\varrho_{NS}) = \langle m_2, B(m_1, m_1) \rangle, \qquad \widehat{g}_{11}(\varrho_{NS}) = \langle m_2, B(m_1, \bar{m}_1) \rangle,$$
$$\widehat{g}_{02}(\varrho_{NS}) = \langle m_2, B(\bar{m}_1, \bar{m}_1) \rangle, \quad \widehat{g}_{21}(\varrho_{NS}) = \langle m_2, C(m_1, m_1, \bar{m}_1) \rangle. \tag{3.13}$$

The sign of first Lyapunov coefficient $l_2(\varrho_{NS})$ determines the direction of NS bifurcation and is defined by

$$l_2\left(\varrho_{\mathrm{NS}}\right) = \mathrm{Re}\left(\frac{\bar{\mu}\widehat{g}_{21}}{2}\right) - \mathrm{Re}\left(\frac{(1-2\mu)\,\bar{\mu}^2}{2\,(1-\mu)}\widehat{g}_{20}\widehat{g}_{11}\right) - \frac{1}{2}\left|\widehat{g}_{11}\right|^2 - \frac{1}{4}\left|\widehat{g}_{02}\right|^2 \tag{3.14}$$

where $\mu, \bar{\mu}$ are pair of complex conjugate eigenvalues.

According to above discussion, the direction and stability of NS bifurcation can be presented in the following theorem.

Theorem 3.6. *Suppose (3.10) holds and $l_2(\varrho_{NS}) \neq 0$, then NS bifurcation at fixed point $E_+(x^+, y^+, z^+)$ for system (1.2) if the ϱ changes its value in small neighbourhood of NSB_{E_+}. Moreover, if $l_2(\varrho_{NS}) < 0 \quad (resp. \quad l_2(\varrho_{NS}) > 0)$, then there exists attracting (resp. repelling) smooth closed invariant curve bifurcate from E_+ and the bifurcation is sub-critical (resp. super-critical).*

4. NUMERICAL SIMULATIONS

In this section, we will validate our theorical results of system (1.2) by using numerical simulations with the help of diagrams of bifurcation, phase portraits, MLEs and FD. For the investigations of bifurcations, we take parameter values provided in table (1).

TABLE 1. Parameter values

Cases	Varying parameter in range	Fixed parameters	System Dynamics
(i)	$0.65 \leq \varrho \leq 0.948$	$a = 0.74, b = 2.5, c = 0.35$	Flip Bifurcation
(ii)	$0.3 \leq \varrho \leq 0.66$	$a = 2, b = 1.5, c = 0.65$	NS Bifurcation
(iii)	$2.6 \leq a \leq 7.5$	$b = 3, c = 1.3, \varrho = 0.228122$	NS Bifurcation
(iv)	$2.6 \leq a \leq 7.5, 3 \leq b \leq 6$	$c = 1.3, \varrho = 0.228122.$	NS Bifurcation

Example 1: We take the values of the parameters as in case (i). We find a fixed point $E_+ = (0.935414, 0.935414, 0.35)$ and bifurcation point for the system (1.2) is evaluated at $\varrho_F = 0.856434$.

Now, at $\varrho = \varrho_F$, the Jacobian matrix of system (1.2) takes the form

$$A(\varrho_F) = \begin{pmatrix} 0.366238 & 0.633762 & 0 \\ -0.299752 & 1.29975 & -0.801122 \\ 0.801122 & 0.801122 & -1.14109 \end{pmatrix}.$$

and the eigenvalues of $A(\varrho_F)$ are $\mu_{1,2} = 0.762452 \pm 0.591875i$ and $\mu_3 = -1$ with $|\mu_{1,2}| = 0.96522$. Moreover,

$1 - \kappa_1 + \kappa_0\,(\kappa_2 - \kappa_0) = 0.856435 > 0,$

$1 + \kappa_1 - \kappa_0\,(\kappa_2 + \kappa_0) = 0.0278016 > 0,$

$1 + \kappa_2 + \kappa_1 + \kappa_0 = 0.813491 > 0,$

$1 - \kappa_2 + \kappa_1 - \kappa_0 = 0,$

$1 + \kappa_0 = 1.93165 > 0,$

$1 - \kappa_0 = 0.0683513 > 0,$

and

$\frac{\kappa_2' - \kappa_1' + \kappa_0'}{3 - 2\kappa_2 + \kappa_1} = 2.33526 \neq 0.$

This shows that all requirements of Lemma 3.2 are validated with $(a, b, c, \varrho) \in FB_{E_+}$. So, the criterion for the existence of flip bifurcation is verified and therefore, system (1.2) experience a flip bifurcation

around E_+ at $\varrho = \varrho_F$.

Next, let the two eigenvectors of $A(\varrho_F)$ corresponding to $\mu_3(\varrho_F) = -1$, be $m_1, m_2 \in \mathbb{R}^3$ respectively. Then, we obtain

$m_1 \sim (-0.143274, 0.308864, 0.940253)^T$ and $m_2 \sim (-0.524191, -0.147704, 0.830694)^T$.

To set $\langle m_1, m_2 \rangle = 1$, we can choose normalized vector as $m_2 = \gamma m_2$ where, $\gamma = 1.22239$. Therefore,

$m_1 \sim (-0.143274, 0.308864, 0.940253)^T$ and $m_2 \sim (-0.640768, -0.180553, 1.02521)^T$.

Then from (3.9), the Lyapunov coefficient $l_1(\varrho_F) = 0.0596487$ is obtained. This guarentees the appropriateness of Theorem 3.3.

The diagrams of bifurcation shown in Figure 1 (a,b,c) express the stability of fixed point E_+ when $\varrho < \varrho_F$, loses its stability at $\varrho = \varrho_F$ and when $\varrho > \varrho_F$, a period-doubling phenomena leads to chaotic dynamics. The MLEs associated with Figure 1 (a) is shown in Figure 1 (d).

Example 2: We select the parameters as in case (ii) . By calculation, we find a fixed point $E_+ = (0.987421, 0.987421, 0.65)$ of system (1.2) and the bifurcation point is obtained as $\varrho_{NS} = 0.33142374934833646$.

The Jacobian matrix is evaluated at E_+ is

$$A(\varrho_{NS}) = \begin{pmatrix} 0.337153 & 0.662847 & 0 \\ -0.215425 & 1.21543 & -0.327255 \\ 0.327255 & 0.327255 & 0.502864 \end{pmatrix},$$

and the eigenvalues of $A(\varrho_{NS})$ are $\mu_{1,2} = 0.906226 \pm 0.422795i$ and $\mu_3 = 0.242991$ with $|\mu_{1,2}| = 1$.

Furthermore,

$1 - \kappa_1 + \kappa_0(\kappa_2 - \kappa_0) = 0$,

$1 + \kappa_1 - \kappa_0(\kappa_2 + \kappa_0) = 1.88191 > 0$,

$1 + \kappa_2 + \kappa_1 + \kappa_0 = 0.141976 > 0$,

$1 - \kappa_2 + \kappa_1 - \kappa_0 = 4.73884 > 0$,

$\frac{d}{d\varrho}(1 - \kappa_1 + \kappa_0(\kappa_2 - \kappa_0)) = -0.350079 \neq 0$

and

$1 - \frac{1 + \kappa_2 + \kappa_1 + \kappa_0}{2(1 + \kappa_0)} = 0.906226$.

From the resonance condition $\cos\left(\frac{2\pi}{l}\right) = 0.906226$, we get $l = \pm 14.3936$.

So, the criterion for the existence of NS bifurcation are fulfilled with $(a, b, c, \varrho) \in NSB_{E_+}$. This confirms the correctness of Lemma 3.5. Therefore, a NS bifurcation occurs around fixed point E_+ if ϱ crosses its critical value ϱ_{NS}.

Let $m_1, m_2 \in \mathbb{C}^3$ be two complex eigenvectors of $A(\varrho_{NS})$ and $A^T(\varrho_{NS})$ corresponding to $\mu_{1,2}$, respectively. Therefore,

$m_1 \sim (0.484688 + 0.251296i, 0.255831 + 0.524901i, 0.600797)^T$ and $m_2 \sim (-0.219176 - 0.348147i, 0.795239, -0.307428 - 0.32224i)^T$.

For $\langle m_1, m_2 \rangle = 1$, we can take normalized vector as $m_2 = \gamma m_2$ where, $\gamma = -0.314824 + 1.30389i$. Then

$m_1 \sim (0.484688 + 0.251296i, 0.255831 + 0.524901i, 0.600797)^T$ and $m_2 \sim (0.522948 - 0.176177i, -0.250361 + 1.03691i, 0.516951 - 0.299404i)^T$.

Also by (3.13) the Taylor coefficients are , $\widehat{g}_{20} = -0.121403 + 0.332837i$, $\widehat{g}_{11} = 0.136013 + 0.250931i$, $\widehat{g}_{02} = 0.212633 + 0.0643138i$, $\widehat{g}_{21} = 0$.

From (3.14), we obtain the Lyapunov coefficient $l_2(\varrho_{NS}) = -0.110501 < 0$. Therefore, the NS bifurcation is super-critical and the requirements of Theorem 3.6 are established.

The NS bifurcation diagrams are displayed in Figure 2 (a,b,c) which reveal that the condition of stability for the positive fixed point E_+ occurs when $\varrho < \varrho_{NS}$, loses its stability at $\varrho = \varrho_{NS}$ and there appears an attracting closed invariant curve when $\varrho > \varrho_{NS}$. The MLEs and FD related to Figure 2 (a) are shown in Figure 2 (d,e). The non stability of system dynamics are justified with the sign of MLEs.

The phase portraits of system (1.2) corresponding to diagram of bifurcation shown in Figure 2 (a) are plotted in Figure 3. This figure explicitly illustrate the mechanism of how an invariant smooth closed curve bifurcates from stable fixed point E_+ when ϱ changes near its critical value. We noticed that NS bifurcations occurs at $\varrho = \varrho_{NS}$ (see in Figure 3(b)). When $\varrho > \varrho_{NS}$, there appears an invariant closed curve and further increasing of ϱ, NS bifurcation instigate a route to chaos.

Example 3: We choose the value of parameters as given in case (iii) and consider a as bifurcation parameter. We obtain a fixed point $E_+ = (1.97484, 1.97484, 1.3)$ of system (1.2) and the bifurcation point is calculated as $a_{NS} = 6.50002$. Moreover,

$1 - \kappa_1 + \kappa_0 (\kappa_2 - \kappa_0) = 0$,

$1 + \kappa_1 - \kappa_0 (\kappa_2 + \kappa_0) = 1.581 > 0$,

$1 + \kappa_2 + \kappa_1 + \kappa_0 = 0.601881 > 0$,

$1 - \kappa_2 + \kappa_1 - \kappa_0 = 1.94525 > 0$,

$\frac{d}{da} (1 - \kappa_1 + \kappa_0 (\kappa_2 - \kappa_0)) = 0.0679625 \neq 0$

and

$1 - \frac{1 + \kappa_2 + \kappa_1 + \kappa_0}{2(1 + \kappa_0)} = 0.793552$.

From the resonance condition $\cos\left(\frac{2\pi}{l}\right) = 0.793552$, we get $l = \pm 9.6048$.

So, the the existence of NS bifurcation criterion are fulfilled. This shows that the correctness of Lemma 3.5. At $a = a_{NS}$ the eigenvalues values are $\mu_{1,2} = 0.793552 \pm 0.608502i$ and $\mu_3 = -0.45771$ with $|\mu_{1,2}| = 1$ and the Lyapunov coefficient $l_2(a_{NS}) = -0.0644966 < 0$ is obtained. Hence, the NS bifurcation is super-critical and it justifies the conditions of Theorem 3.6 . The bifurcation diagrams of system (1.2) with respect to bifurcation parameter a are plotted in Figure 4 (a,b,c). We noticed that the system enters into a chaotic window when $a < a_{NS}$ and the fixed point E_+ is unstable. NS bifurcation takes place at $a = a_{NS}$ and when $a > a_{NS}$, the system dynamics jump to a non-chaotic window. The MLEs and FD related to

Figure 4 (a,b,c) are shown in Figure 4 (d,e). Then the sign of MLEs confirms the occurence of chaotic dynamics in system (1.2).

The phase portraits of system (1.2) associated with Figure 4 (a) are displayed in Figure 5. We noticed that as the values of a increasing the system dynamics jump from a chaotic state to stable state through NS bifurcations.

Example 4: We consider the parameter values in case (iv). Varying two parameters, we observe that the system (1.2) can exhibit complex dynamical behavior. In two-dimensional parameter space, the diagrams of bifurcation are presented in Figure 6(a). Figure 6(b) shows the 2-D projected MLEs onto (a, b) plane. From Figure 6(a), one can notice that the growth of parameter b advances the NS bifurcation with respect to bifurcation parameter a, that is, the system dynamics enter into stable state earlier. The 2-D parameter (a, b) space helps to select parameter values to observe how do system dynamics switch from stable window to unstable window or vice-versa. For example, we observe chaotic trajectories exist in the system for parameters $a = 3, b = 3$ whereas stable trajectories exist for $a = 6.75, b = 3$ (see Figure 4), which are conformable from the sign of MLEs presented in Figure 6 (b). The critical value curves ($\Delta_2 = 0$ and $\Omega_2 = 0$) of NS bifurcation of system (1.2) in (a, b) plane and (ϱ, a) plane are plotted in Figures 7 (a, b) respectively, where

$$\Delta_2 = a^2(0.00483026 - 0.00044834b)b + a(-0.037145 + 0.00483026b)b,$$

and

$$\Omega_2 = a(-0.675 - 1.6575\varrho)\varrho^3 + a^2\varrho^3(1.5 - 4.2\varrho + 5.85\varrho^2 - 3.8025\varrho^3).$$

Figure 7 (a) illustrates that on the left region of the curve, the fixed point E_+ is stable (consistent with Figures 6 (a,b)) whereas on the right region of the curve the fixed point is stable and Figure 7 (b) illustrates that on the region below curve, the fixed point E_+ is unstable whereas on the region above the curve the fixed point is stable. Specially, with the growth of parameter a, bifurcation delays for system (1.2) with respect to bifurcation parameter ϱ.

4.1. **Fractal Dimension.** The chaotic attractors of a system is characterised by the measure of the fractal dimensions (FD) and is defined by [3]

$$\widehat{D}_L = k + \frac{\sum_{j=1}^{k} t_j}{|t_{k+1}|} \tag{4.1}$$

where k is the largest integer such that $\sum_{j=1}^{k} t_j \geq 0$ and $\sum_{j=1}^{k+1} t_j < 0$ and t_j's are Lyapunov exponenets. Now, for the system (1.2) the fractal dimensions takes the form:

$$\widehat{D}_L = 2 + \frac{t_1 + t_2}{|t_3|} \tag{4.2}$$

The chaotic dynamics of the system (1.2) (see Figure 3) are quantified with the sign of FD (see Figure 2 (e)) which guarantees that increasing the values of the parameter ϱ causes unstable system dynamics for discrete-time Lü system.

5. CHAOS CONTROL

Hybrid control strategy [29] is applied to system (1.2) controlling chaos. We rewrite our uncontrolled system (1.2) as

$$X_{n+1} = G(X_n, \varrho) \tag{5.1}$$

where $X_n \in \mathbb{R}^3$, $G(.)$ is non-linear vector function and $\varrho \in \mathbb{R}$ is bifurcation parameter. Applying hybrid control strategy, the controlled system of (5.1) becomes

$$X_{n+1} = \rho G(X_n, \varrho) + (1 - \rho) X_n \qquad (5.2)$$

where $0 < \rho < 1$ is the control parameter. Now, if we implement the above mentioned control strategy to system (1.2), then we get the following controlled system

$$\begin{cases} x_{n+1} = \rho \left(x_n + \varrho \left(a(y_n - x_n) \right) \right) + (1 - \rho) \, x_n, \\ y_{n+1} = \rho \left(y_n + \varrho \left(-x_n z_n + c y_n \right) \right) + (1 - \rho) \, y_n, \\ z_{n+1} = \rho \left(z_n + \varrho \left(x_n y_n - b z_n \right) \right) + (1 - \rho) \, z_n \end{cases} \qquad (5.3)$$

For the controlled system (5.3), the Jacobian matrix at fixed point $E_+(x^+, y^+, z^+)$ (which is a fixed point of system (1.2)) takes the form

$$J(E_+) = \begin{pmatrix} 1 - a\varrho\rho & a\varrho\rho & 0 \\ -z^+ \varrho\rho & 1 + c\varrho\rho & -x^+ \varrho\rho \\ y^+ \varrho\rho & x^+ \varrho\rho & 1 - b\varrho\rho \end{pmatrix},$$

Then, the zeroes of $|\mu I - J(E_+)|$ (eigenvalues of J) satisfy the equation

$$\mu^3 + \varepsilon_2 \mu^2 + \varepsilon_1 \mu + \varepsilon_0 = 0. \qquad (5.4)$$

where,

$$\begin{aligned}
\varepsilon_2 &= -3 + \varrho\rho(a + b - c), \\
\varepsilon_1 &= 3 + 2c\varrho\rho + x^{+2}\varrho^2\rho^2 - b\varrho\rho(2 + c\varrho\rho) + a\varrho\rho(-2 + \varrho\rho(b - c + z^+)), \\
\varepsilon_0 &= -1 - c\varrho\rho - x^{+2}\varrho^2\rho^2 + b\varrho\rho(1 + c\varrho\rho) + \\
&\quad a\varrho\rho(1 + \varrho\rho(c - z^+) + \varrho^2\rho^2(x^{+2} + x^+ y^+) - b\varrho\rho(1 + \varrho\rho(c - z^+)))
\end{aligned} \qquad (5.5)$$

Lemma 5.1. *If the fixed point $E_+(x^+, y^+, z^+)$ of the uncontrolled system (1.2) is unstable, then it is a sink (stable) for the controlled system (5.3) if the roots of (5.4) lie inside open disk satisfying conditions in Lemma 2.1.*

Example 5: To see the effectiveness of hybrid control strategy to control chaotic (unstable) system dynamics, we fix $a = 2, b = 1.5, c = 0.65$ with $\varrho = 0.6 > \varrho_{NS}$. Then, it shows that the fixed point $E_+(0.987421, 0.987421, 0.65)$ of system (1.2) is unstable (see Fig 2), but this fixed point is stable for the controlled system (5.3) iff $0 < \rho < 0.5523729155805587$. Taking $\rho = 0.45$, the unstable system dynamics around E_+ are eliminated showing that E_+ is a sink for the controlled system (5.3) which have been displayed in Figure 8.

6. CONCLUSION

We investigate qualitative analysis of discrte-time chaotic Lü system which is obtained with implementation of Forward Euler scheme. We use explicit Flip-NS bifurcation criterion for the existence of Flip and NS bifurcations of system (1.2) at fixed point E_+. Also, we determine direction of both bifurcations by the applications of center manifold theory and bifurcation theory. Specifically, we observe that system (1.2) emerges a Flip or NS bifurcation around E_+ when ϱ varies in a small vicinity of either the set FB_{E_+} or NSB_{E_+}. Figures (1 and 2) illustrate that for small values of integral step size, ϱ system dynamics are stable, but for larger integral step size system dynamics are unstable producing complex dynamical behavior through NS or Flip bifurcation. For the NS bifurcation diagram, we find closed invariant curve when ϱ passes its critical value and attracting chaotic set for growth of ϱ. Moreover in Figure 4, we notice that for small value of parameter a the system enters chaotic window and as the increase of parameter a the system (1.2) goes to stable window via NS bifurcation. For the mechanism of both bifurcations, system dynamics

switch from stable state to unstable state and trigger a route to chaos and vice-versa. We present the 3D bifurcation diagrams to see how the NS bifurcation advance or delay when parameter varies in two dimensional parameter space. In addition, the chaotic dynamics of system (1.2) is justified with sign of MLEs and FD. Finally, we apply a hybrid control strategy to eliminate unstable system trajectories. For this discrete system, the other properties like complexity, control and synchronization and co-dimension-2 bifurcation need further study.

(a) (b) (c) (d)
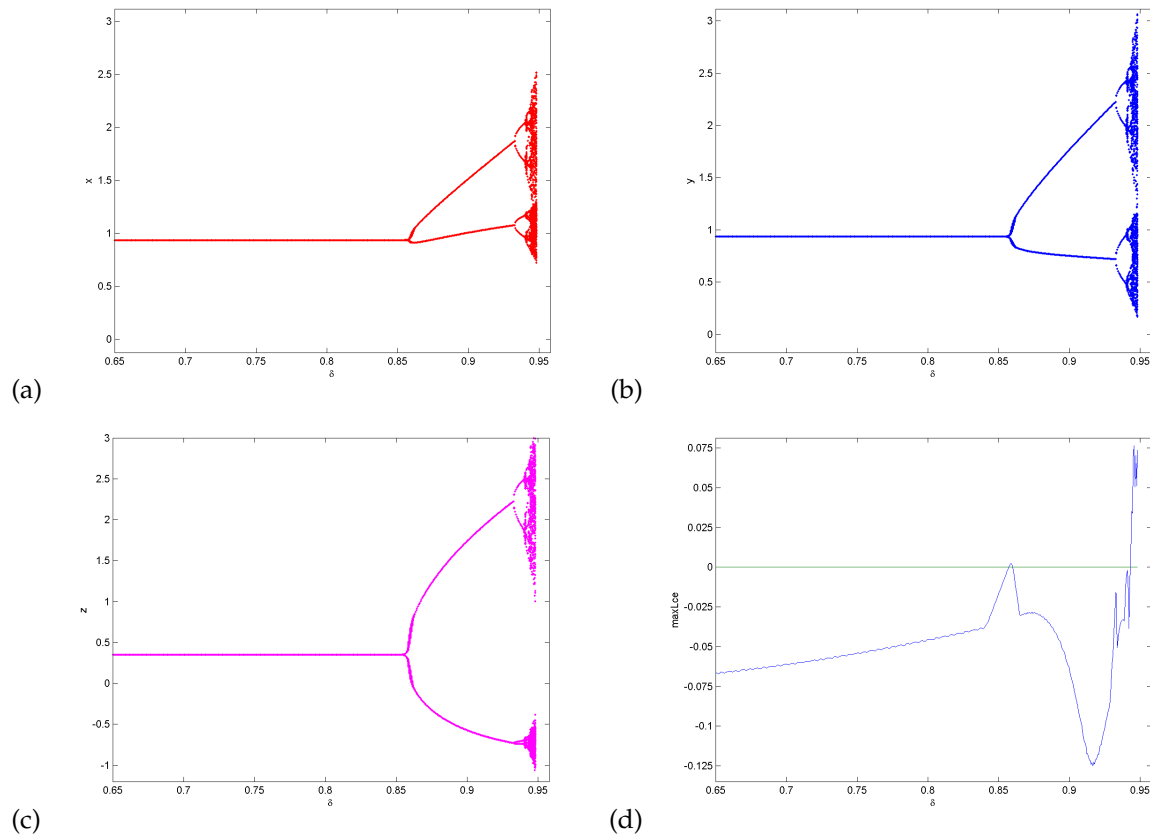
FIGURE 1. **Flip Bifurcation Diagram** in (a) (ϱ, x) plane, (b) (ϱ, y) plane, (c) (ϱ, z) plane(d) MLEs related to (a). $(x_0, y_0, z_0) = (0.93, 0.93, 0.33)$.
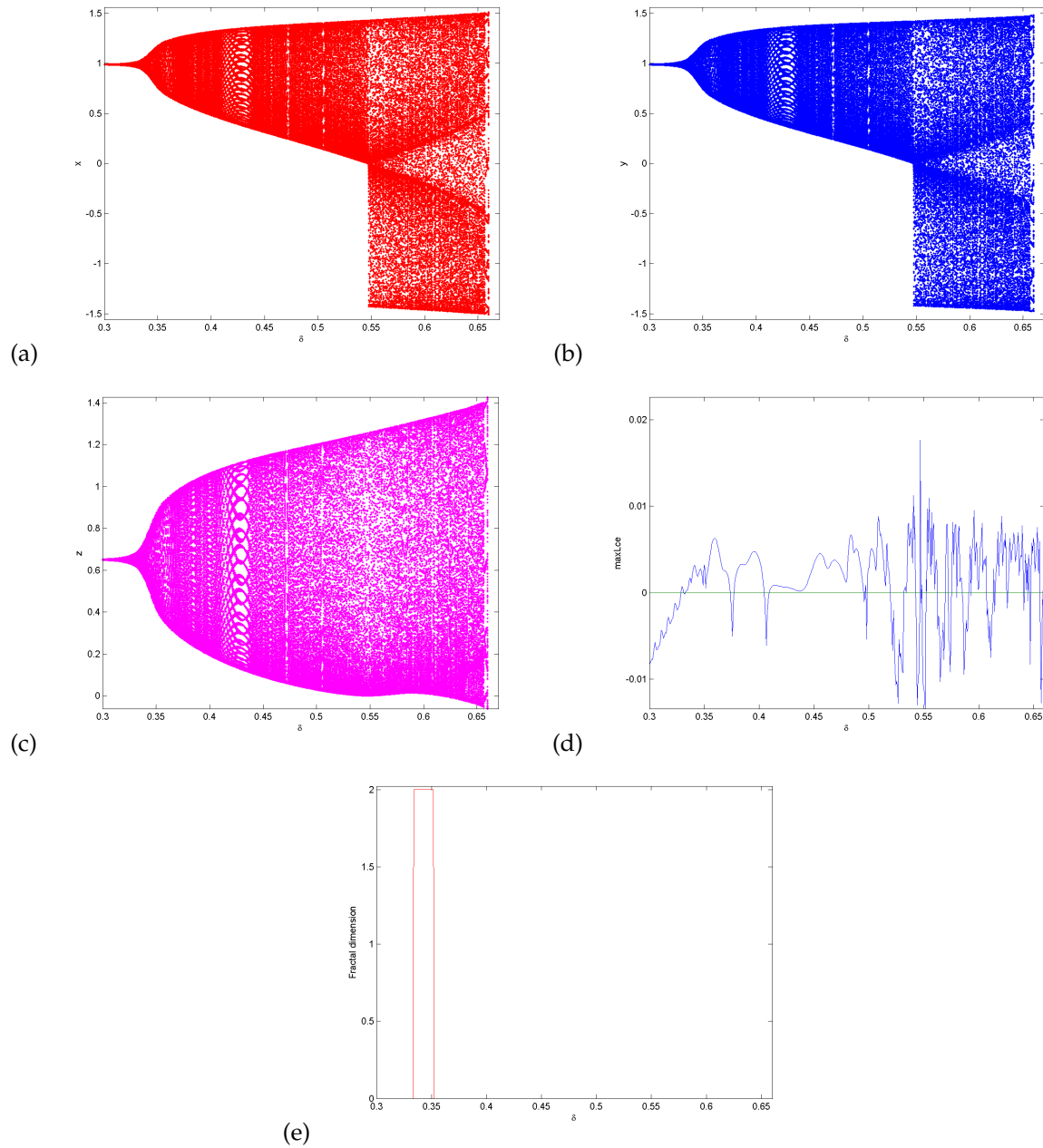
FIGURE 2. **NS Bifurcation diagram in** (a) (ϱ, x) plane, (b) (ϱ, y) plane, (c) (ϱ, z) plane, (d) MLEs (e) FD, $(x_0, y_0, z_0) = (0.98, 0.98, 0.6)$.
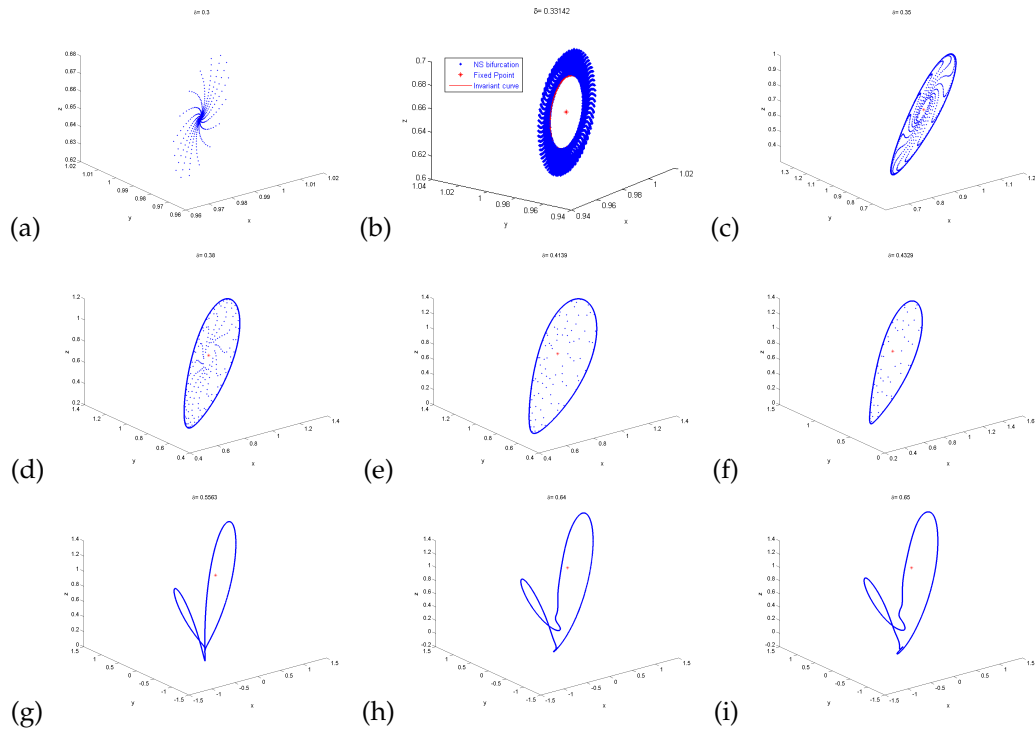
FIGURE 3. **Phase portrait for different values of ϱ corresponding to Figure 2 a,b,c. Red $*$ is the fixed point E_+.**
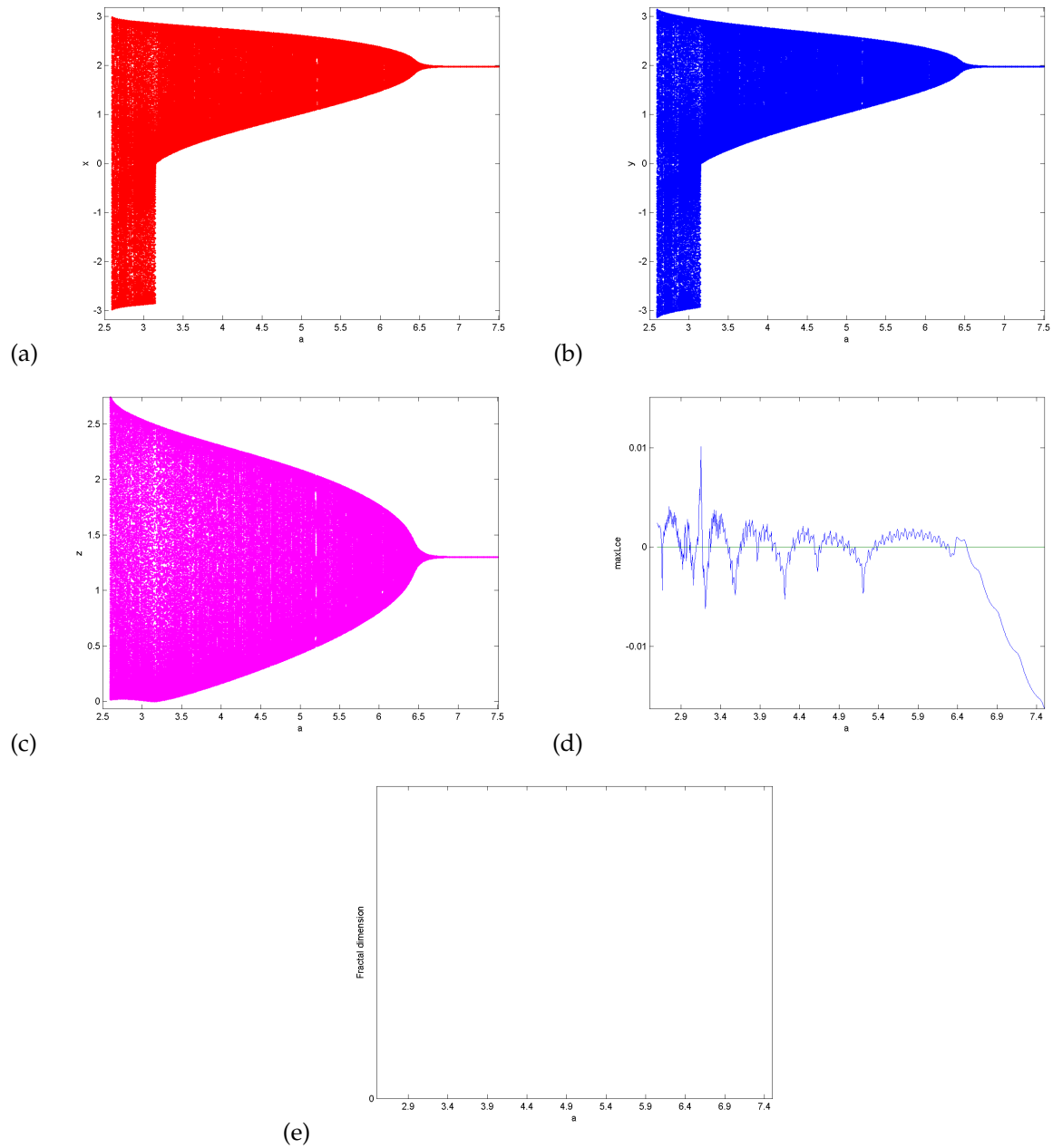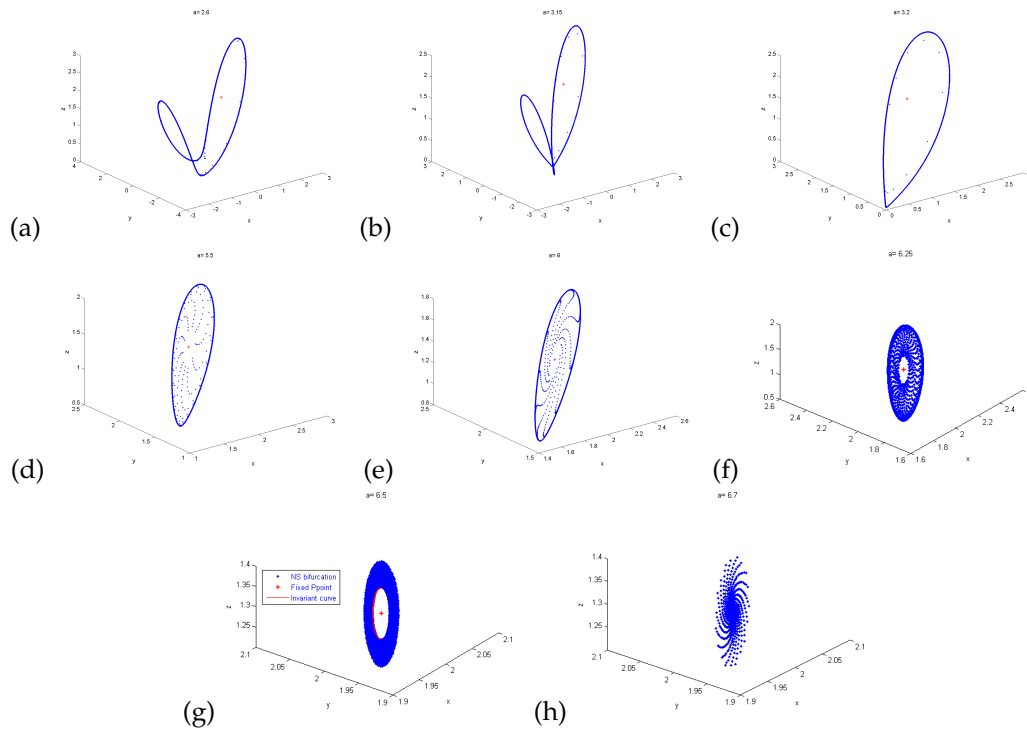
FIGURE 4. **NS Bifurcation diagram in** (a) (a, x) plane, (b) (a, y) plane, (c) (a, z) plane, (d) MLEs (d) FD , $(x_0, y_0, z_0) = (1.95, 1.95, 1.2)$.

FIGURE 5. **Phase portrait for different values of a corresponding to Figure 4 a,b,c. Red $*$ is the fixed point E_+.**
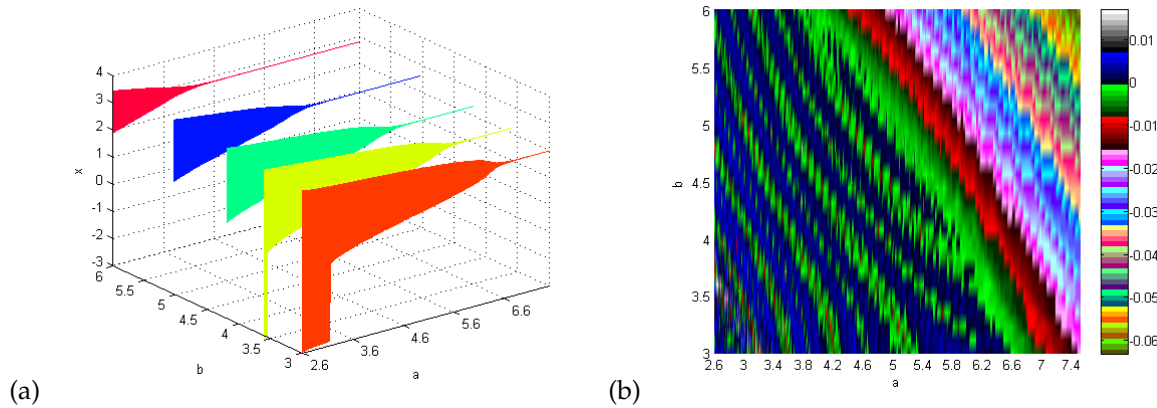

FIGURE 6. **System Dynamics for control parameters a and b (a)NS bifurcation in (a, b, x) space for $a \in [2.6, 7.5]$ and $b = 3, 3.6, 4.2, 5.04, 6 \in [3, 6]$ (b) The projection of MLEs onto (a, b) plane.**
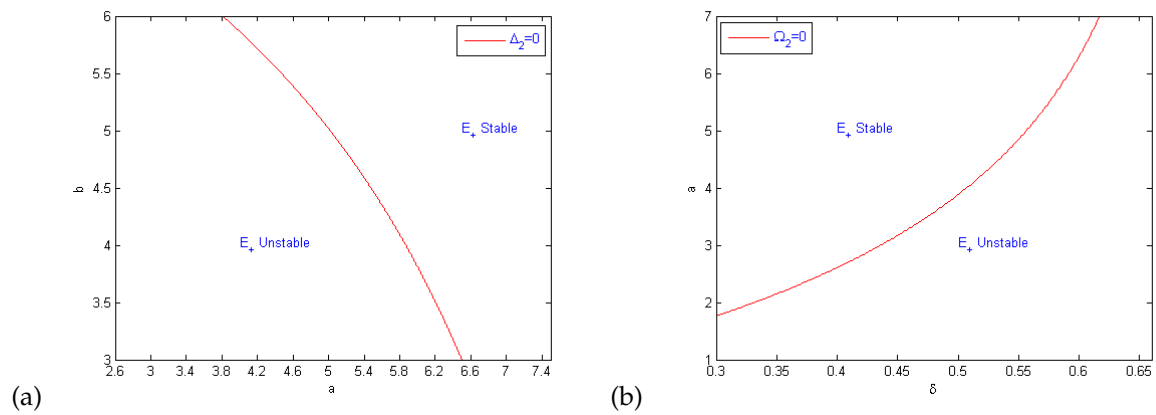
FIGURE 7. **Critical value curve of NS bifurcation.** (a) Critical value curve in parameters (a, b) plane with $c = 1.3$, $\varrho = 0.228122$, (b) Critical value curve in parameters (ϱ, a) plane with $b = 1.5$, $c = 0.65$.
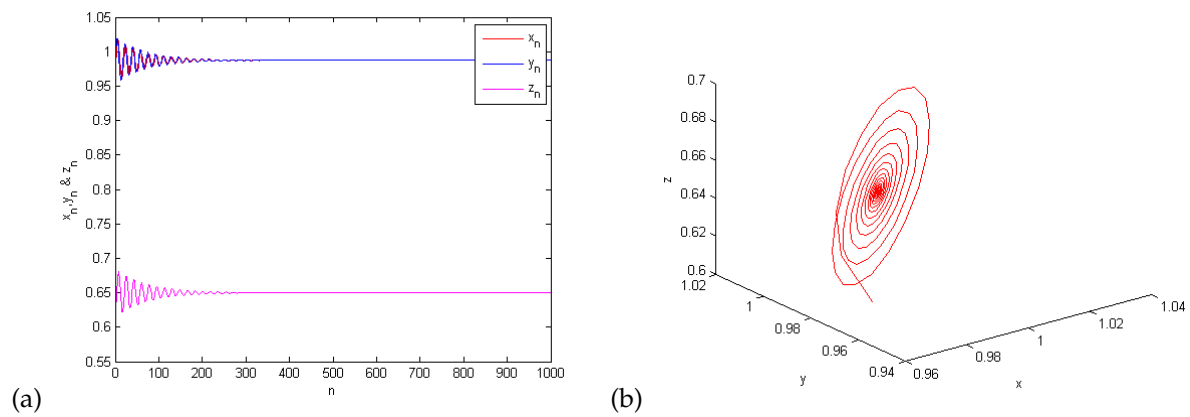

FIGURE 8. Controlling Chaos of system (5.3). (a) Time history of x, y, z (b) Phase diagram of system (5.3).

Funding. Not applicable

Availability of data and materials. Not applicable

Ethics approval and consent to participate. Not applicable

Consent for publication. Not applicable

Authors' contributions. All authors contributed equally to carry out the proof of the main results and approved the final manuscript.

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