ISSN: 2832-4293

Pan-American Journal of Mathematics 4 (2025), 14
https://doi.org/10.28919/cpr-pajm/4-14

A STUDY ON SOME CLASSES OF HYBRID LANGEVIN PANTOGRAPH ψ-CAPUTO FRACTIONAL COUPLED SYSTEMS

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ABSTRACT. This paper focuses on the study of a class of coupled systems of hybrid Langevin fractional pantograph differential equations involving the ψ-Caputo fractional derivative within Banach spaces. By applying Banach's fixed-point theorem, we demonstrate the uniqueness of solutions to our coupled system. The existence of the solution is then shown using Dhage's fixed-point theorem. Additionally, we analyze the stability in the sense of Ulam-Hyers and Ulam-Hyers-Rassias. Finally, we present an example to illustrate our results.

1. INTRODUCTION

Fractional calculus is a very important tool for dealing with the complex structures that appear in various fields, including biology, chemistry, physics, control theory, wave propagation, signal and image processing, etc. This calculus is characterized by an extensive theory and numerous applications, as it involves extending differentiation and integration operations from integer orders to non-integer orders, see [2, 10, 11, 35, 38]. Recently, researchers have developed numerous operators. In a recent publication by Ricardo Almeida [6] in 2017, a novel definition of the fractional operator was introduced, leading to the development of numerous operators by researchers. This definition, known as the ψ-Caputo fractional operator, thereby expands the range of existing operators such as the Caputo, the Caputo-Hadamard, the Caputo-Erdélyi-Kober, and the Caputo-Katugampola, see [23].

Nonlinear coupled systems involving fractional derivatives are an important topic of modern study by researchers, as they arise in various problems in applied mathematics. Consequently, numerous research papers and books have been published in which researchers have discussed the existence, stability, and uniqueness of solutions for different systems involving fractional differential equations and inclusions, using various fractional derivatives and different types of conditions. For further details,

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Submitted on May 02, 2025.

2020 *Mathematics Subject Classification.* Primary 26A33, 34A08; Secondary 34A12, 34B10.

Key words and phrases. coupled systems; ψ-Caputo derivative; uniqueness; existence; Ulam stability; Dhage theorem.

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see [5,8,12,30,31,34]. Many researchers have studied dynamic systems as special cases of fractional differential equations. Hybrid fractional differential equations are among the contemporary topics in scientific research and have been studied by numerous researchers. This class of equations involves the partial differentiation of an unknown hybrid function with a nonlinear dependence on that function. Some recent results related to hybrid differential equations can be found in a series of research papers, see [3,13,17,26,27].

In [25], the authors Matar et al. investigated the following nonlinear fractional differential hybrid system subject to periodic boundary conditions of the form

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1,\psi}\left(\mathrm{x}(\tau)\mathrm{g}_1(\tau)\right) = \mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\tau)), & \tau\in[a,b],\\ {}^{C}D_{0^+}^{\alpha_2,\psi}\left(\mathrm{y}(\tau)\mathrm{g}_2(\tau)\right) = \mathrm{f}_2(\tau,\mathrm{x}(\tau),\mathrm{y}(\tau)), & \tau\in[a,b],\\ \mathrm{x}(a)=\mathrm{x}(b),\ \mathrm{x}'(a)=\mathrm{x}'(b), & \\ \mathrm{y}(a)=\mathrm{y}(b),\ \mathrm{y}'(a)=\mathrm{y}'(b), & \end{cases}$$

where ${}^{C}D_{0^+}^{\alpha_1,\psi}$ and ${}^{C}D_{0^+}^{\alpha_2,\psi}$ are the ψ-Caputo fractional derivatives of order $\alpha_1,\alpha_2 \in (1,2)$, $(\mathrm{g_i})_{i=1,2}:[a,b]\to\mathbb{R}\setminus\{0\}$ and $(\mathrm{f_i})_{i=1,2}:[a,b]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions.

In 1908, Paul Langevin presented in the classic form of the Langevin equation to describe the evolution of physical phenomena in fluctuating environments [24]. subsequently, various generalizations of the Langevin equation were examined by many researchers, we mention here some works, see [18,19].

In [28], Salem et al. study the existence and uniqueness of solutions to dual systems of nonlinear fractional Langevin differential equations of the Caputo type with boundary value conditions given as follows:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1}\left({}^{C}D_{0^+}^{\beta_1}+\lambda_1\right)\mathrm{x}(\tau) = \mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\tau)),\ \tau\in[0,1],\ 0<\beta_1\le 1,\ 1<\alpha_1\le 2,\\ {}^{C}D_{0^+}^{\alpha_2}\left({}^{C}D_{0^+}^{\beta_2}+\lambda_2\right)\mathrm{y}(\tau) = \mathrm{f}_2(\tau,\mathrm{x}(\tau),\mathrm{y}(\tau)),\ \tau\in[0,1],\ 0<\beta_2\le 1,\ 1<\alpha_2\le 2,\\ \mathrm{x}(0)=0,\ \ {}^{C}D_{0^+}^{\beta_1}\mathrm{x}(0)=\Gamma(\beta_1+1)\ {}_{1}^{\gamma}I_{0^+}^{\nu_1}\mathrm{x}(\epsilon_1),\\ \sum_{i=1}^{m_1}\rho_{i_1}\mathrm{x}(\varepsilon_{i_1}) = \mu_1\,{}^{AB}I_{0^+}^{\nu_2}\mathrm{x}(\epsilon_2),\\ \mathrm{y}(0)=0,\ \ {}^{C}D_{0^+}^{\beta_2}\mathrm{y}(0)=\Gamma(\beta_2+1)\ {}_{2}^{\gamma}I_{0^+}^{\nu_3}\mathrm{y}(\epsilon_3),\\ \sum_{i=1}^{m_2}\rho_{i_2}\mathrm{y}(\varepsilon_{i_2}) = \mu_2\,{}^{AB}I_{0^+}^{\nu_4}\mathrm{y}(\epsilon_4), \end{cases}$$

where ${}^{C}D_{0^+}$ is the Caputo fractional derivative of order α_j and β_j for $j=1,2$. ${}^{AB}I_{0^+}$ and ${}^{\gamma}I_{0^+}$ are Atangana-Baleanu, and Katugampola fractional integrals, respectively. $\gamma i>0$ and $\Lambda_i,\mu_i\in\mathbb{R}$ for $=1,2$, $\nu_n\in\mathbb{R}$ for $n=1,2,3,4$. $\rho_{i_j}\in\mathbb{R}$ for $i=1,...,m_i$ and $j=1,2$. $0<\epsilon_n<\varepsilon_1<\varepsilon_2<\varepsilon_3<\cdots<\varepsilon_{m_i}$ for $i=1,2$ and $n=1,2,3,4$. $\mathrm{f}_1,\mathrm{f}_1:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions.

The pantograph equation is an effective differential equation used in various fields, including electrodynamics, astrophysics, and cell growth. This has led to numerous recent works on the fractional-order pantograph equation by several researchers, see [15,22,39].

Additionaly [4], I. Ahmad et al. demonstrated the existence of solutions for a nonlinear coupled system of pantograph fractional differential equations with Caputo fractional derivatives of the form

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1}\mathrm{x}(\tau) = \mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{x}(\vartheta\tau),\mathrm{y}(\tau)),\ \tau\in[0,1],\\ {}^{C}D_{0^+}^{\alpha_2}\mathrm{y}(\tau) = \mathrm{f}_2(\tau,\mathrm{x}(\tau),\mathrm{y}(\tau),\mathrm{y}(\vartheta\tau)),\ \tau\in[0,1],\\ a_1\mathrm{x}(0)-b_1\mathrm{x}(\vartheta_1)-c_1\mathrm{x}(1)=\mathrm{g}_1(\mathrm{x}),\\ a_2\mathrm{x}(0)-b_2\mathrm{x}(\vartheta_2)-c_2\mathrm{y}(1)=\mathrm{g}_2(\mathrm{y}), \end{cases}$$

where ${}^{C}D^{\alpha_i}_{0^+}$ represent the Caputo derivatives of order $\alpha_i \in (0,1)$, $0 < \vartheta_i < 1$, $a_1 \neq b_i + c_i$ which $a_i\, b_i, c_i \in \mathbb{R}$, $\mathrm{f}_i : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $\mathrm{g}_i : C([0,1],\mathbb{R}) \to \mathbb{R}$, $i = 1,2$.

In recent times, substantial attention has been directed towards investigating the Ulam stability of solutions for coupled systems of fractional differential equations. In [32], A. Salim and al. investigate the existence, uniqueness and Ulam stability of differential coupled system involving the Riesz-Caputo derivative with Boundary Conditions of the from

$$\begin{cases} {}^{RC}_{0}D^{\alpha_1}_{T}\mathrm{x}(\tau) = \mathrm{f}_1(\tau, \mathrm{x}(\tau), \mathrm{y}(\tau), {}^{RC}_{0}D^{\alpha_1}_{T}\mathrm{x}(\tau), {}^{RC}_{0}D^{\alpha_2}_{T}\mathrm{y}(\tau)),\ \tau \in [0,T], \\ {}^{RC}_{0}D^{\alpha_2}_{T}\mathrm{y}(\tau) = \mathrm{f}_2(\tau, \mathrm{x}(\tau), \mathrm{y}(\tau), {}^{RC}_{0}D^{\alpha_1}_{T}\mathrm{x}(\tau), {}^{RC}_{0}D^{\alpha_2}_{T}\mathrm{y}(\tau)),\ \tau \in [0,T], \\ \beta_1\mathrm{x}(0) + \beta_2\mathrm{x}(T) = \beta_3, \\ \gamma_1\mathrm{y}(0) + \gamma_2\mathrm{y}(T) = \gamma_3, \end{cases}$$

where ${}^{RC}_{0}D^{\alpha_i}_{T}$ represent the Riesz-Caputo derivatives of order $\alpha_i \in (0,1)$ for $i = 1,2$, $\gamma_j, \beta_j \in \mathbb{R}$ for $j = 1,2,3$, $\gamma_1 + \gamma_2 \neq 0$ and $\beta_1 + \beta_2 \neq 0$. Also $\mathrm{f}_1, \mathrm{f}_2 : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ are given continuous functions.

The study of Ulam stability in fractional differential equations introduces a novel approach for researchers, paving the way for exploring various topics in nonlinear analysis. Moreover, the stability analysis of fractional-order differential equations is more complex than that of classical differential equations, due to the nonlocal nature and weakly singular kernels of fractional derivatives. Consequently, Ulam-type stability issues have attracted considerable interest from numerous researchers, as evidenced in [14,20,21,29,34,37].

Taking motivation from previous mentioned works, this paper introduces nonlinear hybrid Langevin fractional pantograph equations. This study investigates the existence, uniqueness and Ulam-Hyers stability of solutions for ψ-Caputo type to the following problem:

$$\begin{cases} {}^{C}D^{\alpha_1,\psi}_{a_1^+}\left({}^{C}D^{\beta_1,\psi}_{a_1^+}\left(\dfrac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau,\mathrm{x}(\tau))}\right) + \lambda\mathrm{x}(\tau)\right) = \mathrm{f}_1(\tau, \mathrm{x}(\tau), \mathrm{y}(\xi\tau)), \quad \tau \in J = [0,a], \\ \\ {}^{C}D^{\alpha_2,\psi}_{0^+}\left({}^{C}D^{\beta_2,\psi}_{0^+}\left(\dfrac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau,\mathrm{y}(\tau))}\right) + \nu\mathrm{y}(\tau)\right) = \mathrm{f}_2(\tau, \mathrm{y}(\tau), \mathrm{x}(\rho\tau)), \quad \tau \in J = [0,a], \\ \\ \mathrm{x}(\tau)\mid_{\tau=0} = 0,\ \mathrm{x}'(\tau)\mid_{\tau=0} = 0, \mathrm{x}(\tau)\mid_{\tau=\epsilon_1} = 0,\ 0 < \epsilon_1 < a, \\ \mathrm{y}(\tau)\mid_{\tau=0} = 0,\ \mathrm{y}'(\tau)\mid_{\tau=0} = 0, \mathrm{y}(\tau)\mid_{\tau=\epsilon_2} = 0,\ 0 < \epsilon_2 < a, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha_i,\psi}_{0^+}$ and ${}^{C}D^{\beta_i,\psi}_{0^+}$ are the ψ-Caputo fractional derivatives of order $\alpha_1, \alpha_2 \in (0,1]$, $\beta_1, \beta_2 \in (1,2]$, $\lambda, \nu \in \mathbb{R} \setminus \{0\}$ and $0 < \xi, \rho < 1$. The given functions $(\mathrm{f}_i)_{i=1,2} : J \times \mathbb{R}^2 \to \mathbb{R}$ and $(\mathrm{g}_i)_{i=1,2} : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous.

This research stands out for its novelty, as it integrates and generalizes various types of fractional derivatives for multiple values of the function ψ as follows:

1. $\psi(\tau) = \tau$, then the coupled system (1.1) reduce to the Caputo-type system.
2. $\psi(\tau) = \ln \tau$, then the coupled system (1.1) reduce to the Caputo-Hadamard-type system.
3. $\psi(\tau) = \tau^{\sigma}$, then the coupled system (1.1) reduce to the Caputo-Erdélyi-Kober-type system.
4. $\psi(\tau) = \frac{\tau^{\sigma}}{\sigma}$, $\sigma > 0$ coupled system (1.1) reduce to the Caputo-Katugampola-type system.

Thus, we can consider our results about systems with generalized fractional operator as the natural continuation of previous results in the literature.

The structure of our paper is as follows: in section 2, we introduce some important notions and definitions. Section 3 establishes the uniqueness result using Banach's fixed-point theorem, and proves the existence results by applying Dhage's fixed-point theorem for the ψ-Caputo fractional coupled system (1.1). Section 4 covers stability in the sense of Ulam-Hyers and Ulam-Hyers-Rassias, along with its generalizations. Finally, in the concluding part, we provide an example to illustrate the main findings of our study.

2. Preliminaries

In this section, we recall some definitions and basic results about fractional calculus.

We denote By $C(J,\mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|\eta\|_\infty = \sup\limits_{\tau\in J}|\eta(\tau)|$ and let $C^n(J,\mathbb{R})$ represent the space of functions that are n-times continuously differentiable on J.
Now, we consider the following Banach space

$$\Xi := \{(\mathrm{x},\mathrm{y}) :\ \mathrm{x},\mathrm{y}\in C(J,\mathbb{R})\},$$

endowed with the norm

$$\|(\mathrm{x},\mathrm{y})\|_\Xi = \max\{\|\mathrm{x}\|_\infty, \|\mathrm{y}\|_\infty\}.$$

Let $\psi\in C^1(J,\mathbb{R})$ be an increasing differentiable function such that $\psi(\tau)\neq 0$, for all $\tau\in J$.

Definition 2.1 ([7]). For $\alpha > 0$, the ψ-Riemann-Liouville fractional integral of order α for an integrable function $\eta : J\mapsto\mathbb{R}$ is given by :

$$I_{0^+}^{\alpha,\psi}\eta(\tau) = \int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\eta(s)ds, \tag{2.1}$$

where Γ is the classical Euler Gamma function.

Definition 2.2 ([7]). Let $n-1<\alpha<n$, $\eta : J\mapsto\mathbb{R}$ be an integrable function and ψ is defined as in Definition 2.1. The ψ-Riemann Liouville fractional derivative of order α of a function η is defined by

$$\begin{aligned} D_{0^+}^{\alpha,\psi}\eta(\tau) &= \left[\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right]^n I_{0^+}^{n-\alpha,\psi}\eta(\tau) \\ &= \left[\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right]^n \int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\eta(s)ds, \end{aligned} \tag{2.2}$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3 ([7]). Assume that $\eta\in C^n(J,\mathbb{R})$ and ψ be defined as in Definition 2.1. ψ-Caputo fractional derivative of a function η of order $\alpha\in(n-1,n)$, is determined as

$${}^C D_{0^+}^{\alpha,\psi}\eta(\tau) = I_{0^+}^{n-\alpha,\psi}\eta_\psi^{[n]}(\tau),$$

where $\eta_\psi^{[n]}(\tau) = \left[\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right]^n\eta(\tau)$ and $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$, $n=\alpha$ for $\alpha\in\mathbb{N}$. By the definition, we have

$${}^C D_{0^+}^{\alpha,\psi}\eta(\tau) = \begin{cases} \int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\eta_\psi^{[n]}(s)ds, & n\notin\mathbb{N}, \\ \eta_\psi^{[n]}(\tau), & n\in\mathbb{N}, \end{cases}$$

where $\mathbb{N}$ denotes the set of positive integers.

Lemma 2.4 ([7]). *For $\alpha > 0$, we obtain*

I) ${}^{C}D_{0^+}^{\alpha,\psi}\, I_{0^+}^{\alpha,\psi}\eta(\tau) = \eta(\tau)$ *for all functions* $\eta \in C(J,\mathbb{R})$.

II) *If* $\eta \in C^n(J,\mathbb{R})$, *then* $I_{0^+}^{\alpha,\psi}\, {}^{C}D_{0^+}^{\alpha,\psi}\eta(\tau) = \eta(\tau) - \sum_{k=0}^{n-1} \frac{\eta_\psi^{[k]}(0)}{k!}[\psi(\tau)-\psi(0)]^k$.

Lemma 2.5 ([28]). *Consider the functions $\eta,\ \psi \in C(J,\mathbb{R})$ and $\alpha > 0$, we have*

I) $I_{0^+}^{\alpha,\psi}(.)$ *is linear and bounded form* $C(J,\mathbb{R})$ *to* $C(J,\mathbb{R})$.

II) $I_{0^+}^{\alpha,\psi}\eta\,(0) = \lim\limits_{\tau\to 0^+} I_{0^+}^{\alpha,\psi}\eta(\tau) = 0$.

Lemma 2.6 ([12,36]). *For $\alpha,\ \beta > 0$ and $\eta \in C(J,\mathbb{R})$. Then for each $\tau \in J$, we have*

(C1) $I_{0^+}^{\alpha,\psi}[\psi(\tau)-\psi(0)]^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}[\psi(\tau)-\psi(0)]^{\alpha+\beta-1}$,

(C2) *if* $\beta > n \in \mathbb{N}$, *then* ${}^{C}D_{0^+}^{\alpha,\psi}[\psi(\tau)-\psi(0)]^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}[\psi(\tau)-\psi(0)]^{\beta-\alpha-1}$,

(C3) $\forall k \in \{0,1,\cdots,n-1\},\ n$ *is a positive integer, then* ${}^{C}D_{0^+}^{\alpha,\psi}[\psi(\tau)-\psi(0)]^k = 0$,

(C4) $I_{0^+}^{\alpha,\psi}\, I_{0^+}^{\beta,\psi}\eta(\tau) = I_{0^+}^{\alpha+\beta,\psi}\eta(\tau)$,

(C5) *for any constant* ρ, *we always have* ${}^{C}D_{0^+}^{\alpha,\psi}\rho = 0$.

Theorem 2.7 (Banach's fixed point theorem [33]). *Let $\mathbb{E}$ be a Banach space and $D : \mathbb{E} \to \mathbb{E}$ a contraction, i.e. there exists $\Lambda \in [0,1)$ such that*

$$\|\mathrm{D}(x_1) - \mathrm{D}(x_2)\| \le \Lambda \|x_1 - x_2\|,$$

for all $x_1, x_2 \in \mathbb{R}$. Then D has a unique fixed point.

Theorem 2.8 (Dhage fixed point theorem [16]). *Suppose that Σ is a non-empty subset of $C(J,\mathbb{R})$, which closed convex and bounded, $\mathrm{V} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$, and $\mathrm{U} : \Sigma \to C(J,\mathbb{R})$ are two operators satisfying the following conditions:*

D1) V *is Lipschitizian with a constant* C^*,

D2) U *is completely continuous,*

D3) $\eta = \mathrm{V}\eta\mathrm{U}\mu \Rightarrow \eta \in \Sigma, \forall \mu \in \Sigma$, *and*

D4) $\mathbb{A}_{\mathrm{V}}\mathbb{B}_{\mathrm{U}} < 1$, *where* $\mathbb{B}_{\mathrm{U}} = \|\mathbb{B}_{\mathrm{U}}(\Sigma)\| = \sup\{\|\mathrm{U}(\eta)\| : \eta \in \Sigma\}$.

Then the operator equation $\eta = \mathrm{V}\eta\mathrm{U}\eta$ has a solution.

3. Main Results

3.1. Uniqueness and Existence of solutions.

Definition 3.1. A solution to the fractional boundary value problem (3.1), is defined as a function $\eta \in C(J,\mathbb{R})$ that fulfills the equation (3.1) on J along with the specified boundary conditions.

Lemma 3.2. *Let $\alpha \in (0,1]$, $\beta \in (1,2]$ and $(\mathrm{K};\mathrm{H}) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}\setminus\{0\})$. Then, the fractional boundary value problem*

$$(3.1)\qquad \begin{cases} {}^{C}D_{0^+}^{\alpha,\psi}\left[{}^{C}D_{0^+}^{\beta,\psi}\left[\dfrac{\eta(\tau)}{\mathrm{H}(\tau)}\right] + \lambda\eta(\tau)\right] = \mathrm{K}(\tau), \quad \tau \in J = [0,a], \\ \\ \eta(\tau)\mid_{\tau=0} = 0,\ \eta'(\tau)\mid_{\tau=0} = 0, \\ \eta(\tau)\mid_{\tau=\epsilon} = 0,\ 0 < \epsilon < a, \end{cases}$$

has a unique solution defined by

$$\eta(\tau) := \mathrm{H}(\tau)\left[\int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\mathrm{K}(s)ds\right.$$

$$
\begin{aligned}
& - \lambda \int_0^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta-1}}{\Gamma(\beta)} \eta(s) ds \\
& + \frac{\lambda(\psi(\tau)-\psi(0))^{\beta}}{(\psi(\epsilon)-\psi(0))^{\beta}} \int_0^{\epsilon} \frac{\psi'(s)(\psi(\epsilon)-\psi(s))^{\beta-1}}{\Gamma(\beta)} \eta(s) ds \\
& - \frac{(\psi(\tau)-\psi(0))^{\beta}}{(\psi(\epsilon)-\psi(0))^{\beta}} \int_0^{\epsilon} \frac{\psi'(s)(\psi(\epsilon)-\psi(s))^{\alpha+\beta_k-1}}{\Gamma(\alpha+\beta)} \mathrm{K}(s) ds \Bigg]. \qquad (3.2)
\end{aligned}
$$

Proof. Let $\eta \in C(J, \mathbb{R})$ be a solution of the problem (3.1), then by using Lemma 2.4, we have

$$
\eta(\tau) := \mathrm{H}(\tau) \left[I_{0^+}^{\alpha+\beta,\psi} \mathrm{K}(\tau) + c_0 \frac{[\psi(\tau)-\psi(0)]^{\beta}}{\Gamma(\beta+1)} - \lambda I_{0^+}^{\beta,\psi} \eta(\tau) + c_1 + c_2[\psi(\tau)-\psi(0)] \right], \qquad (3.3)
$$

where $c_j \in \mathbb{R}$, with $j = 0, 1, 2$. By the condition $\eta(\tau) \mid_{\tau=0} = 0$ and $\eta'(\tau) \mid_{\tau=0} = 0$, we obtain $c_1 := 0$ and $c_2 := 0$.

On the other hand by $\eta(\tau) \mid_{\tau=\epsilon} = 0$, we have

$$
c_0 := \frac{\Gamma(\beta+1)}{(\psi(\epsilon)-\psi(0))^{\beta}} \left(\lambda I_{0^+}^{\beta,\psi} \eta(\epsilon) - I_{0^+}^{\alpha+\beta,\psi} \mathrm{K}(\epsilon) \right). \qquad (3.4)
$$

Finally, replacing these constants into (3.3), we get (3.2).

Conversely, let us now demonstrate that if (3.2) satisfies Eq (3.1), then the aforementioned equation can be expressed as

$$
\frac{\eta(\tau)}{\mathrm{H}(\tau)} := I_{0^+}^{\alpha+\beta,\psi} \mathrm{K}(\tau) + \frac{[\psi(\tau)-\psi(0)]^{\beta}}{(\psi(\epsilon)-\psi(0))^{\beta}} \left(\lambda I_{0^+}^{\beta,\psi} \eta(\epsilon) - I_{0^+}^{\alpha+\beta,\psi} \mathrm{K}(\epsilon) \right) - \lambda I_{0^+}^{\beta,\psi} \eta(\tau).
$$

Applying the ψ-Caputo derivative, ${}^{C}D_{0^+}^{\beta,\psi}$ on both sides and utilizing Lemma 2.4, we obtain

$$
{}^{C}D_{0^+}^{\beta,\psi} \left(\frac{\eta(\tau)}{\mathrm{H}(\tau)} \right) := I_{0^+}^{\alpha,\psi} \mathrm{K}(\tau) + \frac{\Gamma(\beta+1)}{(\psi(\epsilon)-\psi(0))^{\beta}} \left(\lambda I_{0^+}^{\beta,\psi} \eta(\epsilon) - I_{0^+}^{\alpha+\beta,\psi} \mathrm{K}(\epsilon) \right) - \lambda \eta(\tau).
$$

Reapplying, ${}^{C}D_{0^+}^{\alpha,\psi}$ to the above equation, we obtain

$$
{}^{C}D_{0^+}^{\alpha,\psi} \left[{}^{C}D_{0^+}^{\beta,\psi} \left[\frac{\eta(\tau)}{\mathrm{H}(\tau)} \right] + \lambda \eta(\tau) \right] = \mathrm{K}(\tau).
$$

Lastly, it is clear that the function in (3.2) meets the associated boundary conditions. This completes the proof. □

Now, we define the operator $\mathrm{T} : \Xi \longrightarrow \Xi$ for $\tau \in J$, as follows

$$
\mathrm{T}(\mathrm{x}, \mathrm{y})(\tau) := (\mathrm{T}_1(\mathrm{x}, \mathrm{y})(\tau), \mathrm{T}_2(\mathrm{x}, \mathrm{y})(\tau)),
$$

such as

$$
\mathrm{T}_1(\mathrm{x}, \mathrm{y})(\tau) := \mathrm{g}_1(\tau, \mathrm{x}(\tau)) \left[\Phi(\mathrm{x}, \mathrm{y})(\tau) - \frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}} \Phi(\mathrm{x}, \mathrm{y})(\epsilon_1) \right],
$$

and

$$
\mathrm{T}_2(\mathrm{x}, \mathrm{y})(\tau) := \mathrm{g}_2(\tau, \mathrm{y}(\tau)) \left[\Psi(\mathrm{x}, \mathrm{y})(\tau) - \frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}} \Psi(\mathrm{x}, \mathrm{y})(\epsilon_2) \right],
$$

where

$$
\begin{aligned}
\Phi(\mathrm{x}, \mathrm{y})(\tau) := & \int_0^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} \mathrm{f}_1(s, \mathrm{x}(s), \mathrm{y}(\xi s)) ds \\
& - \lambda \int_0^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)} \mathrm{x}(s) ds,
\end{aligned}
$$

and

$$\Psi(\mathrm{x},\mathrm{y})(\tau) \quad := \quad \int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)}\mathrm{f}_2(s,\mathrm{y}(s),\mathrm{x}(\rho s))ds$$
$$-\nu\int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)}\mathrm{y}(s)ds.$$

The first result is achieved by a theorem derived from Dhages fixed-point theorem [16]. The following hypotheses will be used in the sequel.

Theorem 3.3. *Assume that the following hypotheses hold .*

(cx1) *The functions* $(\mathrm{f}_k)_{k=1,2} : J\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ *are continuous on* J.

(cx2) *The functions* $(\mathrm{g}_k)_{k=1,2} : J\times\mathbb{R} \longrightarrow \mathbb{R}\setminus\{0\}$ *are continuous and there exist functions* $\mathrm{G}_k \in C(J,[0,\infty))$ *that*

$$|\mathrm{g}_k(\tau,\mathrm{x}_1)-\mathrm{g}_k(\tau,\mathrm{y}_1)| \le \mathrm{G}_k(\tau)|\mathrm{x}_1-\mathrm{y}_1|,$$

for any $\mathrm{x}_1,\mathrm{y}_1\in\mathbb{R}$ *and* $\tau\in J$, *with* $k=1,2$.

(cx3) *There exist functions* $\mathrm{P}_k^1,\mathrm{P}_k^2,\mathrm{P}_k^3 \in C(J,\mathbb{R}^+)$ *such that*

$$|\mathrm{f}_k(\tau,\mathrm{x}_1,\mathrm{x}_2)| < \mathrm{P}_k^1(\tau)+\mathrm{P}_k^2(\tau)|\mathrm{x}_1|+\mathrm{P}_k^3(\tau)|\mathrm{x}_2|,$$

for all $\tau\in J$ *and* $(\mathrm{x}_1,\mathrm{x}_2)\in\mathbb{R}^2$, *with* $k=1,2$.

(cx4) *There exists* $\kappa>0$ *such that*

$$\frac{\max\{\mathrm{g}_1^0\Pi_1^\kappa,\mathrm{g}_2^0\Pi_2^\kappa\}}{1-\max\{\|G_1\|_\infty\Pi_1^\kappa,\|G_2\|_\infty\Pi_2^\kappa\}} \le \kappa,$$

and

$$\max\{\|G_1\|_\infty\Pi_1^\kappa,\|G_2\|_\infty\Pi_2^\kappa\} < 1, \tag{3.5}$$

where

$$\Pi_1^\kappa = \left(\frac{2(\psi(a)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\right)\|\mathrm{P}_1^1\|_\infty$$
$$+\left[\left(\frac{2(\psi(a)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\right)(\|\mathrm{P}_1^2\|_\infty+\|\mathrm{P}_1^3\|_\infty)\right.$$
$$\left.+\frac{2|\lambda|(\psi(a)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right]\kappa,$$
$$\Pi_2^\kappa = \left(\frac{2(\psi(a)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\right)\|\mathrm{P}_2^1\|_\infty$$
$$+\left[\left(\frac{2(\psi(a)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\right)(\|\mathrm{P}_2^2\|_\infty+\|\mathrm{P}_2^3\|_\infty)\right.$$
$$\left.+\frac{2|\nu|(\psi(a)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}\right]\kappa,$$

$$\begin{cases}\|G_1\|_\infty = \sup_{\tau\in J}\mathrm{G}_1(\tau),\ \mathrm{g}_1^0=\sup_{\tau\in J}|\mathrm{g}_1(\tau,0)|,\ \|\mathrm{P}_1^i\|_\infty=\sup_{\tau\in J}\mathrm{P}_1^i(\tau),\ i=1,2,3.\\ \|G_2\|_\infty = \sup_{\tau\in J}\mathrm{G}_2(\tau),\ \mathrm{g}_2^0=\sup_{\tau\in J}|\mathrm{g}_2(\tau,0)|,\ \|\mathrm{P}_2^i\|_\infty=\sup_{\tau\in J}\mathrm{P}_2^i(\tau),\ i=1,2,3.\end{cases}$$

Then the coupled system (1.1) has at least one solution on J.

Proof. Define the set

$$\Sigma = \{(\mathrm{x},\mathrm{y}) \in \Xi : \ \|(\mathrm{x},\mathrm{y})\|_{\Xi} \leq \kappa\}.$$

Next, to convert the coupled system (1.1) into the framework of a system of operator equations as

$$(\mathrm{x},\mathrm{y})(\tau) = (\mathrm{V}_1(\mathrm{x},\mathrm{y})(\tau)\mathrm{U}_1(\mathrm{x},\mathrm{y})(\tau), \mathrm{V}_2(\mathrm{x},\mathrm{y})(\tau)\mathrm{U}_2(\mathrm{x},\mathrm{y})(\tau)),$$

we defined the operators $\mathrm{V} = (\mathrm{V}_1, \mathrm{V}_2) : [C(J,\mathbb{R})]^2 \to C(J,\mathbb{R})$ and $\mathrm{U} = (\mathrm{U}_1, \mathrm{U}_2) : \Sigma \to C(J,\mathbb{R})$ as follows

$$\begin{aligned} \mathrm{V}_1(\mathrm{x},\mathrm{y})(\tau) &= \mathrm{g}(\tau, \mathrm{x}(\tau)), \\ \mathrm{V}_2(\mathrm{x},\mathrm{y})(\tau) &= \mathrm{g}(\tau, \mathrm{y}(\tau)), \end{aligned}$$

and

$$\begin{aligned} \mathrm{U}_1(\mathrm{x},\mathrm{y})(\tau) &= \Phi(\mathrm{x},\mathrm{y})(\tau) - \frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_{11}}}\Phi(\mathrm{x},\mathrm{y})(\epsilon_1), \\ \mathrm{U}_2(\mathrm{x},\mathrm{y})(\tau) &= \Psi(\mathrm{x},\mathrm{y})(\tau) - \frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}}\Psi(\mathrm{x},\mathrm{y})(\epsilon_2), \end{aligned}$$

for $\tau \in J$.

In the following steps, we show that the operators V and U satisfy all the conditions of Theorem 2.8.

Step 1: Firstly, we show that $\mathrm{V} = (\mathrm{V}_1, \mathrm{V}_2)$ is Lipschitzian on $[C(J,\mathbb{R})]^2$. Let $(\mathrm{x},\mathrm{y}), (\bar{\mathrm{x}},\bar{\mathrm{y}}) \in [C(J,\mathbb{R})]^2$. Then by $(C2)$ we have

$$\begin{aligned} |\mathrm{V}_1(\mathrm{x},\mathrm{y})(\tau) - \mathrm{V}_1(\bar{\mathrm{x}},\bar{\mathrm{y}})(\tau)| &= |\mathrm{g}_1(\tau,\mathrm{x}(\tau)) - \mathrm{g}_1(\tau,\bar{\mathrm{x}}(\tau)| \\ &\leq \|G_1\|_\infty \|\mathrm{x}-\bar{\mathrm{x}}\|_\infty, \end{aligned}$$

then,

$$\|\mathrm{V}_1(\mathrm{x},\mathrm{y}) - \mathrm{V}_1(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\infty \leq \|G_1\|_\infty \|\mathrm{x}-\bar{\mathrm{x}}\|_\infty. \tag{3.6}$$

As before, we have

$$\|\mathrm{V}_2(\mathrm{x},\mathrm{y}) - \mathrm{V}_2(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\infty \leq \|G_2\|_\infty \|\mathrm{y}-\bar{\mathrm{y}}\|_\infty, \tag{3.7}$$

for all $(\mathrm{x},\mathrm{y}), (\bar{\mathrm{x}},\bar{\mathrm{y}}) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$. Consequently, by the definition of operator V we obtain

$$\begin{aligned} \|\mathrm{V}(\mathrm{x},\mathrm{y}) - \mathrm{V}(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi &= \|(\mathrm{V}_1(\mathrm{x},\mathrm{y}), \mathrm{V}_2(\mathrm{x},\mathrm{y})) - (\mathrm{V}_1(\bar{\mathrm{x}},\bar{\mathrm{y}}), \mathrm{V}_2(\bar{\mathrm{x}},\bar{\mathrm{y}}))\|_\Xi \\ &= \|(\mathrm{V}_1(\mathrm{x},\mathrm{y}) - \mathrm{V}_1(\bar{\mathrm{x}},\bar{\mathrm{y}})), (\mathrm{V}_2(\mathrm{x},\mathrm{y}) - \mathrm{V}_2(\bar{\mathrm{x}},\bar{\mathrm{y}}))\|_\Xi \\ &\leq \max\{\|G_1\|_\infty, \|G_2\|_\infty\} \|\mathrm{x}-\bar{\mathrm{x}}, \mathrm{y}-\bar{\mathrm{y}}\|_\Xi, \end{aligned}$$

Hence, $\mathrm{V} = (\mathrm{V}_1, \mathrm{V}_2)$ is Lipschitzian on $C(J,\mathbb{R}) \times C(J,\mathbb{R})$ with a Lipschitz constant $G^* = \max\{\|G_1\|_\infty, \|G_2\|_\infty\}$.

Step 2: We demonstrate that the operator $\mathrm{U} = (\mathrm{U}_1, \mathrm{U}_2)$ is completely continuous on Σ. To achieve this, we first establish that the operator $\mathrm{U} = (\mathrm{U}_1, \mathrm{U}_2)$ is continuous on $C(J,\mathbb{R})$. Let $\{\mathrm{x}_n, \mathrm{y}_n\}_{n\in\mathbb{N}}$ be a sequence in Σ that converges to a point $(\mathrm{x},\mathrm{y}) \in \Delta$. Applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} &\lim_{n\to+\infty} \mathrm{U}_1(\mathrm{x}_n,\mathrm{y}_n)(\tau) \\ &= \lim_{n\to+\infty} \left\{ \Phi(\mathrm{x}_n,\mathrm{y}_n)(\tau) - \frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}\Phi(\mathrm{x}_n,\mathrm{y}_n)(\epsilon_1) \right\} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to+\infty}\Bigg\{\int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}\mathrm{f}_1(s,\mathrm{x}_n(s),\mathrm{y}_n(\xi s))ds\\
&\quad -\lambda\int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}\mathrm{x}_n(s)ds\\
&\quad -\frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}\left(\int_0^{\epsilon_1} \frac{\psi'(s)(\psi(\epsilon_1)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}\mathrm{f}_1(s,\mathrm{x}_n(s),\mathrm{y}_n(\xi s))ds\right.\\
&\quad \left.-\lambda\int_0^{\epsilon_1} \frac{\psi'(s)(\psi(\epsilon_1)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}\mathrm{x}_n(s)ds\right)\Bigg\}\\
&= \mathrm{U}_1(\mathrm{x},\mathrm{y})(\tau). \qquad (3.8)
\end{aligned}
$$

Hence,

$$\|\mathrm{U}_1(\mathrm{x}_n,\mathrm{y}_n)-\mathrm{U}_1(\mathrm{x},\mathrm{y})\|_\infty \longrightarrow 0,$$

for all $\tau\in J$. Similarly, we also have

$$
\begin{aligned}
&\lim_{n\to+\infty}\mathrm{U}_2(\mathrm{x}_n,\mathrm{y}_n)(\tau)\\
&= \lim_{n\to+\infty}\left\{\Psi(\mathrm{x}_n,\mathrm{y}_n)(\tau)-\frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}}\Psi(\mathrm{x}_n,\mathrm{y}_n)(\epsilon_2)\right\}\\
&= \lim_{n\to+\infty}\Bigg\{\int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)}\mathrm{f}_2(s,\mathrm{y}_n(s),\mathrm{x}_n(\rho s))ds\\
&\quad -\nu\int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)}\mathrm{y}_n(s)ds\\
&\quad -\frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}}\left(\int_0^{\epsilon_2} \frac{\psi'(s)(\psi(\epsilon_2)-\psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)}\mathrm{f}_1(s,\mathrm{y}_n(s),\mathrm{x}_n(\rho s))ds\right.\\
&\quad \left.-\nu\int_0^{\epsilon_2} \frac{\psi'(s)(\psi(\epsilon_2)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)}\mathrm{y}_n(s)ds\right)\Bigg\}\\
&= \mathrm{U}_2(\mathrm{x},\mathrm{y})(\tau). \qquad (3.9)
\end{aligned}
$$

Hence,

$$\|\mathrm{U}_2(\mathrm{x}_n,\mathrm{y}_n)-\mathrm{U}_2(\mathrm{x},\mathrm{y})\|_\infty \longrightarrow 0,$$

Using (3.8)-(3.9), we deduce that

$$\|\mathrm{U}(\mathrm{x}_n,\mathrm{y}_n)-\mathrm{U}(\mathrm{x},\mathrm{y})\|_\Xi \longrightarrow 0,\ n\to 0.$$

Therefore, this shows that $\mathrm{U}(\Sigma)$ is a continuous operator .

Next, we prove that the set $\mathrm{U}(\Sigma)=(\mathrm{U}_1,\mathrm{U}_2)(\Sigma)$ is a uniformly bounded in Σ. For any $(\mathrm{x},\mathrm{y})\in\Sigma$ and $\tau\in J$, we have

$$
\begin{aligned}
|\Phi(\mathrm{x},\mathrm{y})(\tau)| \le &\frac{(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}\left(\|\mathrm{P}_1^1\|_\infty+(\|\mathrm{P}_1^2\|_\infty+\|\mathrm{P}_1^3\|_\infty)\kappa\right)}{\Gamma(\alpha_1+\beta_1+1)}\\
&+\frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}\kappa}{\Gamma(\beta_1+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
|\Psi(\mathrm{x},\mathrm{y})(\tau)| \le &\frac{(\psi(\tau)-\psi(0))^{\alpha_2+\beta_2}\left(\|\mathrm{P}_2^1\|_\infty+(\|\mathrm{P}_2^2\|_\infty+\|\mathrm{P}_2^3\|_\infty)\kappa\right)}{\Gamma(\alpha_2+\beta_2+1)}\\
&+\frac{|\nu|(\psi(\tau)-\psi(0))^{\beta_2}\kappa}{\Gamma(\beta_2+1)},
\end{aligned}
$$

which implies

$$
\begin{aligned}
|\mathrm{U}_1(\mathrm{x},\mathrm{y})(\tau) &\leq |\Phi(\mathrm{x},\mathrm{y})(\tau)| + \frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}|\Phi(\mathrm{x},\mathrm{y})(\epsilon_1)| \\
&\leq \frac{(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}\left(\|\mathrm{P}_1^1\|_\infty + (\|\mathrm{P}_1^2\|_\infty + \|\mathrm{P}_1^3\|_\infty)\kappa\right)}{\Gamma(\alpha_1+\beta_1+1)} \\
&\quad + \frac{2|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}\kappa}{\Gamma(\beta_1+1)} \\
&\quad + \frac{(\psi(\tau)-\psi(0))^{\beta_1}(\psi(\epsilon_1)-\psi(0))^{\alpha_1}\left(\|\mathrm{P}_1^1\|_\infty + (\|\mathrm{P}_1^2\|_\infty + \|\mathrm{P}_1^3\|_\infty)\kappa\right)}{\Gamma(\alpha_1+\beta_1+1)} \\
&\leq \left(\frac{2(\psi(a)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\right)\|\mathrm{P}_1^1\|_\infty \\
&\quad + \left[\left(\frac{2(\psi(a)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\right)(\|\mathrm{P}_1^2\|_\infty + \|\mathrm{P}_1^3\|_\infty)\right. \\
&\quad \left. + \frac{2|\lambda|(\psi(a)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right]\kappa, \qquad (3.10)
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathrm{U}_2(\mathrm{x},\mathrm{y})(\tau)| &\leq |\Psi(\mathrm{x},\mathrm{y})(\tau)| + \frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}}|\Psi(\mathrm{x},\mathrm{y})(\epsilon_2)| \\
&\leq \left(\frac{2(\psi(a)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\right)\|\mathrm{P}_2^1\|_\infty \\
&\quad + \left[\left(\frac{2(\psi(a)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\right)(\|\mathrm{P}_2^2\|_\infty + \|\mathrm{P}_2^3\|_\infty)\right. \\
&\quad \left. + \frac{2|\nu|(\psi(a)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}\right]\kappa. \qquad (3.11)
\end{aligned}
$$

It follows from (3.10)-(3.11) that

$$\|\mathrm{U}(\mathrm{x},\mathrm{y})\|_\Xi \leq \max\{\Pi_1^\kappa, \Pi_2^\kappa\}.$$

Thus,

$$\|\mathrm{U}(\mathrm{x},\mathrm{y})\|_\Xi < \infty,$$

for all $(\mathrm{x},\mathrm{y}) \in \Sigma$. This shows that $\mathrm{U}(\Sigma)$ is uniformly bounded on Σ.

On the other hand, we demonstrate that $\mathrm{U}(\Sigma)$ is an equicontinuous set in Σ. Let $\tau_1, \tau_2 \in [0,a]$ with $\tau_1 < \tau_2$ and $(\mathrm{x},\mathrm{y}) \in \Sigma$. Then we have

$$
\begin{aligned}
&|\mathrm{U}_1(\mathrm{x},\mathrm{y})(\tau_2) - \mathrm{U}_1(\mathrm{x},\mathrm{y})(\tau_1)| \\
&\leq \int_0^{\tau_1} \frac{\psi'(s)[(\psi(\tau_2)-\psi(s))^{\alpha_1+\beta_1-1} - (\psi(\tau_1)-\psi(s))^{\alpha_1+\beta_1-1}]}{\Gamma(\alpha_1+\beta_1)}|\mathrm{f}_1(s,\mathrm{x}(s),\mathrm{y}(\xi s))|ds \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}|\mathrm{f}_1(s,\mathrm{x}(s),\mathrm{y}(\xi s))|ds \\
&\quad + \int_0^{\tau_1} \frac{|\lambda|\psi'(s)[(\psi(\tau_2)-\psi(s))^{\beta_1-1} - (\psi(\tau_1)-\psi(s))^{\beta_1-1}]}{\Gamma(\beta_1)}|\mathrm{x}(s)|ds \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{|\lambda|\psi'(s)(\psi(\tau_2)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}|\mathrm{x}(s)|ds \\
&\quad + \frac{[(\psi(\tau_2)-\psi(0))^{\beta_1} - (\psi(\tau_1)-\psi(0))^{\beta_1}]}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}
\end{aligned}
$$

$$\times\left(\int_0^{\epsilon_1}\frac{\psi'(s)(\psi(\epsilon_1)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}|\mathrm{f}_1(s,\mathrm{x}(s),\mathrm{y}(\xi s))|ds\right.$$
$$\left.+|\lambda|\int_0^{\epsilon_1}\frac{\psi'(s)(\psi(\epsilon_1)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}|\mathrm{x}(s)|ds\right).$$

Therefore,

$$\begin{aligned}
&|\mathrm{U}_1(\mathrm{x},\mathrm{y})(\tau_2)-\mathrm{U}_1(\mathrm{x},\mathrm{y})(\tau_1)|\\
&\leq|\Phi(\mathrm{x},\mathrm{y})(\tau_2)-\Phi(\mathrm{x},\mathrm{y})(\tau_1)|+\frac{[(\psi(\tau_2)-\psi(0))^{\beta_1}-(\psi(\tau_1)-\psi(0))^{\beta_1}]}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}|\Phi(\mathrm{x},\mathrm{y})(\epsilon_1)|\\
&\leq\frac{[(\psi(\tau_1)-\psi(0))^{\alpha_1+\beta_1}-(\psi(\tau_2)-\psi(0))^{\alpha_1+\beta_1}]}{\Gamma(\alpha_1+\beta_1+1)}[\|\mathrm{P}_1^1\|_\infty+(\|\mathrm{P}_1^2\|_\infty+\|\mathrm{P}_1^3\|_\infty)\kappa]\\
&\quad+\left(\frac{(\psi(\tau_2)-\psi(\tau_1))^{\beta_1}-(\psi(\tau_2)-\psi(0))^{\beta_1}+(\psi(\tau_1)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right)|\lambda|\kappa\\
&\quad+\frac{[(\psi(\tau_2)-\psi(0))^{\beta_1}-(\psi(\tau_1)-\psi(0))^{\beta_1}]}{2}\Pi_1^\kappa.
\end{aligned}\tag{3.12}$$

We also have

$$\begin{aligned}
&|\mathrm{U}_2(\mathrm{x},\mathrm{y})(\tau_2)-\mathrm{U}_2(\mathrm{x},\mathrm{y})(\tau_1)|\\
&\leq\int_0^{\tau_1}\frac{\psi'(s)[(\psi(\tau_2)-\psi(s))^{\alpha_2+\beta_2-1}-(\psi(\tau_1)-\psi(s))^{\alpha_2+\beta_2-1}]}{\Gamma(\alpha_2+\beta_2)}|\mathrm{f}_2(s,\mathrm{y}(s),\mathrm{x}(\rho s))|ds\\
&\quad+\int_{\tau_1}^{\tau_2}\frac{\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)}|\mathrm{f}_2(s,\mathrm{y}(s),\mathrm{x}(\rho s))|ds\\
&\quad+\int_0^{\tau_1}\frac{|\nu|\psi'(s)[(\psi(\tau_2)-\psi(s))^{\beta_2-1}-(\psi(\tau_1)-\psi(s))^{\beta_2-1}]}{\Gamma(\beta_2)}|\mathrm{y}(s)|ds\\
&\quad+\int_{\tau_1}^{\tau_2}\frac{|\nu|\psi'(s)(\psi(\tau_2)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)}|\mathrm{y}(s)|ds\\
&\quad+\frac{[(\psi(\tau_2)-\psi(0))^{\beta_2}-(\psi(\tau_1)-\psi(0))^{\beta_2}]}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}}\\
&\quad\times\left(\int_0^{\epsilon_2}\frac{\psi'(s)(\psi(\epsilon_2)-\psi(s))^{\alpha_1+\beta_2-1}}{\Gamma(\alpha_1+\beta_2)}|\mathrm{f}_2(s,\mathrm{y}(s),\mathrm{x}(\rho s))|ds\right.\\
&\quad\left.+|\nu|\int_0^{\epsilon_2}\frac{\psi'(s)(\psi(\epsilon_2)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)}|\mathrm{y}(s)|ds\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&|\mathrm{U}_2(\mathrm{x},\mathrm{y})(\tau_2)-\mathrm{U}_2(\mathrm{x},\mathrm{y})(\tau_1)|\\
&\leq\frac{[(\psi(\tau_1)-\psi(0))^{\alpha_2+\beta_2}-(\psi(\tau_2)-\psi(0))^{\alpha_2+\beta_2}]}{\Gamma(\alpha_2+\beta_2+1)}[\|\mathrm{P}_2^1\|_\infty+(\|\mathrm{P}_2^2\|_\infty+\|\mathrm{P}_2^3\|_\infty)\kappa]\\
&\quad+\left(\frac{(\psi(\tau_2)-\psi(\tau_1))^{\beta_2}-(\psi(\tau_2)-\psi(0))^{\beta_2}+(\psi(\tau_1)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}\right)|\lambda_2|\kappa\\
&\quad+\frac{[(\psi(\tau_2)-\psi(0))^{\beta_2}-(\psi(\tau_1)-\psi(0))^{\beta_2}]}{2}\Pi_2^\kappa.
\end{aligned}\tag{3.13}$$

Using (3.12)-(3.13), we deduce that

$$\|\mathrm{U}(\mathrm{x},\mathrm{y})(\tau_2)-\mathrm{U}(\mathrm{x},\mathrm{y})(\tau_1)\|_\Xi\longrightarrow_{\tau_1\to\tau_2}0,\ \text{uniformly for all } (\mathrm{x},\mathrm{y})\in\Delta.$$

Thus $\mathrm{U}(\Sigma)$ has the equicontinuity specification on the Banach space $C(J,\mathbb{R})$. As a consequence, U is relatively compact, and thus the Arzelà-Ascoli theorem yields that U is completely continuous and finally U is compact on Σ.

Step 3: We now show that the hypothesis $(D3)$ of Theorem 2.8 is satisfied. For $(\mathrm{x},\mathrm{y}),(\bar{\mathrm{x}},\bar{\mathrm{y}})\in\Sigma$ such that

$$(\mathrm{x},\mathrm{y})=\left(\mathrm{V}_1(\mathrm{x},\mathrm{y})\mathrm{U}_1(\bar{\mathrm{x}},\bar{\mathrm{y}}),\mathrm{V}_2(\mathrm{x},\mathrm{y})\mathrm{U}_2(\bar{\mathrm{x}},\bar{\mathrm{y}})\right).$$

Then

$$\begin{aligned}|\mathrm{x}(\tau)| &= |\mathrm{V}_1(\mathrm{x},\mathrm{y})\mathrm{U}_1(\bar{\mathrm{x}},\bar{\mathrm{y}})|\\ &\leq [|\mathrm{g}_1(\tau,\mathrm{x}(\tau))+\mathrm{g}_1(\tau,0)|+|\mathrm{g}_1(\tau,0)|]\Pi_1^{\kappa}\\ &\leq [\|G_1\|_\infty\|\mathrm{x}\|_\infty+\mathrm{g}_1^0]\Pi_1^{\kappa},\end{aligned}$$

and

$$\begin{aligned}|\mathrm{y}(\tau)| &= |\mathrm{V}_2(\mathrm{x},\mathrm{y})\mathrm{U}_2(\bar{\mathrm{x}},\bar{\mathrm{y}})|\\ &\leq [|\mathrm{g}_2(\tau,\mathrm{y}(\tau))+\mathrm{g}_2(\tau,0)|+|\mathrm{g}_2(\tau,0)|]\Pi_2^{\kappa}\\ &\leq [\|G_2\|_\infty\|\mathrm{y}\|_\infty+\mathrm{g}_2^0]\Pi_2^{\kappa},\end{aligned}$$

which implies that

$$\|(\mathrm{x},\mathrm{y})\|_\Xi\leq\frac{\max\{\mathrm{g}_1^0\Pi_1^{\kappa},\mathrm{g}_2^0\Pi_2^{\kappa}\}}{1-\max\{\|G_1\|_\infty\Pi_1^{\kappa},\|G_2\|_\infty\Pi_2^{\kappa}\}}=\kappa,$$

under the condition that (3.5) is fulfilled. This shows that condition $(D3)$ of Theorem 2.8 is satisfied.

step 4: Finally, we have

$$\begin{aligned}\mathbb{B}_{\mathrm{U}}=\|\mathbb{B}_{\mathrm{U}}(\Sigma)\|_\Xi &= \sup\left\{\|\mathrm{U}(\mathrm{x},\mathrm{y})\|_\Xi:(\mathrm{x},\mathrm{y})\in\Sigma\right\}\leq\Pi_\kappa\\ &\leq \max\{\Pi_1^{\kappa},\Pi_2^{\kappa}\},\end{aligned}$$

and

$$\mathbb{A}_{\mathrm{V}}\leq\max\{\|G_1\|_\infty,\|G_2\|_\infty\}.$$

Fromabove estimate, we obtain

$$\mathbb{A}_{\mathrm{V}}\mathbb{B}_{\mathrm{U}}\leq\max\{\|G_1\|_\infty\Pi_1^{\kappa},\|G_2\|_\infty\Pi_2^{\kappa}\}<1,$$

hence, all the conditions of Theorem 2.8 are satisfied, and therefore, the operator equation $(\mathrm{x},\mathrm{y})=\mathrm{V}(\mathrm{x},\mathrm{y})\mathrm{U}(\mathrm{x},\mathrm{y})$ has a solution in Σ. Consequently, coupled system (1.1) has a solution on J. □

The following result is based on Banach's fixed-point theorem. Moreover, to establish uniqueness, given the nature of our problem, we need to assume the following stronger conditions:

$(C1)$ The functions $(\mathrm{f}_k)_{k=1,2}:J\times\mathbb{R}^2\to\mathbb{R}$ and $(\mathrm{g}_k)_{k=1,2}:J\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ are continuous.

$(C2)$ There exist positive functions $\mathrm{P}_k,\mathrm{G}_k\in C(J,\mathbb{R}^+)$ such that

$$|\mathrm{f}_k(\tau,\mathrm{x}_1,\mathrm{x}_2)-\mathrm{f}_k(\tau,\mathrm{y}_1,\mathrm{y}_2)|\leq\mathrm{P}_{k,1}(\tau)|\mathrm{x}_1-\mathrm{y}_1|+\mathrm{P}_{k,2}(\tau)|\mathrm{x}_2-\mathrm{y}_2|,$$

and

$$|\mathrm{g}_k(\tau,\mathrm{x}_1)-\mathrm{g}_k(\tau,\mathrm{y}_1)|\leq\mathrm{G}_k(\tau)|\mathrm{x}_1-\mathrm{y}_1|,$$

for all $\tau\in J$ and $\mathrm{x}_k,\ \mathrm{y}_k\in\mathbb{R},\ k=1,2$, where $\|\mathrm{P}_{k,1}\|_\infty=\sup_{\tau\in J}\mathrm{P}_{k,1}(\tau)$, $\|\mathrm{P}_{k,2}\|_\infty=\sup_{\tau\in J}\mathrm{P}_{k,2}(\tau)$ and $\|G_k\|_\infty=\sup_{\tau\in J}\mathrm{G}_k(\tau)$, with $k=1,2$.

$(C3)$ There exist positive constants $\mathcal{L}_k$ and $\mathcal{M}_k$, such that

$$|\mathrm{f}_k(\tau,\mathrm{x}_1,\mathrm{x}_2)|<\mathcal{L}_k\text{ and }|\mathrm{g}_k(\tau,\mathrm{x}_1,\mathrm{x}_2)|<\mathcal{M}_k,$$

for all $\tau\in J$ and $\mathrm{x}_k\in\mathbb{R},\ k=1,2$.

For the sake of clarity, we denote

$$\begin{cases} \Delta_1 = \dfrac{[(\psi(a)-\psi(0))^{\alpha_1+\beta_1} + (\psi(a)-\psi(0))^{\beta_1}(\psi(\epsilon_1)-\psi(0))^{\alpha_1}]}{\Gamma(\alpha_1+\beta_1+1)}, \\ \Delta_2 = \dfrac{2|\lambda|(\psi(a)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}, \\ \Delta_3 = \dfrac{(\psi(a)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}, \\ \nabla_1 = \dfrac{[(\psi(a)-\psi(0))^{\alpha_2+\beta_2} + (\psi(a)-\psi(0))^{\beta_2}(\psi(\epsilon_2)-\psi(0))^{\alpha_2}]}{\Gamma(\alpha_2+\beta_2+1)}, \\ \nabla_2 = \frac{2|\nu|(\psi(a)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}, \\ \nabla_3 = \dfrac{(\psi(a)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}, \\ \Lambda_1 = (\mathrm{P}_{\mathrm{f}_1}M_1 + \|G_1\|_\infty\mathcal{L}_1)\Delta_3 + (M_1 + \|G_1\|_\infty\delta)\frac{\Delta_2}{2}, \\ \Lambda_2 = (\mathrm{P}_{\mathrm{f}_2}M_2 + \|G_2\|_\infty\mathcal{L}_2)\nabla_3 + (M_2 + \|G_2\|_\infty\delta)\frac{\nabla_2}{2}, \\ \Omega_1 = (\mathrm{P}_{\mathrm{f}_1}M_1 + \|G_1\|_\infty\mathcal{L}_1)\Delta_1 + (M_1 + \|G_1\|_\infty\delta)\Delta_2, \\ \Omega_2 = (\mathrm{P}_{\mathrm{f}_2}M_2 + \|G_2\|_\infty\mathcal{L}_2)\nabla_1 + (M_2 + \|G_2\|_\infty\delta)\nabla_2. \end{cases}$$

Theorem 3.4. *Suppose that $(C1)-(C3)$ holds. If*

$$\max\{\Omega_1, \Omega_2\} < 1, \tag{3.14}$$

then the coupled system (1.1) have a unique solution on $J = [0, a]$.

Proof. Setting $\delta \geq \max\left(\dfrac{\mathcal{M}_1\mathcal{L}_1\Delta_1}{1-\mathcal{M}_1\Delta_2}, \dfrac{\mathcal{M}_2\mathcal{L}_2\nabla_1}{1-\mathcal{M}_2\nabla_2}\right)$, with $0 \leq \mathcal{M}_1\Delta_2, \mathcal{M}_2\nabla_2 < 1$. we show that $\mathrm{T}\mathcal{B}_\delta \subset \mathcal{B}_\delta$, where

$$\mathcal{B}_\delta = \{(\mathrm{x},\mathrm{y}) \in \Xi : \|\mathrm{x},\mathrm{y}\|_\Xi \leq \delta\}.$$

For $\mathrm{x}, \mathrm{y} \in \mathcal{B}_\delta$ and for each $t \in J$, from the definition of T and hypothesis $(C1)$- $(C3)$, we get

$$|\Phi(\mathrm{x},\mathrm{y})(\tau)| \leq \frac{\mathcal{L}_1(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\delta, \tag{3.15}$$

and

$$|\Psi(\mathrm{x},\mathrm{y})(\tau)| \leq \frac{\mathcal{L}_2(\psi(\tau)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\nu|(\psi(\tau)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}\delta. \tag{3.16}$$

On the other hand, we obtain

$$\begin{aligned} &|\mathrm{T}_1(\mathrm{x},\mathrm{y})(\tau)| \\ &\leq \left|\mathrm{g}_1(\tau,\mathrm{x}(\tau))\left[\Phi(\mathrm{x},\mathrm{y})(\tau) - \frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}\Phi(\mathrm{x},\mathrm{y})(\epsilon_1)\right]\right| \\ &\leq |\mathrm{g}_1(\tau,\mathrm{x}(\tau))|\left[\frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}\right. \\ &\quad\times\left(\frac{\mathcal{L}_1(\psi(\epsilon_1)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|(\psi(\epsilon_1)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\delta\right) \\ &\quad\left.+\frac{\mathcal{L}_1(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\delta\right] \\ &\leq \mathcal{M}_1\left[\mathcal{L}_1\Delta_1 + \Delta_2\delta\right] \\ &\leq \delta. \end{aligned}$$

and

$$|\mathrm{T}_2(\mathrm{x},\mathrm{y})(\tau)|$$

$$
\begin{aligned}
&\leq \left| g_2(\tau, x(\tau)) \left[\Psi(x,y)(\tau) - \frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}} \Psi(x,y)(\epsilon_2) \right] \right| \\
&\leq |g_2(\tau, x(\tau))| \left[\frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}} \right. \\
&\quad \times \left(\frac{\mathcal{L}_2(\psi(\epsilon_2)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\lambda|(\psi(\epsilon_2)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)} \delta \right) \\
&\quad \left. + \frac{\mathcal{L}_2(\psi(\tau)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\nu|(\psi(\tau)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)} \delta \right] \\
&\leq \mathcal{M}_2 \left[\mathcal{L}_2 \nabla_1 + \nabla_2 \delta \right] \\
&\leq \delta.
\end{aligned}
$$

Hence,

$$
\|T(x,y)\|_{\Xi} \leq \delta,
$$

which implies that $T\mathcal{B}_\delta \subset \mathcal{B}_\delta$.

We shall now prove that T is contractive. Let for $(x,y), (\bar{x},\bar{y}) \in \mathcal{B}_\delta \subset \Xi$ and for any $\tau \in J$, by condition $(C2)$, we get

$$
\begin{aligned}
&|\Phi(x,y)(\tau) - \Phi(\bar{x},\bar{y})(\tau)| \\
&\leq \int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} |f_1(s,x(s),y(\xi s)) - f_1(s,\bar{x}(s),\bar{y}(\xi s))| ds \\
&\quad + |\lambda| \int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)} |x(s)-\bar{x}(s)| ds \\
&\leq \left(\frac{P_{f_1}(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)} \right) \|x-\bar{x}, y-\bar{y}\|_{\Xi},
\end{aligned} \tag{3.17}
$$

and

$$
\begin{aligned}
&|\Psi(x,y)(\tau) - \Psi(\bar{x},\bar{y})(\tau)| \\
&\leq \int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} |f_2(s,x(s),y(\rho s)) - f_2(s,\bar{y}(s),\bar{x}(\rho s))| ds \\
&\quad + |\nu| \int_0^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)} |y(s)-\bar{y}(s)| ds \\
&\leq \left(\frac{P_{f_2}(\psi(\tau)-\psi(0))^{\alpha_2+\beta_1}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\nu|(\psi(\tau)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)} \right) \|x-\bar{x}, y-\bar{y}\|_{\Xi},
\end{aligned} \tag{3.18}
$$

where $P_{f_1} = \|P_{1,1}\|_\infty + \|P_{1,2}\|_\infty$ and $P_{f_2} = \|P_{2,1}\|_\infty + \|P_{2,2}\|_\infty$. By applying the triangle inequality (3.17)-(3.15), we obtain

$$
\begin{aligned}
&|T_1(x,y)(\tau) - T_1(\bar{x},\bar{y})(\tau)| \\
&\quad \leq |g_1(\tau,x(\tau))\Phi(x,y)(\tau) - g_1(\tau,\bar{x}(\tau))\Phi(\bar{x},\bar{y})(\tau)| \\
&\quad\quad + \frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}} |g_1(\tau,x(\tau))\Phi(x,y)(\epsilon_1) - g_1(\tau,\bar{x}(\tau))\Phi(\bar{x},\bar{y})(\epsilon_1)|,
\end{aligned} \tag{3.19}
$$

and, by (3.18)-(3.16) we find that

$$
\begin{aligned}
&|T_2(x,y)(\tau) - T_2(\bar{x},\bar{y})(\tau)| \\
&\quad \leq |g_2(\tau,y(\tau))\Psi(x,y)(\tau) - g_2(\tau,\bar{y}(\tau))\Psi(\bar{x},\bar{y})(\tau)| \\
&\quad\quad + \frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}} |g_2(\tau,y(\tau))\Psi(x,y)(\epsilon_2) - g_1(\tau,\bar{y}(\tau))\Psi(\bar{x},\bar{y})(\epsilon_2)|,
\end{aligned} \tag{3.20}
$$

we conclude that

$$\begin{aligned}
&|\mathrm{T}_1(\mathrm{x},\mathrm{y})(\tau)-\mathrm{T}_1(\bar{\mathrm{x}},\bar{\mathrm{y}})(\tau)| \\
&\leq M_1\left(\frac{\mathrm{P}_{\mathrm{f}_1}(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right)\|\mathrm{x}-\bar{\mathrm{x}},\mathrm{y}-\bar{\mathrm{y}}\|_{\Xi} \\
&\quad+\|G_1\|_\infty\left(\frac{\mathcal{L}_1(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\delta\right)\|\mathrm{x}-\bar{\mathrm{x}},\mathrm{y}-\bar{\mathrm{y}}\|_{\Xi} \\
&\quad+M_1(\psi(\tau)-\psi(0))^{\beta_1}\left(\frac{\mathrm{P}_{\mathrm{f}_1}(\psi(\epsilon_1)-\psi(0))^{\alpha_1}}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|}{\Gamma(\beta_1+1)}\right)\|\mathrm{x}-\bar{\mathrm{x}},\mathrm{y}-\bar{\mathrm{y}}\|_{\Xi} \\
&\quad+\|G_1\|_\infty(\psi(\tau)-\psi(0))^{\beta_1}\left(\frac{\mathcal{L}_1(\psi(\epsilon_1)-\psi(0))^{\alpha_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda|}{\Gamma(\beta_1+1)}\delta\right)\|\mathrm{x}-\bar{\mathrm{x}},\mathrm{y}-\bar{\mathrm{y}}\|_{\Xi}.
\end{aligned}$$

After taking the supremum over J and simplifying, we get:

$$\|\mathrm{T}_1(\mathrm{x},\mathrm{y})-\mathrm{T}_1(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\infty \leq \Omega_1\|\mathrm{x}-\bar{\mathrm{x}},\mathrm{y}-\bar{\mathrm{y}}\|_{\Xi}. \tag{3.21}$$

With the same arguments as before, we can show that

$$\|\mathrm{T}_2(\mathrm{x},\mathrm{y})-\mathrm{T}_2(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\infty \leq \Omega_2\|\mathrm{x}-\bar{\mathrm{x}},\mathrm{y}-\bar{\mathrm{y}}\|_{\Xi}. \tag{3.22}$$

Thanks to (3.21)-(3.22), we get

$$\|\mathrm{T}(\mathrm{x},\mathrm{y})(\tau)-\mathrm{T}(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_{\Xi} \leq \max\{\Omega_1,\Omega_2\}\|\mathrm{x}-\bar{\mathrm{x}},\mathrm{y}-\bar{\mathrm{y}}\|_{\Xi}. \tag{3.23}$$

Consequently, by (3.14), T is a contraction, and by utilizing Banach's fixed point theorem, the coupled system (1.1) has a unique solution. □

4. Stability Results

This section discusses the Ulam stability of the coupled system (1.1). Let $(\mathrm{x},\mathrm{y})\in\Xi$, $\varrho_1,\varrho_2>0$, and $\mathfrak{J}_1,\mathfrak{J}_2:[0,a]\longrightarrow\mathbb{R}^+$ be continuous functions. We consider the following inequalities:

$$\begin{cases}
\left|{}^C D_{0^+}^{\alpha_1,\psi}\left({}^C D_{0^+}^{\beta_1,\psi}\left(\dfrac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau,\mathrm{x}(\tau))}\right)+\lambda\mathrm{x}(\tau)\right)-\mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau))\right|<\varrho_1, & \tau\in J,\\
\left|{}^C D_{0^+}^{\alpha_2,\psi}\left({}^C D_{0^+}^{\beta_2,\psi}\left(\dfrac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau,\mathrm{y}(\tau))}\right)+\nu\mathrm{y}(\tau)\right)-\mathrm{f}_2(\tau,\mathrm{y}(\tau),\mathrm{x}(\rho\tau))\right|<\varrho_2, & \tau\in J,
\end{cases} \tag{4.1}$$

$$\begin{cases}
\left|{}^C D_{0^+}^{\alpha_1,\psi}\left({}^C D_{0^+}^{\beta_1,\psi}\left(\dfrac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau,\mathrm{x}(\tau))}\right)+\lambda\mathrm{x}(\tau)\right)-\mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau))\right|<\varrho_1\mathfrak{J}_1(\tau), & \tau\in J,\\
\left|{}^C D_{0^+}^{\alpha_2,\psi}\left({}^C D_{0^+}^{\beta_2,\psi}\left(\dfrac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau,\mathrm{y}(\tau))}\right)+\nu\mathrm{y}(\tau)\right)-\mathrm{f}_2(\tau,\mathrm{y}(\tau),\mathrm{x}(\rho\tau))\right|<\varrho_2\mathfrak{J}_2(\tau), & \tau\in J,
\end{cases} \tag{4.2}$$

and

$$\begin{cases}
\left|{}^C D_{0^+}^{\alpha_1,\psi}\left({}^C D_{0^+}^{\beta_1,\psi}\left(\dfrac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau,\mathrm{x}(\tau))}\right)+\lambda\mathrm{x}(\tau)\right)-\mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau))\right|<\mathfrak{J}_1(\tau), & \tau\in J,\\
\left|{}^C D_{0^+}^{\alpha_2,\psi}\left({}^C D_{0^+}^{\beta_2,\psi}\left(\dfrac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau,\mathrm{y}(\tau))}\right)+\nu\mathrm{y}(\tau)\right)-\mathrm{f}_2(\tau,\mathrm{y}(\tau),\mathrm{x}(\rho\tau))\right|<\mathfrak{J}_2(\tau), & \tau\in J.
\end{cases} \tag{4.3}$$

Definition 4.1 ([32,34]). The coupled system (1.1) is Ulam-Hyers stable if there exists a real number $\varphi^*=\max\{\varphi_1,\varphi_2\}>0$ such that for each $\varrho^*=\max\{\varrho_1,\varrho_2\}>0$ and for each solution $(\mathrm{x},\mathrm{y})\in\Xi$ to the previous inequality (4.1), there exists a solution $(\bar{\mathrm{x}},\bar{\mathrm{y}})\in\Xi$ of the system (1.1) with

$$\|(\mathrm{x},\mathrm{y})(\tau)-(\bar{\mathrm{x}},\bar{\mathrm{y}})(\tau)\|_{\Xi}\leq\varrho^*\varphi^*.$$

Definition 4.2 ([32, 34]). The coupled system (1.1) is generalized Ulam-Hyers stable if there exists $\mathfrak{Z} \in C(,\mathbb{R}^+,\mathbb{R}^+)$, such that $\mathfrak{Z}(0) = 0$ for any $\varrho^* > 0$, and for each solution $(\mathrm{x}, \mathrm{y}) \in \Xi$ to the inequality (4.1), there exists a solution $(\bar{\mathrm{x}}, \bar{\mathrm{y}}) \in \Xi$ of the system (1.1) with

$$\|(\mathrm{x}, \mathrm{y})(\tau) - (\bar{\mathrm{x}}, \bar{\mathrm{y}})(\tau)\|_\Xi \leq \mathfrak{Z}(\varrho^*).$$

Definition 4.3 ([32, 34]). System (1.1) is Ulam-Hyers-Rassias stable with respect to $\mathfrak{J} = \max\{\mathfrak{J}_1, \mathfrak{J}_2\} \in C(J,\mathbb{R}^+)$, if there exists a positive real number $\theta = \max\{\theta_{\mathfrak{J}_1}, \theta_{\mathfrak{J}_2}\} > 0$ such that for each $\varrho^* = \max\{\varrho_1, \varrho_2\} > 0$ and for each solution $(\mathrm{x}, \mathrm{y}) \in \Xi$ to the inequality (4.2), there exists a solution $(\bar{\mathrm{x}}, \bar{\mathrm{y}}) \in \Xi$ of the system (1.1) with

$$\|(\mathrm{x}, \mathrm{y})(\tau) - (\bar{\mathrm{x}}, \bar{\mathrm{y}})(\tau)\|_\Xi \leq \varrho^*\theta\mathfrak{J}(\tau).$$

Definition 4.4 ([32, 34]). System (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\mathfrak{J} = \max\{\mathfrak{J}_1, \mathfrak{J}_2\} \in C(J,\mathbb{R}^+)$, if there exists a positive real number $\theta = \max\{\theta_{\mathfrak{J}_1}, \theta_{\mathfrak{J}_2}\} > 0$ such that for each solution $(\mathrm{x}, \mathrm{y}) \in \Xi$ to the inequality (4.3), there exists a solution $(\bar{\mathrm{x}}, \bar{\mathrm{y}}) \in \Xi$ of the system (1.1) with

$$\|(\mathrm{x}, \mathrm{y})(\tau) - (\bar{\mathrm{x}}, \bar{\mathrm{y}})(\tau)\|_\Xi \leq \theta\mathfrak{J}(\tau).$$

Remark 4.5. It is clear that:

1. Definition 4.1⇒ Definition 4.2.
2. Definition 4.3⇒ Definition 4.4.
3. Definition 4.3⇒ Definition 4.1, (for taking $\mathfrak{J}(.) = 1$).

A function $(\mathrm{x}, \mathrm{y}) \in \Xi$ is a solution of the inequality (4.1) if and only if there exist a function $\mathrm{W}_k \in C(J,\mathbb{R})$ such that for all $k = 1, 2$:

I) $|\mathrm{W}_1(\tau)| \leq \varrho_1$ and $|\mathrm{W}_2(\tau)| \leq \varrho_2$, for all $\tau \in J$.

II) ${}^C D_{0^+}^{\alpha_1,\psi}\left({}^C D_{0^+}^{\beta_1,\psi}\left(\dfrac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau,\mathrm{x}(\tau))}\right) + \lambda\mathrm{x}(\tau)\right) - \mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau)) = \mathrm{W}_1(\tau)$, for all $\tau \in J$,

III) ${}^C D_{0^+}^{\alpha_2,\psi}\left({}^C D_{0^+}^{\beta_2,\psi}\left(\dfrac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau,\mathrm{y}(\tau))}\right) + \nu\mathrm{y}(\tau)\right) - \mathrm{f}_2(\tau,\mathrm{y}(\tau),\mathrm{x}(\rho\tau)) = \mathrm{W}_2(\tau)$, for all $\tau \in J$.

We now present the Ulam stability of the solution to coupled system (1.1).

Theorem 4.6. *Assume that* $(C1)$, $(C2)$ *and* $(C3)$ *are satisfied. then the problem (1.1) is Ulam-Hyers stable and hence generalizes Ulam-Hyers stability under the condition* (3.14).

Proof. Assume $\varrho^* > 0$ and $(\mathrm{x}, \mathrm{y}) \in \mathcal{B}_\delta \subset \Xi$ is a function that fulfills the inequality (4.1), and let $(\bar{\mathrm{x}}, \bar{\mathrm{y}}) \in \mathcal{B}_\delta$ is the sole solution of the coupled system (1.1). Since $(\mathrm{x}, \mathrm{y}) \in \mathcal{B}_\delta$ is a function satisfes the inequality (4.1). It follows from Remark 4.5 that

$$\begin{cases} {}^C D_{0^+}^{\alpha_1,\psi}\left({}^C D_{0^+}^{\beta_1,\psi}\left(\dfrac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau,\mathrm{x}(\tau))}\right) + \lambda\mathrm{x}(\tau)\right) - \mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau)) = \mathrm{W}_1(\tau),\ \tau \in J, \\ {}^C D_{0^+}^{\alpha_2,\psi}\left({}^C D_{0^+}^{\beta_2,\psi}\left(\dfrac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau,\mathrm{y}(\tau))}\right) + \nu\mathrm{y}(\tau)\right) - \mathrm{f}_2(\tau,\mathrm{y}(\tau),\mathrm{x}(\rho\tau)) = \mathrm{W}_2(\tau),\ \tau \in J, \\ \mathrm{x}(\tau)\mid_{\tau=0} = 0,\ \mathrm{x}'(\tau)\mid_{\tau=0} = 0, \mathrm{x}(\tau)\mid_{\tau=\epsilon_1} = 0,\ 0 < \epsilon_1 < a_2\ , \\ \mathrm{y}(\tau)\mid_{\tau=0} = 0,\ \mathrm{y}'(\tau)\mid_{\tau=0} = 0, \mathrm{y}(\tau)\mid_{\tau=\epsilon_2} = 0,\ 0 < \epsilon_2 < a_2\ . \end{cases}$$

Using Lemma 3.2 once more, we have

$$\mathrm{x}(\tau) \quad := \quad \mathrm{g}(\tau,\mathrm{x}(\tau))\left[\aleph(\mathrm{x},\mathrm{y})(\tau) - \frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}\aleph(\mathrm{x},\mathrm{y})(\epsilon_1)\right],$$

and

$$\mathrm{y}(\tau) \quad := \quad \mathrm{g}(\tau,\mathrm{y}(\tau))\left[\mathrm{Q}(\mathrm{x},\mathrm{y})(\tau) - \frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_2)-\psi(0))^{\beta_2}}\mathrm{Q}(\mathrm{x},\mathrm{y})(\epsilon_2)\right],$$

where

$$\aleph(\mathrm{x},\mathrm{y})(\tau) := I_{0^+}^{\alpha_1+\beta_1,\psi}[\mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau)) + \mathrm{W}_1(\tau)] - \lambda I_{0^+}^{\alpha_1,\psi}\mathrm{x}(\tau),$$

and

$$\mathrm{Q}(\mathrm{x},\mathrm{y})(\tau) := I_{0^+}^{\alpha_2+\beta_2,\psi}[\mathrm{f}_2(\tau,\mathrm{y}(\tau),\mathrm{x}(\rho\tau)) + \mathrm{W}_2(\tau)] - \nu I_{0^+}^{\alpha_2,\psi}\mathrm{y}(\tau),$$

Moreover, using part (I) of Remark 4.5 and $(C2)$, we can obtain the following formula for each $\tau \in J$.

$$\begin{aligned}
&|\aleph(\mathrm{x},\mathrm{y})(\tau) - \Phi(\bar{\mathrm{x}},\bar{\mathrm{y}})(\tau)| \\
&\leq I_{0^+}^{\alpha_1+\beta_1,\psi}|\mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau)) - \mathrm{f}_1(\tau,\bar{\mathrm{x}}(\tau),\bar{\mathrm{y}}(\xi\tau))| + \lambda I_{0^+}^{\alpha_1,\psi}|\mathrm{x}(\tau) - \bar{\mathrm{x}}(\tau)| \\
&\quad + I_{0^+}^{\alpha_1+\beta_1,\psi}|\mathrm{W}_1(\tau)| \\
&\leq \left(\frac{\mathrm{P}_{\mathrm{f}_1}(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right)\|(\mathrm{x},\mathrm{y}) - (\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi \\
&\quad + \frac{\varrho_1(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)},
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&|\mathrm{Q}(\mathrm{x},\mathrm{y})(\tau) - \Psi(\bar{\mathrm{x}},\bar{\mathrm{y}})(\tau)| \\
&\leq I_{0^+}^{\alpha_2+\beta_2,\psi}|\mathrm{f}_2(\tau,\mathrm{y}(\tau),\mathrm{x}(\rho\tau)) - \mathrm{f}_2(\tau,\bar{\mathrm{y}}(\tau),\bar{\mathrm{x}}(\rho\tau))| + \nu I_{0^+}^{\alpha_2,\psi}|\mathrm{y}(\tau) - \bar{\mathrm{y}}(\tau)| \\
&\quad + I_{0^+}^{\alpha_2+\beta_2,\psi}|\mathrm{W}_2(\tau)| \\
&\leq \left(\frac{\mathrm{P}_{\mathrm{f}_2}(\psi(\tau)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\nu|(\psi(\tau)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}\right)\|(\mathrm{x},\mathrm{y}) - (\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi \\
&\quad + \frac{\varrho_2(\psi(\tau)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_2+1)},
\end{aligned} \tag{4.5}$$

in addition to,

$$\begin{cases} |\aleph(\mathrm{x},\mathrm{y})(\tau)| & \leq & \frac{(\mathcal{L}_1+\varrho_1)(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\delta, \\ |\Phi(\mathrm{x},\mathrm{y})(\tau)| & \leq & \frac{\mathcal{L}_1(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\delta, \end{cases} \tag{4.6}$$

$$\begin{cases} |\mathrm{Q}(\mathrm{x},\mathrm{y})(\tau)| & \leq & \frac{(\mathcal{L}_2+\varrho_2)(\psi(\tau)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\nu|(\psi(\tau)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}\delta, \\ |\Psi(\mathrm{x},\mathrm{y})(\tau)| & \leq & \frac{\mathcal{L}_2(\psi(\tau)-\psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\nu|(\psi(\tau)-\psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}\delta. \end{cases} \tag{4.7}$$

Applying the triangle inequality,(4.4)-(4.6), we have

$$\begin{aligned}
&|\mathrm{x}(\tau) - \bar{\mathrm{x}}(\tau)| \\
&\leq |\mathrm{g}_1(\tau,\mathrm{x}(\tau))\aleph(\mathrm{x},\mathrm{y})(\tau) - \mathrm{g}_1(\tau,\bar{\mathrm{x}}(\tau))\Phi(\bar{\mathrm{x}},\bar{\mathrm{y}})(\tau)| \\
&\quad + \frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}|\mathrm{g}_1(\tau,\mathrm{x}(\tau))\aleph(\mathrm{x},\mathrm{y})(\epsilon_1) - \mathrm{g}_1(\tau,\bar{\mathrm{x}}(\tau))\Phi(\bar{\mathrm{x}},\bar{\mathrm{y}})(\epsilon_1)| \\
&\leq \mathcal{M}_1\left(\frac{\mathrm{P}_{\mathrm{f}_1}(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right)\|(\mathrm{x},\mathrm{y}) - (\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi \\
&\quad + \|G_1\|_\infty\left(\frac{\mathcal{L}_1(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\delta\right)\|(\mathrm{x},\mathrm{y}) - (\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi \\
&\quad + \mathcal{M}_1(\psi(\tau)-\psi(0))^{\beta_1}\left(\frac{\mathrm{P}_{\mathrm{f}_1}(\psi(\epsilon_1)-\psi(0))^{\alpha_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|}{\Gamma(\beta_1+1)}\right)\|(\mathrm{x},\mathrm{y}) - (\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi \\
&\quad + \|G_1\|_\infty(\psi(\tau)-\psi(0))^{\beta_1}\left(\frac{\mathcal{L}_1(\psi(\epsilon_1)-\psi(0))^{\alpha_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|}{\Gamma(\beta_1+1)}\delta\right)\|(\mathrm{x},\mathrm{y}) - (\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi
\end{aligned}$$

$$+\frac{\mathcal{M}_1[(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}+((\psi(\tau)-\psi(0))^{\beta_1}((\psi(\epsilon_1)-\psi(0))^{\alpha_1}]}{\Gamma(\alpha_1+\beta_1+1)}\varrho_1.$$

we obtain

$$\|\mathrm{x}-\bar{\mathrm{x}}\|_\infty \leq \Omega_1\|(\mathrm{x},\mathrm{y})-(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi+\varpi_1\varrho_1. \tag{4.8}$$

On the other hand, we have

$$\|\mathrm{y}-\bar{\mathrm{y}}\|_\infty \leq \Omega_2\|(\mathrm{x},\mathrm{y})-(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi+\varpi_2\varrho_2, \tag{4.9}$$

where $\varpi_1 := \mathcal{M}_1\Delta_1$, and $\varpi_2 := \mathcal{M}_2\nabla_1$.

Combining the two last inequalities (4.8) and (4.10), we get

$$\|(\mathrm{x},\mathrm{y})-(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Xi \leq [1-\max\{\Omega_1,\Omega_2\}]^{-1}\max\{\varpi_1,\varpi_2\}\varrho^*. \tag{4.10}$$

Let is put $\varrho^*=\max\{\varrho_1,\varrho_2\}$; $\varphi^*=[1-\max\{\Omega_1,\Omega_2\}]^{-1}\max\{\varpi_1,\varpi_2\}$. Taking into account $\max\{\Omega_1,\Omega_2\}<1$, we notice that $\varphi^*>0$. Thus, we have

$$\|(\mathrm{x},\mathrm{y})(\tau)-(\bar{\mathrm{x}},\bar{\mathrm{y}})(\tau)\|_\Xi \leq \varrho^*\varphi^*.$$

Consequently, the coupled system (1.1) is stable in the sense of Ulam-Hyers. This completes the proof using Ulam-Hyers definition. □

Theorem 4.7. *Suppose the conditions of Theorem 4.6 hold. If there exists $\mathfrak{Z}\in C(\mathbb{R}^+,\mathbb{R}^+)$, such that $\mathfrak{Z}(0)=0$ with $\varrho^*>0$. Therefore, the coupled system (1.1) is generalized Ulam-Hyers stable.*

Proof. For $\mathfrak{Z}(\varrho^*)=\varphi^*\varrho^*$; $\mathfrak{Z}(0)=0$. We prove that the solution to the system (1.1) is also generalized Ulam-Hyers stable. □

We are now interested in the Ulam-Hyers-Rassias stability of our system.

Theorem 4.8. *Consider the hypotheses $(C1)-(C3)$ and let (3.14) hold. Assume*

$(C4)$ *There exist nondecreasing functions $\mathfrak{J}_1,\mathfrak{J}_2\in C(J,\mathbb{R}^+)$ and $Q_{\mathfrak{J}_1},Q_{\mathfrak{J}_2}>0$, such that for any $\tau\in J$,*

$$I_{0^+}^{\alpha_1+\beta_1,\psi}\mathfrak{J}_1(\tau)\leq Q_{\mathfrak{J}_1}\mathfrak{J}_1(\tau), \tag{4.11}$$

and

$$I_{0^+}^{\alpha_2+\beta_2,\psi}\mathfrak{J}_1(\tau)\leq Q_{\mathfrak{J}_2}\mathfrak{J}_2(\tau). \tag{4.12}$$

Then, the coupled system (1.1) is Ulam-Hyers-Rassias stabile concerning $\mathfrak{J}$.

Proof. Let $(\mathrm{x},\mathrm{y})\in\mathcal{B}_\delta$ is a solution of the inequality (4.2), we have

$$\left|{}^C D_{0^+}^{\alpha_1,\psi}\left[{}^C D_{0^+}^{\beta_1,\psi}\left[\frac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau,\mathrm{x}(\tau))}\right]+\lambda_1\mathrm{x}(\tau)\right]-\mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau))\right|\leq\varrho_1\mathfrak{J}_1(\tau),$$

and

$$\left|{}^C D_{0^+}^{\alpha_2,\psi}\left[{}^C D_{0^+}^{\beta_2,\psi}\left[\frac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau,\mathrm{y}(\tau))}\right]+\lambda_2\mathrm{y}(\tau)\right]-\mathrm{f}_2(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau))\right|\leq\varrho_2\mathfrak{J}_2(\tau).$$

Let $(\bar{x},\bar{y}) \in \mathcal{B}_\delta$ be the unique solution of the system

$$\begin{cases} {}^C D_{0^+}^{\alpha_1,\psi}\left({}^C D_{0^+}^{\beta_1,\psi}\left(\dfrac{\bar{x}(\tau)}{g_1(\tau,\bar{x}(\tau))}\right)+\lambda\bar{x}(\tau)\right)-f_1(\tau,\bar{x}(\tau),\bar{y}(\xi\tau))=W_1(\tau),\ \tau\in J,\\ {}^C D_{0^+}^{\alpha_2,\psi}\left({}^C D_{0^+}^{\beta_2,\psi}\left(\dfrac{\bar{y}(\tau)}{g_2(\tau,\bar{y}(\tau))}\right)+\nu\bar{y}(\tau)\right)-f_2(\tau,\bar{y}(\tau),\bar{x}(\rho\tau))=W_2(\tau),\ \tau\in J,\\ \bar{x}(\tau)\mid_{\tau=0}=x(\tau)\mid_{\tau=0}=0,\ \bar{x}'(\tau)\mid_{\tau=0}=x'(\tau)\mid_{\tau=0}=0,\\ \bar{x}(\tau)\mid_{\tau=\epsilon_1}=x(\tau)\mid_{\tau=\epsilon_1}=0,\ 0<\epsilon_1<a_2\ ,\\ \bar{y}(\tau)\mid_{\tau=0}=y(\tau)\mid_{\tau=0}=0,\ \bar{y}'(\tau)\mid_{\tau=0}=y'(\tau)\mid_{\tau=0}=0,\\ \bar{y}(\tau)\mid_{\tau=\epsilon_2}=y(\tau)\mid_{\tau=\epsilon_2}=0,\ 0<\epsilon_2<a_2. \end{cases}$$

So by Lemma 3.2, we have

$$\begin{cases} \bar{x}(\tau) &:= g_1(\tau,\bar{x}(\tau))\left[\aleph(\bar{x},\bar{y})(\tau)-\frac{(\psi(\tau)-\psi(0))^{\beta_1}}{(\psi(\epsilon_1)-\psi(0))^{\beta_1}}\aleph(\bar{x},\bar{y})(\epsilon_1)\right],\\ \bar{y}(\tau) &:= g_2(\tau,\bar{y}(\tau))\left[Q(\bar{x},\bar{y})(\tau)-\frac{(\psi(\tau)-\psi(0))^{\beta_2}}{(\psi(\epsilon_1)-\psi(0))^{\beta_2}}Q(\bar{x},\bar{y})(\epsilon_2)\right], \end{cases}$$

where

$$\begin{cases} \aleph(\bar{x},\bar{y})(\tau):=I_{0^+}^{\alpha_1+\beta_1,\psi}[f_1(\tau,\bar{x}(\tau),\bar{y}(\xi\tau))]-\lambda I_{0^+}^{\alpha,\psi}x(\tau),\\ Q(\bar{x},\bar{y})(\tau):=I_{0^+}^{\alpha_2+\beta_2,\psi}[f_2(\tau,\bar{y}(\tau),\bar{x}(\rho\tau))]-\nu I_{0^+}^{\alpha_2,\psi}y(\tau), \end{cases}$$

From (3.14) and (4.11) we can write

$$\begin{aligned} \|x-\bar{x}\|_\infty \le{}& \|G_1\|\left(\frac{\mathcal{L}_1(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\delta\right)\|(x,y)-(\bar{x},\bar{y})\|_\Xi\\ &+\mathcal{M}_1\left(\frac{P_{f_1}(\psi(\tau)-\psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda|(\psi(\tau)-\psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right)\|(x,y)-(\bar{x},\bar{y})\|_\Xi\\ &+\mathcal{M}_1\varrho_1 Q_1\mathfrak{J}_1(\tau). \end{aligned} \tag{4.13}$$

Therefor,

$$\|x-\bar{x}\|_\infty \le \Lambda_1\|(x,y)-(\bar{x},\bar{y})\|_\Xi+\mathcal{M}_1\varrho_1 Q_1\mathfrak{J}_1(\tau).$$

On the other hand, by (3.14) and (4.12), we have

$$\|y-\bar{y}\|_\infty \le \Lambda_2\|(x,y)-(\bar{x},\bar{y})\|_\Xi+\mathcal{M}_2\varrho_2 Q_2\mathfrak{J}_1(\tau).$$

Then, we may obtain

$$\|(x,y)-(\bar{x},\bar{y})\|_\Xi \le \max\{\Lambda_1,\Lambda_2\}\|(x,y)-(\bar{x},\bar{y})\|_\Xi+\mathcal{M}\varrho^* Q_{\mathfrak{J}}\mathfrak{J}(\tau), \tag{4.14}$$

where

$$\mathcal{M}=\max\{\mathcal{M}_1,\mathcal{M}_2\},\quad Q_{\mathfrak{J}}=\max\{Q_{\mathfrak{J}_1},Q_{\mathfrak{J}_2}\},$$

Then

$$\|(x,y)-(\bar{x},\bar{y})\|_\Xi \le \varrho^*\frac{\mathcal{M}Q_{\mathfrak{J}}}{1-\max\{\Lambda_1,\Lambda_2\}}\mathfrak{J}(\tau), \tag{4.15}$$

$$= \varrho^*\theta\mathfrak{J}(\tau). \tag{4.16}$$

Hence, the system (1.1) is Ulam-Hyers-Rassias stable concerning $\mathfrak{J}$. Similarly, by following the same procedure as before, substituting $\varrho^*=1$, we can easily see that

$$\|(x,y)-(\bar{x},\bar{y})\|_\Xi \le \theta\mathfrak{J}(\tau), \tag{4.17}$$

for each $\tau\in J$. Hence, system (1.1) is generalized Ulam-Hyers-Rassias stable. □

5. Illustratives Examples

Example 5.1. Let us consider the following system.

$$(5.1)\qquad \begin{cases} {}^{C}D_{0^+}^{0.75,\psi}\left({}^{C}D_{0^+}^{1.45,\psi}\left(\dfrac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau,\mathrm{x}(\tau))}\right)+\frac{33}{100}\mathrm{x}(\tau)\right)=\mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau)),\\ {}^{C}D_{0^+}^{0.45,\psi}\left({}^{C}D_{0^+}^{1.75,\psi}\left(\dfrac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau,\mathrm{y}(\tau))}\right)+\frac{73}{100}\mathrm{y}(\tau)\right)=\mathrm{f}_2(\tau,\mathrm{x}(\tau),\mathrm{y}(\rho\tau)),\\ \mathrm{x}(\tau)\mid_{\tau=1}=0,\ \mathrm{x}'(\tau)\mid_{\tau=1}=0,\ \mathrm{x}(\tau)\mid_{\tau=\frac{3}{2}}=0,\ 1<\epsilon_1=\frac{3}{2}<2,\\ \mathrm{y}(\tau)\mid_{\tau=1}=0,\ \mathrm{y}'(\tau)\mid_{\tau=1}=0,\ \mathrm{y}(\tau)\mid_{\tau=\frac{5}{4}}=0,\ 1<\epsilon_2=\frac{5}{4}<2, \end{cases}$$

where

$$J=[0,2],\ a=2,\ \alpha_1=0.75,\ \beta_1=1.45,\ \lambda=0.33,\ \epsilon_1=\tfrac{3}{2},$$
$$\alpha_2=0.45,\ \beta_2=1.75,\ \nu=0.73,\ \epsilon_2=\tfrac{5}{4},\ \xi=\rho=\tfrac{9}{10},\ \psi(\tau)=\tau^2.$$

Also,

$$(5.2)\qquad \mathrm{f}_1(\tau,\mathrm{x}(\tau),\mathrm{y}(\xi\tau))=\frac{\sin(\tau)\left[\mathrm{y}(\dfrac{9}{10}\tau)+\cos(\tau)\mathrm{x}(\tau)\right]+\sqrt{\tau}+0.001}{e^{2+\tau}+32\pi},$$

$$(5.3)\qquad \mathrm{f}_2(\tau,\mathrm{y}(\tau),\mathrm{x}(\rho\tau))=\frac{\tau^2+1}{32\pi^2 e}\left[\cos(\mathrm{y}(\frac{9}{10}\tau))+\cos(\mathrm{x}(\tau))\right]+0.002.$$

For $\tau\in[0;2]$ and $(\mathrm{x},\mathrm{y});(\bar{\mathrm{x}},\bar{\mathrm{y}})\in\mathbb{R}^2$, we have

$$|\mathrm{f}_1(\tau,\mathrm{x},\mathrm{y})-\mathrm{f}_1(\tau,\bar{\mathrm{x}},\bar{\mathrm{y}})|\le\frac{1}{e^3}(|\mathrm{x}-\bar{\mathrm{x}}|+|\mathrm{y}-\bar{\mathrm{y}}|),$$

and

$$|\mathrm{f}_2(\tau,\mathrm{x},\mathrm{y})-\mathrm{f}_2(\tau,\bar{\mathrm{x}},\bar{\mathrm{y}})|\le\frac{5}{32e}(|\mathrm{x}-\bar{\mathrm{x}}|+|\mathrm{y}-\bar{\mathrm{y}}|).$$

So, we can take

$$\|\mathrm{P}_{\mathrm{f}_1}\|_\infty\approx 0.099574,\ \mathcal{L}_1\approx 0.01406744,$$
$$\|\mathrm{P}_{\mathrm{f}_2}\|_\infty\approx 0.0574811,\ \mathcal{L}_2\approx 0.0574811.$$

We also have

$$\mathrm{g}_1(\tau,\mathrm{x}(\tau))=\frac{\cos(\mathrm{x}(\tau))}{12e^{\tau^2+\pi}}\ and\ \mathrm{g}_2(\tau,\mathrm{y}(\tau))=\frac{\sin(\tau)\mathrm{y}(\tau)}{e^{\pi}},$$

For $\tau\in[0;2]$ and $(\mathrm{x},\mathrm{y});(\bar{\mathrm{x}},\bar{\mathrm{y}})\in\mathbb{R}^2$, we can write

$$|\mathrm{g}_1(\tau,\mathrm{x})-\mathrm{g}_1(\tau,\bar{\mathrm{x}})|\le\frac{1}{12e^1}|\mathrm{x}-\bar{\mathrm{x}}|,$$

and

$$|\mathrm{g}_2(\tau,\mathrm{y})-\mathrm{g}_2(\tau,\bar{\mathrm{y}})|\le\frac{1}{e^{\pi}}|\mathrm{y}-\bar{\mathrm{y}}|.$$

Hence,

$$G_1^{\mathrm{g}}\approx 0.0306566,\ M_1\approx 0.0306566,$$
$$G_2^{\mathrm{g}}\approx 0.0432139,\ M_2\approx 0.0432139.$$

and

$$\Delta_1=14.367,\ \Delta_2=11.625,\ \Delta_3=8.7096,$$
$$\nabla_1=15.432,\ \nabla_2=14.069,\ \nabla_3=8.7096.$$

therefore,

$$\delta\ge\max\left(9.6267\times10^{-3},9.7782\times10^{-2}\right).$$

So, let's assume that

$$\delta = \frac{1}{2}.$$

We also have

$$\Omega_1 = 0.58463,\ \Omega_2 \approx 0.98863.$$

Thus,

$$\max\{\Omega_1, \Omega_2\} < 1,$$

and by Theorem 4.6, we conclude that the coupled system (5.1) has a unique solution on $[0, 2]$.

Let $\varrho_1 = \frac{1}{5} > 0$ and $\varrho_2 = \frac{3}{7} > 0$, as illustrated Theorem 4.6 and by (4.1). If $\mathrm{x}, \mathrm{y} \in C([0, 2], \mathbb{R})$ complies with

$$|{}^{C}D_{0^+}^{0.75,\psi}\left({}^{C}D_{0^+}^{1.45,\psi}\left(\frac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau, \mathrm{x}(\tau))}\right) + \frac{33}{100}\mathrm{x}(\tau)\right) - \mathrm{f}_1(\tau, \mathrm{x}(\tau), \mathrm{y}(\xi\tau))| < \frac{1}{5}, \tag{5.4}$$

and

$$|{}^{C}D_{0^+}^{0.45,\psi}\left({}^{C}D_{0^+}^{1.75,\psi}\left(\frac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau, \mathrm{y}(\tau))}\right) + \frac{73}{100}\mathrm{y}(\tau)\right) - \mathrm{f}_2(\tau, \mathrm{x}(\tau), \mathrm{y}(\xi\tau))| < \frac{3}{7}, \tag{5.5}$$

there exists a solution $\bar{\mathrm{x}}, \bar{\mathrm{y}} \in C([0, 2], \mathbb{R})$ of the coupled system (5.1) with

$$\|(\nu_1, \nu_2) - (\bar{\mathrm{x}}, \bar{\mathrm{y}})\|_{\Xi} \leq \frac{3}{7}\varphi^*,$$

which

$$\varphi^* = [1 - \max\{\Omega_1, \Omega_2\}]^{-1} \max\{\varpi_1, \varpi_2\} \approx 58.653 > 0.$$

where,

$$\left.\begin{array}{l}\varpi_1 \approx 0.47309\\ \varpi_2 \approx 0.66688\end{array}\right\} \Rightarrow \max\{\varpi_1, \varpi_2\} \approx 0.66688. \tag{5.6}$$

Consequently, the coupled system (5.1) is Ulam-Hyers stable on $[0, 2]$. Finally, we assume that $\varrho^* = 0$, we obtain $\mathfrak{Z}(0) = 0$. Hence, the system (5.1) is generalized Ulam-Hyers stable.

Lastly, we consider $\mathfrak{J}_1(\tau) = \mathfrak{J}_2(\tau) = e^{\tau}$. To confirm the condition

$$I_{0^+}^{\alpha_k+\beta_k,\psi}\mathfrak{J}_k(\tau) \leq Q_{\mathfrak{J}_k}\mathfrak{J}_k(\tau),$$

for $Q_{\mathfrak{J}_k} > 0$ and $k = 1, 2$.

Using fractional integration and simple calculations, we obtain

$$I_{0^+}^{\alpha_k+\beta_k,\psi}\mathfrak{J}_k(\tau) = \frac{1}{\Gamma(\alpha_k + \beta_k)}\int_{0^+}^{\tau} 2s(\tau^2 - s^2)^{\alpha_k+\beta_k}\mathfrak{J}_k(s)ds \tag{5.7}$$

$$\leq \frac{\mathfrak{J}_k(\tau)}{\Gamma(\alpha_k + \beta_k)}\int_{0^+}^{\tau} 2s(\tau^2 - s^2)^{\alpha_k+\beta_k}ds \leq 5\mathfrak{J}_k(\tau), \tag{5.8}$$

with $Q_{\mathfrak{J}_1} = Q_{\mathfrak{J}_2} = 5 > 0$ and $k = 1, 2$. Hence, hypothesis $(C4)$ is satisfied and for every $\varrho_1 = \frac{1}{5}$ and $\varrho_2 = \frac{3}{7}$. If $\mathrm{x}, \mathrm{y} \in C([0, 2], \mathbb{R})$ satisfied

$$|{}^{C}D_{0^+}^{0.75,\psi}\left({}^{C}D_{0^+}^{1.45,\psi}\left(\frac{\mathrm{x}(\tau)}{\mathrm{g}_1(\tau, \mathrm{x}(\tau))}\right) + \frac{33}{100}\mathrm{x}(\tau)\right) - \mathrm{f}_1(\tau, \mathrm{x}(\tau), \mathrm{y}(\xi\tau))| < \frac{1}{5}e^{\tau}, \tag{5.9}$$

and

$$|{}^{C}D_{0^+}^{0.45,\psi}\left({}^{C}D_{0^+}^{1.75,\psi}\left(\frac{\mathrm{y}(\tau)}{\mathrm{g}_2(\tau, \mathrm{y}(\tau))}\right) + \frac{73}{100}\mathrm{y}(\tau)\right) - \mathrm{f}_2(\tau, \mathrm{x}(\tau), \mathrm{y}(\xi\tau))| < \frac{3}{7}e^{\tau}, \tag{5.10}$$

for $\tau \in [0, 2]$, there exists a solution $\bar{\mathrm{x}}, \bar{\mathrm{y}} \in \mathbb{R}^2$ of the system (5.1) such that

$$\|(\mathrm{x}, \mathrm{y}) - (\bar{\mathrm{x}}, \bar{\mathrm{y}})\|_{\Xi} \leq \frac{3}{7}\theta e^{\tau},$$

where

$$\theta = \left[1 - \max\{\Lambda_1, \Lambda_2\}\right]^{-1} Q_{\mathfrak{J}}\mathcal{M} = 0.43149 > 0.$$

This implies that system (5.1) is Ulam-Hyers-Rassias stable. Finally, we assume that $\varrho^* = 1$. We can easily see that

$$\|(\mathrm{x}, \mathrm{y}) - (\bar{\mathrm{x}}, \bar{\mathrm{y}})\|_{\Xi} \leq \theta e^{\tau},$$

for each $\tau \in [0, 2]$. Hence, coupled system (5.1) is generalized Ulam-Hyers-Rassias stable.

6. Conclusion

The current research focuses on exploring the existence, uniqueness and Ulam stability results for a class of coupled systems for hybrid Langevin fractional pantograph differential equations involving the ψ-Caputo fractional derivative subject to non-local and boundary conditions within a Banach space. Our method for achieve a results the existence of solutions the coupled system relies on the application of Dhage and Banach's fixed-point theorems. On the other hand, we explore stability in the sense of Ulam-Hyers then that of Ulam-Hyers-Rassias our probleme. To illustrate the practical application of the main findings, we provide a practical example. As research in this area continues to progress, we recommend further exploration using generalized fractional derivatives. Furthermore, we have the opportunity to extend our work by incorporating several recently introduced fractional operators, including the ψ-Hilfer fractional derivative, reduced fractional operators, and the compact Caputo derivative. We hope that this article will serve as a starting point for deeper exploration in these fields. We believe that there are many potential avenues for further research, including differential inclusion, delay problems, impulsive systems, and many others.

Author's contributions. The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

Competing interests. The authors declare no competing interests.

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