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UNIQUE COMMON FIXED POINT FOR OCCASIONALLY WEAKLY BIASED MAPPINGS OF TYPE $(\mathbb{A})$ IN ULTRAMETRIC SPACE

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ABSTRACT. In this paper, we have established the existence and uniqueness of a common fixed point for occasionally weakly biased mappings of type $(\mathbb{A})$ in an ultrametric space employing an implicit function. Our result accredits us to ameliorate some fixed point theorems specifically Alinejad and Mirmostafaee [2]. In the sequel, we have provided a novel explanation to a problem of Rhoades [B. E. Rhoades, Contractive definitions and continuity, Fixed Point Theory and its Applications (Berkeley 1986), Contemp. Math. (Amer. Math. Soc.), 72 (1988), 233-245.] on the question of the existence of contractive mapping having a fixed point at the point of discontinuity in a non-complete ultrametric space via implicit relations. Our theorems and corollaries are improved and enhanced versions of renowned conclusions wherein completeness and continuity have not been utilized.

1. INTRODUCTION AND PRELIMINARIES

It is noted that the Banach contraction principle is an elementary result in fixed point theory. Later, many authors have broadened, generalized and enhanced given fundamental result in different ways. Further, recently, fixed and common fixed point results in different types of spaces have been established.

One of the significant extensions of metric space [9] is an ultrametric space [2] in which the triangle inequality is enhanced to

$$d(\mu_1, \mu_3) \leq \max\{d(\mu_1, \mu_2), d(\mu_2, \mu_3)\}$$

and validate it by suitable example. Occasionally the related metric is also called a non-Archimedean metric.

The discrete metric on any set M is defined by setting $d(\mu_1, \mu_2) = 1$ when $\mu_1 \neq \mu_2$. This is an ultrametric, and there are also more interesting examples.

Definition 1.1. Let $\mathcal{X}$ be a nonempty set. A mapping $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ is an ultrametric on $\mathcal{X}$ if, for all μ_1, μ_2, $\mu_3 \in \mathcal{X}$:

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- U_1: $d(\mu_1, \mu_2) \geq 0$;
- U_2: $d(\mu_1, \mu_2) = d(\mu_2, \mu_1)$ (symmetry);
- U_3: $d(\mu_1, \mu_1) = 0$;
- U_4: $d(\mu_1, \mu_2) = 0$ then $\mu_1 = \mu_2$;
- U_5: $d(\mu_1, \mu_3) \leq \max\{d(\mu_1, \mu_2), d(\mu_2, \mu_3)\}$ (ultrametric inequality).

Then the pair $(\mathcal{X}, d)$ is known as an ultrametric space.

Equivalently, the ultrametric version of the triangle inequality says that $d(\mu_1, \mu_2)$ and $d(\mu_2, \mu_3)$ cannot both be strictly less than $d(\mu_1, \mu_3)$ for any μ_1, μ_2, $\mu_3 \in M$.

In particular, the standard metric on the real line does not have this property.

Example 1.2. ([10]) For $a \in \mathbb{R}$ let $[a]$ be the entire part of a: By

$$d(x, y) = \inf\{2^{-n} : n \in \mathbb{Z}, [2^n(x - e)] = [2^n(y - e)]\}$$

(here e is any irrational number) an ultrametric d on $\mathbb{Q}$ is defined which determines the usual topology on $\mathbb{Q}$.

Example 1.3. Let $X = \mathbb{R}$ endowed with the metric $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ such that

$$d(x, y) = \max\{|x|, |y|\},$$

then, $(\mathcal{X}, d)$ is an ultrametric space.

Example 1.4. Endow the nonempty $\mathcal{X}$ with the discrete metric

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

Then, $(\mathcal{X}, d)$ is an ultrametric space.

In this paper, we prove the existence and uniqueness of a common fixed point for occasionally weakly biased mappings of type $(\mathbb{A})$ in an ultrametric space using an implicit function. We start our work by giving the definition of commuting mappings in a metric space.

Definition 1.5. Two self-mappings $\mathcal{P}$ and $\mathcal{Q}$ of a metric space $(\mathcal{X}, d)$ are said to be commuting if and only if

$$\mathcal{P}\mathcal{Q}\mu_1 = \mathcal{Q}\mathcal{P}\mu_1$$

for all μ_1 in $\mathcal{X}$.

In 1982, Sessa [18] relaxed the commutativity to the weak commutativity.

Definition 1.6. ([18]) Two self-mappings $\mathcal{P}$ and $\mathcal{Q}$ of a metric space $(\mathcal{X}, d)$ are called weakly commuting if and only if

$$d(\mathcal{P}\mathcal{Q}\mu_1, \mathcal{Q}\mathcal{P}\mu_1) \leq d(\mathcal{P}\mu_1, \mathcal{Q}\mu_1)$$

for all μ_1 in $\mathcal{X}$.

In 1986, Jungck [11] enhanced the concept of weak commutativity by introducing the notion of compatible mappings.

Definition 1.7. ([11]) Two self-mappings $\mathcal{P}$ and $\mathcal{Q}$ of a metric space $(\mathcal{X}, d)$ are called compatible if and only if

$$\lim_{n\to\infty} d(\mathcal{P}\mathcal{Q}\mu_n, \mathcal{Q}\mathcal{P}\mu_n) = 0,$$

whenever $\{\mu_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{P}\mu_n = \lim_{n\to\infty} \mathcal{Q}\mu_n = t$ for some $t \in \mathcal{X}$.

In 1995, Jungck and Pathak [14] gave a generalization of the above concept of compatible mappings called biased mappings.

Definition 1.8. ([14]) Let $\mathcal{P}$ and $\mathcal{Q}$ be self-mappings of a metric space $(\mathcal{X}, d)$. The pair $(\mathcal{P}, \mathcal{Q})$ is $\mathcal{Q}$-biased if and only if whenever $\{\mu_n\}$ is a sequence in $\mathcal{X}$ and $\mathcal{P}\mu_n$, $\mathcal{Q}\mu_n \to t \in \mathcal{X}$, then

$$\alpha d(\mathcal{Q}\mathcal{P}\mu_n, \mathcal{Q}\mu_n) \leq \alpha d(\mathcal{P}\mathcal{Q}\mu_n, \mathcal{P}\mu_n)$$

if $\alpha = \lim inf$ and $\alpha = \lim sup$.

Again, the same authors [14], introduced the concept of weakly biased mappings which represents a convenient generalization of biased mappings.

Definition 1.9. ([14]) Let $\mathcal{P}$ and $\mathcal{Q}$ be self-mappings of a metric space $(\mathcal{X}, d)$. The pair $(\mathcal{P}, \mathcal{Q})$ is weakly $\mathcal{Q}$-biased if and only if $\mathcal{P}\mu_1 = \mathcal{Q}\mu_1$ implies

$$d(\mathcal{Q}\mathcal{P}\mu_1, \mathcal{Q}\mu_1) \leq d(\mathcal{P}\mathcal{Q}\mu_1, \mathcal{P}\mu_1).$$

In 2012, Bouhadjera and Djoudi [6] coined the term occasionally weakly biased mappings which is a legitimate generalization of weakly biased mappings given by Jungck and Pathak in [14].

Definition 1.10. ([6]) Let $\mathcal{P}$ and $\mathcal{Q}$ be self-mappings of a set $\mathcal{X}$. The pair $(\mathcal{P}, \mathcal{Q})$ is said to be occasionally weakly $\mathcal{P}$-biased and $\mathcal{Q}$-biased, respectively, if and only if, there exists a point μ_1 in $\mathcal{X}$ such that $\mathcal{P}\mu_1 = \mathcal{Q}\mu_1$ implies

$$d(\mathcal{P}\mathcal{Q}\mu_1, \mathcal{P}\mu_1) \leq d(\mathcal{Q}\mathcal{P}\mu_1, \mathcal{Q}\mu_1),$$
$$d(\mathcal{Q}\mathcal{P}\mu_1, \mathcal{Q}\mu_1) \leq d(\mathcal{P}\mathcal{Q}\mu_1, \mathcal{P}\mu_1),$$

respectively.

Let us return back to 1993, Jungck et al. [13] introduced the concept of compatible mappings of type $(\mathbb{A})$ which is equivalent to compatible mappings under the continuity condition.

Definition 1.11. ([13]) Self-mappings $\mathcal{P}$ and $\mathcal{Q}$ of a metric space $(\mathcal{X}, d)$ are said to be compatible of type $(\mathbb{A})$ if

$$\lim_{n\to\infty} d(\mathcal{Q}\mathcal{P}\mu_n, \mathcal{P}\mathcal{P}\mu_n) = 0, \lim_{n\to\infty} d(\mathcal{P}\mathcal{Q}\mu_n, \mathcal{Q}\mathcal{Q}\mu_n) = 0$$

whenever $\{\mu_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{P}\mu_n$ and $\mathcal{Q}\mu_n \to t \in \mathcal{X}$.

After two years, Pathak et al. [16] generalized the above notion by giving the concept of biased mappings of type $(\mathbb{A})$.

Definition 1.12. ([16]) Let $\mathcal{P}$ and $\mathcal{Q}$ be self-mappings of a metric space $(\mathcal{X}, d)$. The pair $(\mathcal{P}, \mathcal{Q})$ is said to be $\mathcal{Q}$-biased and $\mathcal{P}$-biased of type $(\mathbb{A})$, respectively, if, whenever $\{\mu_n\}$ is a sequence in $\mathcal{X}$ and $\mathcal{P}\mu_n$, $\mathcal{Q}\mu_n \to t \in \mathcal{X}$,

$$\alpha d(\mathcal{Q}\mathcal{Q}\mu_n, \mathcal{P}\mu_n) \leq \alpha d(\mathcal{P}\mathcal{Q}\mu_n, \mathcal{Q}\mu_n),$$
$$\alpha d(\mathcal{P}\mathcal{P}\mu_n, \mathcal{Q}\mu_n) \leq \alpha d(\mathcal{Q}\mathcal{P}\mu_n, \mathcal{P}\mu_n),$$

respectively, where $\alpha = \lim inf_{n\to\infty}$ and if $\alpha = \lim sup_{n\to\infty}$.

And in the same paper [16], the authors gave the definition of weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$ as follows:

Definition 1.13. ([16]) Let $\mathcal{P}$ and $\mathcal{Q}$ be self-mappings of a metric space $(\mathcal{X}, d)$. The pair $(\mathcal{P}, \mathcal{Q})$ is said to be weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$ if $\mathcal{P}\mu_1 = \mathcal{Q}\mu_1$ implies

$$d(\mathcal{Q}\mathcal{Q}\mu_1, \mathcal{P}\mu_1) \leq d(\mathcal{P}\mathcal{Q}\mu_1, \mathcal{Q}\mu_1).$$

In 1996, the notion of compatible mappings was again generalized in [12] by Jungck.

Definition 1.14. ([12]) Two self-mappings $\mathcal{P}$ and $\mathcal{Q}$ of a metric space $(\mathcal{X}, d)$ are called weakly compatible if and only if $\mathcal{P}$ and $\mathcal{Q}$ commute on the set of coincidence points.

In 2008, Al-Thagafi and Shahzad [3] introduced the notion of occasionally weakly compatible (owc) mappings as a generalization of weakly compatible mappings. While the paper [3] was under review, Jungck and Rhoades [15] used the concept of owc and proved several results under different contractive conditions (see [1], [21]).

Definition 1.15. ([3]) Two self-mappings $\mathcal{P}$ and $\mathcal{Q}$ of a set $\mathcal{X}$ are occasionally weakly compatible if and only if, there is a point ν in $\mathcal{X}$ which is a coincidence point of $\mathcal{P}$ and $\mathcal{Q}$ at which $\mathcal{P}$ and $\mathcal{Q}$ commute.

Recently, in 2022, Bouhadjera [5] introduced the concept of weakly $\mathcal{P}$-biased of type $(\mathbb{A})$, and the concepts of occasionally weakly $\mathcal{P}$-biased of type $(\mathbb{A})$ and occasionally weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$, and showed that the two last new definitions coincide with our concepts; occasionally weakly $\mathcal{P}$-biased and occasionally weakly $\mathcal{Q}$-biased respectively given in [6].

Definition 1.16. ([5]) Let $\mathcal{P}$ and $\mathcal{Q}$ be self-mappings of a metric space $(\mathcal{X}, d)$. The pair $(\mathcal{P}, \mathcal{Q})$ is said to be **weakly $\mathcal{P}$-biased of type** $(\mathbb{A})$ if $\mathcal{P}\mu_1 = \mathcal{Q}\mu_1$ implies

$$d(\mathcal{P}\mathcal{P}\mu_1, \mathcal{Q}\mu_1) \leq d(\mathcal{Q}\mathcal{P}\mu_1, \mathcal{P}\mu_1).$$

Definition 1.17. ([5]) Let $\mathcal{P}$ and $\mathcal{Q}$ be self-mappings of a non-empty set $\mathcal{X}$. The pair $(\mathcal{P}, \mathcal{Q})$ is said to be **occasionally weakly $\mathcal{P}$-biased of type** $(\mathbb{A})$ **and occasionally weakly $\mathcal{Q}$-biased of type** $(\mathbb{A})$, respectively, if and only if, there exists a point μ_1 in $\mathcal{X}$ such that $\mathcal{P}\mu_1 = \mathcal{Q}\mu_1$ implies

$$d(\mathcal{P}\mathcal{P}\mu_1, \mathcal{Q}\mu_1) \leq d(\mathcal{Q}\mathcal{P}\mu_1, \mathcal{P}\mu_1),$$
$$d(\mathcal{Q}\mathcal{Q}\mu_1, \mathcal{P}\mu_1) \leq d(\mathcal{P}\mathcal{Q}\mu_1, \mathcal{Q}\mu_1),$$

respectively.

In addition that weakly $\mathcal{P}$-biased of type $(\mathbb{A})$ and weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$ are occasionally weakly $\mathcal{P}$-biased of type $(\mathbb{A})$ and occasionally weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$, respectively, it is also clear from the definitions that if $\mathcal{P}$ and $\mathcal{Q}$ are occasionally weakly compatible or weakly compatible then $\mathcal{P}$, $\mathcal{Q}$ are both occasionally weakly $\mathcal{P}$-biased and $\mathcal{Q}$-biased of type $(\mathbb{A})$. Therefore, occasionally weakly compatible and weakly compatible mappings are subclasses of occasionally weakly biased mappings of type $(\mathbb{A})$. The next example confirms.

Example 1.18. Let $\mathcal{X} = [0, \infty)$ with the usual metric $d(\mu_1, \mu_2) = |\mu_1 - \mu_2|$. Define $\mathcal{P}$, $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{P}\mu_1 = \begin{cases} \mu_1^2 & \text{if } \mu_1 \in [0, 1] \\ \frac{1}{\mu_1} & \text{if } \mu_1 \in (1, \infty), \end{cases} \quad \mathcal{Q}\mu_1 = \begin{cases} \frac{1}{9} & \text{if } \mu_1 \in [0, 1] \\ \frac{\mu_1}{4} & \text{if } \mu_1 \in (1, \infty). \end{cases}$$

We have $\mathcal{P}\mu_1 = \mathcal{Q}\mu_1$ if and only if $\mu_1 = \frac{1}{3}$ or $\mu_1 = 2$ and

$$0 = d\left(\mathcal{Q}\mathcal{Q}(\frac{1}{3}), \mathcal{P}(\frac{1}{3})\right) \leq d\left(\mathcal{P}\mathcal{Q}(\frac{1}{3}), \mathcal{Q}(\frac{1}{3})\right) = \frac{8}{81};$$

that is, the pair $(\mathcal{P}, \mathcal{Q})$ is occasionally weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$. However,

$$\frac{7}{18} = d\left(\mathcal{Q}\mathcal{Q}(2), \mathcal{P}(2)\right) > d\left(\mathcal{P}\mathcal{Q}(2), \mathcal{Q}(2)\right) = \frac{1}{4},$$

then, $\mathcal{P}$ and $\mathcal{Q}$ are not weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$.

On the other hand we have

$$\frac{1}{4} = d\left(\mathcal{P}\mathcal{P}(2), \mathcal{Q}(2)\right) \leq d\left(\mathcal{Q}\mathcal{P}(2), \mathcal{P}(2)\right) = \frac{7}{18};$$

that is, the pair $(\mathcal{P}, \mathcal{Q})$ is occasionally weakly $\mathcal{P}$-biased of type $(\mathbb{A})$. But, as

$$\frac{8}{81} = d\left(\mathcal{P}\mathcal{P}(\frac{1}{3}), \mathcal{Q}(\frac{1}{3})\right) > d\left(\mathcal{Q}\mathcal{P}(\frac{1}{3}), \mathcal{P}(\frac{1}{3})\right) = 0;$$

that is, the pair $(\mathcal{P}, \mathcal{Q})$ is not weakly $\mathcal{P}$-biased of type $(\mathbb{A})$.

Again, we have

$$\mathcal{P}\mathcal{Q}(\frac{1}{3}) = \frac{1}{81} \neq \frac{1}{9} = \mathcal{Q}\mathcal{P}(\frac{1}{3}),$$
$$\mathcal{P}\mathcal{Q}(2) = \frac{1}{4} \neq \frac{1}{9} = \mathcal{Q}\mathcal{P}(2),$$

that is, $\mathcal{P}$ and $\mathcal{Q}$ are neither occasionally weakly compatible nor weakly compatible.

Now, recall that a non-Archimedean metric space is a special kind of metric space in which the triangle inequality is replaced with

$$d(\mu_1, \mu_2) \leq \max\{d(\mu_1, \mu_3), d(\mu_3, \mu_2)\}.$$

Sometimes the associated metric is also called a non-Archimedean metric or an ultra-metric (see [2]).

Example 1.19. Let $\mathcal{X} = [0, \infty)$ with the ultra metric space

$$d(\vartheta_1, \vartheta_2) = \begin{cases} 0 & \text{if } \vartheta_1 = \vartheta_2 \\ 1 & \text{if } \vartheta_1 \neq \vartheta_2. \end{cases}$$

Define $\mathcal{P}, \mathcal{Q} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{P}\mu_1 = \begin{cases} \frac{1}{16} & \text{if } \mu_1 \in [0, 1) \\ \mu_1 & \text{if } \mu_1 \in [1, \infty), \end{cases}$$

$$\mathcal{Q}\mu_1 = \begin{cases} 9 & \text{if } \mu_1 \in [0, \frac{1}{16}], \\ \frac{\mu_1}{4} & \text{if } \mu_1 \in (\frac{1}{16}, 1), \\ \mu_1^2 & \text{if } \mu_1 \in [1, \infty). \end{cases}$$

Now, we have $\mathcal{P}\mu_1 = \mathcal{Q}\mu_1$ if $\mu_1 = 1$ or $\mu_1 = \frac{1}{4}$

$$d(\mathcal{Q}\mathcal{Q}(1), \mathcal{P}(1)) = d(1, 1) = 0$$

$$d(\mathcal{P}\mathcal{Q}(1), \mathcal{Q}(1)) = d(1, 1) = 0$$

$d(\mathcal{Q}\mathcal{Q}(1), \mathcal{P}(1)) \leq d(\mathcal{P}\mathcal{Q}(1), \mathcal{Q}(1))$. But, for $\mu_1 = 3$

$$d\left(\mathcal{Q}\mathcal{Q}(\frac{1}{4}), \mathcal{P}(\frac{1}{4})\right) = d\left(9, \frac{1}{16}\right) = 1$$

$$d\left(\mathcal{P}\mathcal{Q}(\frac{1}{4}), \mathcal{Q}(\frac{1}{4})\right) = d\left(\frac{1}{16}, \frac{1}{16}\right) = 0.$$

Hence $d(\mathcal{Q}\mathcal{Q}(\frac{1}{4}), \mathcal{P}(\frac{1}{4})) > d(\mathcal{P}\mathcal{Q}(\frac{1}{4}), \mathcal{Q}(\frac{1}{4}))$ then, $\mathcal{P}$ and $\mathcal{Q}$ are not weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$; that is, the pair $(\mathcal{P}, \mathcal{Q})$ is occasionally weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$.

Now, we are ready to present our main results.

2. The Main Results

let $\mathfrak{F}$ be a family of all continuous increasing functions $\mathcal{F} : [0,\infty)^6 \to [0,\infty)$ such that $\mathcal{F}$ satisfies the following conditions:

- $\mathcal{F}(1,1,0,0,1,1)$, $\mathcal{F}(1,1,1,0,1,1)$ and $\mathcal{F}(1,1,0,1,1,1) \in [0,1)$,
- $\mathcal{F}(\lambda\omega) = \lambda\mathcal{F}(\omega)$ where $\omega = (\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6)$ and $\lambda \geq 0$.

2.1. **A Unique Common Fixed Point Theorem for Four Maps.**

Theorem 2.1. *Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ be self-mappings of an ultrametric space $(\mathcal{X}, d)$ satisfying the following condition:*

$$d(\mathcal{P}\mu_1,\mathcal{Q}\mu_2) \leq \mathcal{F}(d(\mathcal{P}\mu_1,\mathcal{Q}\mu_2), d(\mathcal{R}\mu_1,\mathcal{S}\mu_2), d(\mathcal{P}\mu_1,\mathcal{R}\mu_1), d(\mathcal{Q}\mu_2,\mathcal{S}\mu_2), d(\mathcal{R}\mu_1,\mathcal{Q}\mu_2), d(\mathcal{P}\mu_1,\mathcal{S}\mu_2)) \tag{2.1}$$

for all μ_1, $\mu_2 \in \mathcal{X}$, where $\mathcal{F} \in \mathfrak{F}$. If $\mathcal{P}$ and $\mathcal{R}$ as well as $\mathcal{Q}$ and $\mathcal{S}$ are occasionally weakly $\mathcal{R}$-biased of type $(\mathbb{A})$ and occasionally weakly $\mathcal{S}$-biased of type $(\mathbb{A})$, respectively, then $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ have a unique common fixed point.

Proof. By hypotheses, there are two points ϑ_1 and ϑ_2 in $\mathcal{X}$ such that $\mathcal{P}\vartheta_1 = \mathcal{R}\vartheta_1$ implies

$$d(\mathcal{R}\mathcal{R}\vartheta_1,\mathcal{P}\vartheta_1) \leq d(\mathcal{P}\mathcal{R}\vartheta_1,\mathcal{R}\vartheta_1)$$

and $\mathcal{Q}\vartheta_2 = \mathcal{S}\vartheta_2$ implies

$$d(\mathcal{S}\mathcal{S}\vartheta_2,\mathcal{Q}\vartheta_2) \leq d(\mathcal{Q}\mathcal{S}\vartheta_2,\mathcal{S}\vartheta_2).$$

First, we are going to prove that $\mathcal{P}\vartheta_2 = \mathcal{Q}\vartheta_2$. From inequality (2.1) we have

$$\begin{aligned} d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2) &\leq \mathcal{F}(d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{R}\vartheta_1,\mathcal{S}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{R}\vartheta_1), d(\mathcal{Q}\vartheta_2,\mathcal{S}\vartheta_2),\\ &\quad d(\mathcal{R}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{S}\vartheta_2)) \end{aligned}$$

$$\begin{aligned} d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2) &\leq \mathcal{F}(d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{R}\vartheta_1,\mathcal{S}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{R}\vartheta_1), d(\mathcal{Q}\vartheta_2,\mathcal{S}\vartheta_2),\\ &\quad d(\mathcal{R}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{S}\vartheta_2)) \end{aligned}$$

$$\begin{aligned} \Rightarrow d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2) &\leq \mathcal{F}(d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{P}\vartheta_1), d(\mathcal{Q}\vartheta_2,\mathcal{Q}\vartheta_2),\\ &\quad d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2)) \end{aligned}$$

$$\Rightarrow d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2) \leq d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2)\mathcal{F}(1,1,0,0,1,1).$$

As we know that $\mathcal{F}(1,1,0,0,1,1) \in [0,1)$ which implies that $d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2) < d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2)$, a contradiction. Hence, $\mathcal{P}\vartheta_1 = \mathcal{Q}\vartheta_2$ implies that $\mathcal{P}\vartheta_1 = \mathcal{R}\vartheta_1 = \mathcal{Q}\vartheta_2 = \mathcal{S}\vartheta_2$.

Now, we assert that $\mathcal{P}\mathcal{P}\vartheta_1 = \mathcal{P}\vartheta_1$. If not, then the use of condition (2.1) gives

$$\begin{aligned} d(\mathcal{P}\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2) &\leq \mathcal{F}(d(\mathcal{P}\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{R}\mathcal{P}\vartheta_1,\mathcal{S}\vartheta_2), d(\mathcal{P}\mathcal{P}\vartheta_1,\mathcal{R}\mathcal{P}\vartheta_1),\\ &\quad d(\mathcal{Q}\vartheta_2,\mathcal{S}\vartheta_2), d(\mathcal{R}\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{P}\mathcal{P}\vartheta_1,\mathcal{S}\vartheta_2)) \end{aligned}$$

$$\begin{aligned} \Rightarrow d(\mathcal{P}\mathcal{P}\vartheta_1,\mathcal{P}\vartheta_1) &\leq \mathcal{F}(d(\mathcal{P}\mathcal{P}\vartheta_1,\mathcal{P}\vartheta_1), d(\mathcal{R}\mathcal{P}\vartheta_1,\mathcal{P}\vartheta_1), d(\mathcal{P}\mathcal{P}\vartheta_1,\mathcal{R}\mathcal{P}\vartheta_1),\\ &\quad d(\mathcal{P}\vartheta_1,\mathcal{R}\vartheta_1), d(\mathcal{R}\mathcal{P}\vartheta_1,\mathcal{P}\vartheta_1), d(\mathcal{P}\mathcal{P}\vartheta_1,\mathcal{P}\vartheta_1)). \end{aligned}$$

$$\begin{aligned} d(\mathcal{R}\mathcal{P}\vartheta_1,\mathcal{P}\vartheta_1) &= d(\mathcal{R}\mathcal{R}\vartheta_1,\mathcal{P}\vartheta_1)\\ &\leq d(\mathcal{P}\mathcal{R}\vartheta_1,\mathcal{R}\vartheta_1)\\ &= d(\mathcal{P}\mathcal{P}\vartheta_1,\mathcal{P}\vartheta_1). \end{aligned}$$

Also, we have

$$\begin{aligned} d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{R}\mathcal{P}\vartheta_1) &\leq \max\{d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\vartheta_1, \mathcal{R}\mathcal{P}\vartheta_1)\} \\ &= \max\{d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\vartheta_1, \mathcal{R}\mathcal{R}\vartheta_1)\} \\ &\leq \max\{d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{R}\vartheta_1, \mathcal{P}\mathcal{R}\vartheta_1)\} \\ &= d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1). \end{aligned}$$

Therefore

$$\begin{aligned} d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1) \leq\ & \mathcal{F}(d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), 0, \\ & d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1)) \end{aligned}$$

$$\Rightarrow d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1) \leq d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1)\mathcal{F}(1, 1, 1, 0, 1, 1)$$

as we know that $\mathcal{F}(1, 1, 1, 0, 1, 1) \in [0, 1)$ which implies that $d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1) < d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1)$, a contradiction. Hence, $\mathcal{P}\mathcal{P}\vartheta_1 = \mathcal{P}\vartheta_1$.

Suppose that $\mathcal{Q}\mathcal{Q}\vartheta_2 \neq \mathcal{Q}\vartheta_2$. Using inequality (2.1) we obtain

$$\begin{aligned} d(\mathcal{P}\vartheta_1, \mathcal{Q}\mathcal{Q}\vartheta_2) \leq\ & \mathcal{F}(d(\mathcal{P}\vartheta_1, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{R}\vartheta_1, \mathcal{S}\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1, \mathcal{R}\vartheta_1), \\ & d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2), d(\mathcal{R}\vartheta_1, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1, \mathcal{S}\mathcal{Q}\vartheta_2)) \end{aligned}$$

$$\begin{aligned} \Rightarrow d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2) \leq\ & \mathcal{F}(d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2), 0, d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2), \\ & d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2)). \end{aligned}$$

As the pair $(\mathcal{Q}, \mathcal{S})$ is occasionally weakly $\mathcal{S}$-biased of type $(\mathbb{A})$, we have

$$\begin{aligned} d(\mathcal{S}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2) &= d(\mathcal{S}\mathcal{S}\vartheta_2, \mathcal{Q}\vartheta_2) \\ &\leq d(\mathcal{Q}\mathcal{S}\vartheta_2, \mathcal{S}\vartheta_2) \\ &= d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2). \end{aligned}$$

Again we have

$$\begin{aligned} d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2) &\leq \max\{d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2)\} \\ &= \max\{d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{S}\vartheta_2)\} \\ &\leq \max\{d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2), d(\mathcal{S}\vartheta_2, \mathcal{Q}\mathcal{S}\vartheta_2)\} \\ &= d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2), \end{aligned}$$

$$\begin{aligned} \Rightarrow d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2) \leq\ & \mathcal{F}(d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), 0, d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), \\ & d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2)) \end{aligned}$$

$$\Rightarrow d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2) \leq d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2)\mathcal{F}(1, 1, 0, 1, 1, 1).$$

As we know that $\mathcal{F}(1, 1, 0, 1, 1, 1) \in [0, 1)$ $d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2) < d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2)$, a contradiction. Hence, $\mathcal{Q}\vartheta_2 = \mathcal{Q}\mathcal{Q}\vartheta_2$. So $\mathcal{S}\mathcal{Q}\vartheta_2 = \mathcal{Q}\vartheta_2$; that is, $\mathcal{Q}\mathcal{P}\vartheta_1 = \mathcal{P}\vartheta_1$ and $\mathcal{S}\mathcal{P}\vartheta_1 = \mathcal{P}\vartheta_1$. Put $\mathcal{P}\vartheta_1 = \mathcal{R}\vartheta_1 = \mathcal{Q}\vartheta_2 = \mathcal{S}\vartheta_2 = \vartheta_3$, therefore ϑ_3 is a common fixed point of mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$.

Finally, let ϑ_3 and ϑ_4 be two distinct common fixed points of mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$. Then, $\vartheta_3 = \mathcal{P}\vartheta_3 = \mathcal{Q}\vartheta_3 = \mathcal{R}\vartheta_3 = \mathcal{S}\vartheta_3$ and $\vartheta_4 = \mathcal{P}\vartheta_4 = \mathcal{Q}\vartheta_4 = \mathcal{R}\vartheta_4 = \mathcal{S}\vartheta_4$. From (2.1) we have

$$\begin{aligned} d(\mathcal{P}\vartheta_3, \mathcal{Q}\vartheta_4) \quad &\leq \quad \mathcal{F}(d(\mathcal{P}\vartheta_3, \mathcal{Q}\vartheta_4), d(\mathcal{R}\vartheta_3, \mathcal{S}\vartheta_4), d(\mathcal{P}\vartheta_3, \mathcal{R}\vartheta_3), d(\mathcal{Q}\vartheta_4, \mathcal{S}\vartheta_4), \\ &\quad\quad d(\mathcal{R}\vartheta_3, \mathcal{Q}\vartheta_4), d(\mathcal{P}\vartheta_3, \mathcal{S}\vartheta_4)) \end{aligned}$$

$$\begin{aligned} \Rightarrow d(\mathcal{P}\vartheta_3, \mathcal{Q}\vartheta_4) \quad &\leq \quad \mathcal{F}(d(\mathcal{P}\vartheta_3, \mathcal{Q}\vartheta_4), d(\mathcal{P}\vartheta_3, \mathcal{Q}\vartheta_4), 0, 0, d(\mathcal{P}\vartheta_3, \mathcal{Q}\vartheta_4), \\ &\quad\quad d(\mathcal{P}\vartheta_3, \mathcal{Q}\vartheta_4)) \end{aligned}$$

$$\Rightarrow d(\mathcal{P}\vartheta_3, \mathcal{Q}\vartheta_4) \leq d(\mathcal{P}\vartheta_3, \mathcal{Q}\vartheta_4)\mathcal{F}(1, 1, 0, 0, 1, 1)$$

which implies that $\vartheta_3 = \vartheta_4$. Hence $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ have a unique common fixed point. □

Example 2.2. Let $\mathcal{X} = [0, \infty)$ with the ultra metric space

$$d(\vartheta_1, \vartheta_2) = \begin{cases} 0 & \text{if } \vartheta_1 = \vartheta_2 \\ 1 & \text{if } \vartheta_1 \neq \vartheta_2. \end{cases}$$

Define $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{P}\mu_1 = \begin{cases} \frac{1}{25} & \text{if } \mu_1 \in [0, \frac{1}{5}] \\ \frac{\mu_1}{3} & \text{if } \mu_1 \in (\frac{1}{5}, 1) \\ \mu_1 & \text{if } \mu_1 \in [1, \infty), \end{cases} \quad \mathcal{Q}\mu_1 = \begin{cases} \frac{\mu_1}{5} & \text{if } \mu_1 \in [0, \frac{1}{5}], \\ \frac{1}{9} & \text{if } \mu_1 \in [\frac{1}{5}, 1), \\ \mu_1 & \text{if } \mu_1 \in [1, \infty). \end{cases}$$

$$\mathcal{R}\mu_1 = \begin{cases} 2 & \text{if } \mu_1 \in [0, \frac{1}{25}], \\ \frac{\mu_1}{5} & \text{if } \mu_1 \in (\frac{1}{25}, 1), \\ \mu_1^3 & \text{if } \mu_1 \in [1, \infty), \end{cases} \quad \mathcal{S}\mu_1 = \begin{cases} 4 & \text{if } \mu_1 \in [0, \frac{1}{9}] \\ \frac{\mu_1}{3} & \text{if } \mu_1 \in (\frac{1}{9}, 1) \\ \mu_1^2 & \text{if } \mu_1 \in [1, \infty). \end{cases}$$

and let $\mathcal{F}(t_1, t_2, t_3, t_4, t_5, t_6) = \frac{t_1}{3} + \frac{t_2}{5} + \frac{11t_3}{18} + \frac{11t_4}{18} + \frac{t_5}{8} + \frac{t_6}{9}$.

Now, we have $\mathcal{P}\mu_1 = \mathcal{R}\mu_1$ if $\mu_1 = 1$ or $\mu_1 = \frac{1}{5}$ and $\mathcal{Q}\mu_1 = \mathcal{S}\mu_1$ if $\mu_1 = 1$ or $\mu_1 = \frac{1}{3}$

$$d(\mathcal{R}\mathcal{R}(1), \mathcal{P}(1)) = d(1, 1) = 0$$

$$d(\mathcal{P}\mathcal{R}(1), \mathcal{R}(1)) = d(1, 1) = 0$$

$d(\mathcal{R}\mathcal{R}(1), \mathcal{P}(1)) \leq d(\mathcal{P}\mathcal{R}(1), \mathcal{R}(1))$. But, for $\mu_1 = \frac{1}{5}$

$$d\left(\mathcal{R}\mathcal{R}(\frac{1}{5}), \mathcal{P}(\frac{1}{5})\right) = d\left(2, \frac{1}{25}\right) = 1$$

$$d\left(\mathcal{P}\mathcal{R}(\frac{1}{5}), \mathcal{R}(\frac{1}{5})\right) = d\left(\frac{1}{25}, \frac{1}{25}\right) = 0.$$

Hence $d\left(\mathcal{R}\mathcal{R}(\frac{1}{5}), \mathcal{P}(\frac{1}{5})\right) > d\left(\mathcal{P}\mathcal{R}(\frac{1}{5}), \mathcal{R}(\frac{1}{5})\right)$ then, $\mathcal{P}$ and $\mathcal{R}$ are not weakly $\mathcal{R}$-biased of type $(\mathbb{A})$; that is, the pair $(\mathcal{P}, \mathcal{R})$ is occasionally weakly $\mathcal{R}$-biased of type $(\mathbb{A})$. Similarly, for $\mu_1 = 1$

$$d(\mathcal{S}\mathcal{S}(1), \mathcal{Q}(1)) = d(1, 1) = 0$$

$$d(\mathcal{Q}\mathcal{S}(1), \mathcal{S}(1)) = d(1, 1) = 0.$$

Now for $\mu_1 = \frac{1}{3}$

$$d\left(\mathcal{S}\mathcal{S}(\frac{1}{3}), \mathcal{R}(\frac{1}{3})\right) = d\left(4, \frac{1}{9}\right) = 1$$

$$d\left(\mathcal{Q}\mathcal{S}(\frac{1}{3}), \mathcal{S}(\frac{1}{3})\right) = d\left(\frac{1}{9}, \frac{1}{9}\right) = 0.$$

Hence $d\left(\mathcal{SS}(\frac{1}{3}),\mathcal{R}(\frac{1}{3})\right) > d\left(\mathcal{QS}(\frac{1}{3}),\mathcal{S}(\frac{1}{3})\right)$ then, $\mathcal{Q}$ and $\mathcal{S}$ are not weakly $\mathcal{Q}$-biased of type $(\mathbb{A})$; that is, the pair $(\mathcal{Q},\mathcal{S})$ is occasionally weakly $\mathcal{S}$-biased of type $(\mathbb{A})$. As we can clearly see that $\mathcal{P}$, $\mathcal{Q}$ $\mathcal{R}$ and $\mathcal{S}$ satisfy the given condition (2.1). Hence $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ have a unique common fixed point at $\mu_1 = 1$.

Remark 2.3. We have evidenced common fixed point theorems for two pairs of self mappings in a non-complete ultrametric space via implicit relations without utilizing continuity or its variant like reciprocal continuity, weak reciprocal continuity, conditional reciprocal continuity, sub-sequential continuity, sequential continuity of type $(A\mathcal{P})$, or $(A\mathcal{R})$, and so on (see, Tomar and Karapinar [20]). Further, the weaker variant of commutativity (see, Singh and Tomar [19]), namely the notion of occasionally weakly $\mathcal{R}$-biased (k-biased) of type $(\mathbb{A})$ has been utilized for the existence of common fixed point at the point of discontinuity of self mappings. Consequently, we respond to open question about the survival of contractive condition which assures the common fixed point at the discontinuity of self mappings in an ultrametric space (see, Rhoades [17]). Our conclusions extend, generalize, and refine the conclusions of Banach [4], Chatterjea [7], Ćirić [8] and so on to the ultrametric space.

Corollary 2.4. *Let $\mathcal{P}$ and $\mathcal{R}$ be self-mappings of an ultrametric space $(\mathcal{X},d)$ satisfying the following condition:*

$$\begin{aligned} d(\mathcal{P}\mu_1,\mathcal{P}\mu_2) \quad &\leq \quad \mathcal{F}(d(\mathcal{P}\mu_1,\mathcal{P}\mu_2), d(\mathcal{R}\mu_1,\mathcal{R}\mu_2), d(\mathcal{P}\mu_1,\mathcal{R}\mu_1),\\ &\qquad d(\mathcal{P}\mu_2,\mathcal{R}\mu_2), d(\mathcal{R}\mu_1,\mathcal{P}\mu_2), d(\mathcal{P}\mu_1,\mathcal{R}\mu_2)). \end{aligned} \tag{2.2}$$

If $\mathcal{P}$ and $\mathcal{R}$ are occasionally weakly $\mathcal{R}$-biased of type $(\mathbb{A})$, then $\mathcal{P}$ and $\mathcal{R}$ have a unique common fixed point.

2.2. **Implicit Relations.** let Φ be a family of all functions $\varphi : \mathbb{R}_+^6 \to \mathbb{R}$ such that φ is non-increasing in variables t_2, t_3, t_4, t_5 and t_6, and satisfies the next condition:

$$\varphi(t,t,0,0,t,t) \text{ and } \varphi(t,t,t,0,t,t) \text{ and } \varphi(t,t,0,t,t,t)$$

are positive for all t positive.

Theorem 2.5. *Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ be self-mappings of an ultrametric space $(\mathcal{X},d)$ satisfying the following condition:*

$$\begin{aligned} &\varphi(d(\mathcal{P}\mu_1,\mathcal{Q}\mu_2), d(\mathcal{R}\mu_1,\mathcal{S}\mu_2), d(\mathcal{P}\mu_1,\mathcal{R}\mu_1),\\ &d(\mathcal{Q}\mu_2,\mathcal{S}\mu_2), d(\mathcal{R}\mu_1,\mathcal{Q}\mu_2), d(\mathcal{P}\mu_1,\mathcal{S}\mu_2)) \leq 0 \end{aligned} \tag{2.3}$$

for all μ_1, $\mu_2 \in \mathcal{X}$, where $\varphi \in \Phi$. If $\mathcal{P}$ and $\mathcal{R}$ as well as $\mathcal{Q}$ and $\mathcal{S}$ are occasionally weakly $\mathcal{R}$-biased of type $(\mathbb{A})$ and occasionally weakly $\mathcal{S}$-biased of type $(\mathbb{A})$, respectively, then $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ have a unique common fixed point.

Proof. By hypotheses, there are two points ϑ_1 and ϑ_2 in $\mathcal{X}$ such that $\mathcal{P}\vartheta_1 = \mathcal{R}\vartheta_1$ implies

$$d(\mathcal{RR}\vartheta_1,\mathcal{P}\vartheta_1) \leq d(\mathcal{PR}\vartheta_1,\mathcal{R}\vartheta_1)$$

and $\mathcal{Q}\vartheta_2 = \mathcal{S}\vartheta_2$ implies

$$d(\mathcal{SS}\vartheta_2,\mathcal{Q}\vartheta_2) \leq d(\mathcal{QS}\vartheta_2,\mathcal{S}\vartheta_2).$$

First, we are going to prove that $\mathcal{P}\vartheta_1 = \mathcal{Q}\vartheta_2$. From inequality (2.3) we have

$$\begin{aligned} &\varphi(d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{R}\vartheta_1,\mathcal{S}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{R}\vartheta_1), d(\mathcal{Q}\vartheta_2,\mathcal{S}\vartheta_2), d(\mathcal{R}\vartheta_1,\mathcal{Q}\vartheta_2),\\ &d(\mathcal{P}\vartheta_1,\mathcal{S}\vartheta_2))\\ = \quad &\varphi(d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), 0, 0, d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1,\mathcal{Q}\vartheta_2)) \leq 0, \end{aligned}$$

which implies that $\mathcal{P}\vartheta_1 = \mathcal{Q}\vartheta_2$.

Now, we assert that $\mathcal{P}\mathcal{P}\vartheta_1 = \mathcal{P}\vartheta_1$. If not, then the use of condition (2.3) gives

$$\begin{aligned} & \varphi(d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{Q}\vartheta_2), d(\mathcal{R}\mathcal{P}\vartheta_1, \mathcal{S}\vartheta_2), d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{R}\mathcal{P}\vartheta_1), d(\mathcal{Q}\vartheta_2, \mathcal{S}\vartheta_2), \\ & d(\mathcal{R}\mathcal{P}\vartheta_1, \mathcal{Q}\vartheta_2), d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{S}\vartheta_2)) \\ = \; & \varphi(d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{R}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{R}\mathcal{P}\vartheta_1), 0, d(\mathcal{R}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), \\ & d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1)) \leq 0. \end{aligned}$$

Since the pair $(\mathcal{P}, \mathcal{R})$ is occasionally weakly $\mathcal{R}$-biased of type $(\mathbb{A})$ we have $d(\mathcal{R}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1) = d(\mathcal{R}\mathcal{R}\vartheta_1, \mathcal{P}\vartheta_1) \leq d(\mathcal{P}\mathcal{R}\vartheta_1, \mathcal{R}\vartheta_1) = d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1)$. Also, we have

$$\begin{aligned} d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{R}\mathcal{P}\vartheta_1) & \leq \max\{d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\vartheta_1, \mathcal{R}\mathcal{P}\vartheta_1)\} \\ & = \max\{d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\vartheta_1, \mathcal{R}\mathcal{R}\vartheta_1)\} \\ & \leq \max\{d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{R}\vartheta_1, \mathcal{P}\mathcal{R}\vartheta_1)\} \\ & = d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), \end{aligned}$$

and as φ is non-increasing in t_2, t_3 and t_5, we get

$$\begin{aligned} & \varphi(d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), 0, d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), \\ & d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1)) \\ \leq \; & \varphi(d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{R}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{R}\mathcal{P}\vartheta_1), 0, d(\mathcal{R}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1), \\ & d(\mathcal{P}\mathcal{P}\vartheta_1, \mathcal{P}\vartheta_1)) \leq 0, \end{aligned}$$

which implies that $\mathcal{P}\mathcal{P}\vartheta_1 = \mathcal{P}\vartheta_1$ and so $\mathcal{R}\mathcal{P}\vartheta_1 = \mathcal{P}\vartheta_1$.

Suppose that $\mathcal{Q}\mathcal{Q}\vartheta_2 \neq \mathcal{Q}\vartheta_2$. Using inequality (2.3) we obtain

$$\begin{aligned} & \varphi(d(\mathcal{P}\vartheta_1, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{R}\vartheta_1, \mathcal{S}\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1, \mathcal{R}\vartheta_1), d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2), \\ & d(\mathcal{R}\vartheta_1, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{P}\vartheta_1, \mathcal{S}\mathcal{Q}\vartheta_2)) \\ = \; & \varphi(d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2), 0, d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), \\ & d(\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2)) \leq 0. \end{aligned}$$

As the pair $(\mathcal{Q}, \mathcal{S})$ is occasionally weakly $\mathcal{S}$-biased of type $(\mathbb{A})$, we have $d(\mathcal{S}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2) = d(\mathcal{S}\mathcal{S}\vartheta_2, \mathcal{Q}\vartheta_2) \leq d(\mathcal{Q}\mathcal{S}\vartheta_2, \mathcal{S}\vartheta_2) = d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2)$. Again we have

$$\begin{aligned} d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2) & \leq \max\{d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2)\} \\ & = \max\{d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{S}\vartheta_2)\} \\ & \leq \max\{d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2), d(\mathcal{S}\vartheta_2, \mathcal{Q}\mathcal{S}\vartheta_2)\} \\ & = d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{Q}\vartheta_2), \end{aligned}$$

and since φ is non-increasing in t_2, t_4 and t_6, we get

$$\begin{aligned} & \varphi(d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), 0, d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), \\ & d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2)) \\ \leq \; & \varphi(d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2), 0, d(\mathcal{Q}\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2), d(\mathcal{Q}\vartheta_2, \mathcal{Q}\mathcal{Q}\vartheta_2), \\ & d(\mathcal{Q}\vartheta_2, \mathcal{S}\mathcal{Q}\vartheta_2)) \leq 0, \end{aligned}$$

which implies that $\mathcal{Q}\mathcal{Q}\vartheta_2 = \mathcal{Q}\vartheta_2$ and so $\mathcal{S}\mathcal{Q}\vartheta_2 = \mathcal{Q}\vartheta_2$; that is, $\mathcal{Q}\mathcal{P}\vartheta_1 = \mathcal{P}\vartheta_1$ and $\mathcal{S}\mathcal{P}\vartheta_1 = \mathcal{P}\vartheta_1$. Put $\mathcal{P}\vartheta_1 = \mathcal{R}\vartheta_1 = \mathcal{Q}\vartheta_2 = \mathcal{S}\vartheta_2 = \vartheta_3$, therefore ϑ_3 is a common fixed point of mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$.

Finally, let ϑ_3 and ϑ_4 be two distinct common fixed points of mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$. Then, $\vartheta_3 = \mathcal{P}\vartheta_3 = \mathcal{Q}\vartheta_3 = \mathcal{R}\vartheta_3 = \mathcal{S}\vartheta_3$ and $\vartheta_4 = \mathcal{P}\vartheta_4 = \mathcal{Q}\vartheta_4 = \mathcal{R}\vartheta_4 = \mathcal{S}\vartheta_4$. From (2.3) we have

$$\begin{aligned}&\varphi(d(\mathcal{P}\vartheta_4, \mathcal{Q}\vartheta_3), d(\mathcal{R}\vartheta_4, \mathcal{S}\vartheta_3), d(\mathcal{P}\vartheta_4, \mathcal{R}\vartheta_4), d(\mathcal{Q}\vartheta_3, \mathcal{S}\vartheta_3), d(\mathcal{R}\vartheta_4, \mathcal{Q}\vartheta_3),\\ &d(\mathcal{P}\vartheta_4, \mathcal{S}\vartheta_3))\\ &= \varphi(d(\vartheta_4, \vartheta_3), d(\vartheta_4, \vartheta_3), 0, 0, d(\vartheta_4, \vartheta_3), d(\vartheta_4, \vartheta_3)) \leq 0\end{aligned}$$

which implies that $\vartheta_3 = \vartheta_4$. □

If $\mathcal{P} = \mathcal{Q}$ and $\mathcal{R} = \mathcal{S}$ then

Corollary 2.6. *Let $\mathcal{P}$ and $\mathcal{R}$ be self-mappings of an ultrametric space $(\mathcal{X}, d)$ satisfying the following condition*

$$\begin{aligned}&\varphi(d(\mathcal{P}\mu_1, \mathcal{P}\mu_2), d(\mathcal{R}\mu_1, \mathcal{R}\mu_2), d(\mathcal{P}\mu_1, \mathcal{R}\mu_1),\\ &d(\mathcal{P}\mu_2, \mathcal{R}\mu_2), d(\mathcal{R}\mu_1, \mathcal{P}\mu_2), d(\mathcal{P}\mu_1, \mathcal{R}\mu_2)) \leq 0\end{aligned} \tag{2.4}$$

for all μ_1, $\mu_2 \in \mathcal{X}$, where $\varphi \in \Phi$. If $\mathcal{P}$ and $\mathcal{R}$ are occasionally weakly $\mathcal{R}$-biased of type $(\mathbb{A})$, then $\mathcal{P}$ and $\mathcal{R}$ have a unique common fixed point.

Conclusion. We have established the existence and uniqueness of a common fixed point of two pairs of occasionally weakly biased of type (A) mappings which are not even continuous at a common fixed point in a non-complete ultrametric space via implicit relations. It is interesting to see that an ultrametric space is fascinating, generalized, and more distinct than a usual metric space due to the enhanced triangle inequality which says that $d(\mu_1, \mu_2)$ and $d(\mu_2, \mu_3)$ can not both be strictly less than $d(\mu_1, \mu_3)$ for any μ_1, μ_2, $\mu_3 \in M$. However, the standard metric on the real line does not have this property. Our theorems and corollaries are sharpened versions of the well-known results, wherein completeness and continuity have not been utilized to prove a common fixed point. Example 3.1 substantiates the utility of these extensions. Consequently, we have provided a novel explanation to the Rhoades problem [17] on the question of the existence of a contractive mapping having the fixed point at the point of discontinuity in a non-complete ultrametric space.

Competing interests. The authors declare no competing interests.

References

[1] R.P. Agarwal, R.K. Bisht, N. Shahzad, A Comparison of Various Noncommuting Conditions in Metric Fixed Point Theory and Their Applications, Fixed Point Theory Appl. 2014 (2014), 38. https://doi.org/10.1186/1687-1812-2014-38.

[2] Z. Alinejad, A.K. Mirmostafaee, Common Fixed Points of Commuting Mappings in Ultrametric Spaces, Mat. Vesnik 68 (2016), 204–214.

[3] M.A. Al-Thagafi, N. Shahzad, Generalized I-nonexpansive Selfmaps and Invariant Approximations, Acta Math. Sin. Engl. Ser. 24 (2008), 867–876. https://doi.org/10.1007/s10114-007-5598-x.

[4] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux éQuations Intégrales, Fundam. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[5] H. Bouhadjera, Fixed Points for Occasionally Weakly Biased Mappings of Type (A), Math. Moravica 26 (2022), 113–122. https://doi.org/10.5937/matmor2201113b.

[6] H. Bouhadjera, A. Djoudi, Fixed Point for Occasionally Weakly Biased Maps, Southeast Asian. Bull. Math. 36 (2012), 489–500.

[7] S.K. Chatterjea, Fixed-point theorems, C. R. Acad., Bulgare Sci. 25 (1972), 727–730.

[8] L.B. Ciric, A Generalization of Banach's Contraction Principle, Proc. Amer. Math. Soc. 45 (1974), 267–273. https://doi.org/10.2307/2040075.

[9] R.M. Fréchet, Sur Quelques Points du Calcul Fonctionnel, Rend. Circ. Mat. Palermo 22 (1906), 1–74.

[10] L. Gajić, On Ultrametric Space, Novi Sad J. Math. 31 (2001), 69–71.

[11] G. Jungck, Compatible Mappings and Common Fixed Points, Int. J. Math. Math. Sci. 9 (1986), 771–779. https://doi.org/10.1155/s0161171286000935.

[12] G. Jungck, Common Fixed Points for Noncontinuous Nonself Maps on Nonmetric Spaces, Far East J. Math. Sci. 4 (1996), 199–215.

[13] G. Jungck, P.P. Murthy, Y.J. Cho, Compatible Mappings of Type $(\mathbb{A})$ and Common Fixed Points, Math. Japon. 38 (1993), 381–390.

[14] G. Jungck, H.K. Pathak, Fixed Points Via "Biased Maps", Proc. Am. Math. Soc. 123 (1995), 2049–2060. https://doi.org/10.2307/2160940.

[15] G. Jungck, B.E. Rhoades, Fixed Point Theorems for Occasionally Weakly Compatible Mappings, Fixed Point Theory 7 (2006), 287–296.

[16] H.K. Pathak, Y.J. Cho, S.M. Kang, Common Fixed Points of Biased Maps of Type $(\mathbb{A})$ and Applications, Int. J. Math. Math. Sci. 21 (1997), 681–693. https://doi.org/10.1155/s0161171298000945.

[17] B.E. Rhoades, Contractive Definitions and Continuity, Contemp. Math. (1988), 233–245. https://doi.org/10.1090/conm/072/956495.

[18] S. Sessa, On a Weak Commutativity Condition in Fixed Point Considerations, Publ. Inst. Math. (Beograd) (N.S.) 32 (1982), 149–153.

[19] S.L. Singh, A. Tomar, Weaker Forms of Commuting Maps and Existence of Fixed Points, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. 10 (2003), 145–161.

[20] A. Tomar, E. Karapinar, On Variants of Continuity and Existence of Fixed Point via Meir-Keeler Contractions in MC-Spaces, J. Adv. Math. Stud. 9 (2016), 348–359.

[21] H. Qawaqneh, Fractional Analytic Solutions and Fixed Point Results With Some Applications, Adv. Fixed Point Theory 14 (2024), 1. https://doi.org/10.28919/afpt/8279.