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TIME-FREQUENCY CONCENTRATION ASSOCIATED WITH THE MULTIDIMENSIONAL HANKEL-WAVELET TRANSFORM

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ABSTRACT. The main goal of this paper is to define a new integral transform called the multidimensional Hankel-wavelet transform and to give some new results related to this transform as generalized Parseval's, Plancherel's, inversion and Calderon's reproducing formulas. Next, we analyse the concentration of this transform on set of finite measure and we give uncertainty principle for orthonormal sequences. Last, we introduce a new class of pseudo-differential operator $\mathscr{L}_{u,v}(\sigma)$ called localization operator which depend on a symbol σ and two functions u and v, we give a criteria in terms of the symbol σ for its boundedness and compactness, we also show that this operator belongs to the Schatten-Von Neumann classes S^p for all $p \in [1; +\infty]$ and we give a trace formula.

1. INTRODUCTION

Time-frequency analysis [13] and uncertainty principles [10, 14, 15] play a fundamental role in field of mathematics and physics, these principes appear in harmonic analysis and signal theory in a various different forms involving not only the signal f and its Fourier transform $\hat{f}$, but also every representation of a signal in the time-frequency space.

The uncertainty principles are mathematical results that gives limitations on the simultaneous concentration of a signal and its Fourier transform and they have implications in signal analysis and quantum physics. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies of signal consists of.

Timelimited and bound limited functions are basic tools in signal analysis and imaging processing. In quantum physics they tell us that a particule's speed and position cannot both of them be measured with infinite presicion, the mathematical formulation of this principle is given by the Heisenberg-Pauli-Weyl sharp inequality see [20]. Other uncertainty relations have been investigated among them, we refer to the papers of Benedick's [1], Donoho-Stark's [10], Jaming's [14].

The multidimensional Bessel operator is an elliptic partial differential operator denoted by $\Delta_{\alpha,d}$ defined for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+, \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d, \alpha_k > -\frac{1}{2}; k = 1, \ldots, d$, by

$$\Delta_{\alpha,d} = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} + \frac{2\alpha_k + 1}{x_k} \frac{\partial}{\partial x_k}. \tag{1.1}$$

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The eigenfunctions of the operator (1.1) are related to the Bessel functions and they satisfies a product formula which permits to develop a new harmonic analysis associated to this operator for more information we refer the reader to [2,7,8].
The wavelet transform has been successfully used to analyse signals in numerous applications such as seismic recording, ground vibrations, geophysics, medical imaging, hydrology, gravitational waves, power system analysis and many other areas see [9,12,13].
For its importance, the mathematical theory of this transform is under development in different directions and many extensions of this transform have been proposed in recent years, for example in the Riemann-Liouville setting [3], the Laguerre-Bessel setting [4]. As harmonic analysis associated with the multidimensional Bessel operator (1.1), has known remarkable development, it is natural to ask whether there exist the equivalent of the time-frequency analysis for wavelet transform in the multidimensional Bessel setting.
The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the multidimensional Bessel operator (1.1), in section 3 we define the wavelet transform associated with the multidimensional Bessel operator (1.1) and we give new results related to this transform, in section 4 we give some uncertainty principles associated with the multidimensional Hankel-wavelet transform, the last section is devoted to introduce a new class of pseudo-differential operator $\mathscr{L}_{u,v}(\sigma)$ called localization operator which depend on a symbol σ and two functions u and v, we give a criteria in terms of the symbol σ for its boundedness and compactness, we also show that this operator belongs to the Schatten-Von Neumann classes S^p for all $p \in [1;+\infty]$ and we give a trace formula.

2. Harmonic Analysis Associated With the Multidimensional Hankel Transform

In this section we set some notations and we recall some results in harmonic analysis related to the multidimensional Hankel transform, for more details, we refer the reader to [2,7,8].
In the following we denote by

- $\mathbb{R}_+^d = \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d, x_1 > 0, x_2 > 0, \ldots, x_n > 0\right\}$, equipped with the weighted Lebesgue measure μ_α given by

$$\mathrm{d}\mu_\alpha(x) = \prod_{k=1}^{d} \frac{x_k^{2\alpha_k+1}}{2^{|\alpha|}\Gamma(\alpha_k+1)} dx_k, \quad \alpha_k > -1/2. \tag{2.1}$$

- $L_\alpha^p\left(\mathbb{R}_+^d\right), 1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}_+^d$ such that

$$\|f\|_{p,\mu_\alpha} = \left(\int_{\mathbb{R}_+^d} |f(x)|^p \,\mathrm{d}\mu_\alpha(x)\right)^{1/p} < \infty, \quad p \in [1,\infty),$$
$$\|f\|_{\infty,\mu_\alpha} = \operatorname{ess}\sup_{x\in\mathbb{R}_+^d} |f(x)| < \infty$$

2.1. The Eigenfunctions of the multidimensional Bessel operator.

The main purpose of this subsection is to define the eigenfunctions of the multidimensional Bessel operator $\Delta_{\alpha,d}$ which will be used to define the multidimensional Hankel transform.

2.1.1. *One dimensional case.* For $\alpha > -1/2$, the one dimensional Bessel operator is defined by

$$B_\alpha = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} \tag{2.2}$$

We recall that the normalized Bessel function of the first kind and order α is defined on $\mathbb{C}$ as follows

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty} \frac{(-1)^k (z)^{2k}}{2^{2k}k!\Gamma(\alpha+k+1)}, \tag{2.3}$$

we will use this function to define the eigenfunctions of the multidimensional Bessel operator.

2.1.2. *The multidimensional case.* For x, λ, we put

$$\psi_{\alpha,d}(\lambda x) = \prod_{k=1}^{d} j_{\alpha_k}\left(\lambda_k x_k\right), \tag{2.4}$$

where j_{α_k} is the Bessel function given by (2.3), from [2,7] we have the following results

Proposition 2.1. *the function $\psi_\alpha(\lambda.)$ is the unique solution of the following Cauchy problem*

$$\begin{cases} \Delta_{\alpha,d} u = -\|\lambda\|^2 u, \\ u\left(0_{\mathbf{R}^d}\right) = 1, \\ \frac{\partial}{\partial x_i} u(x)\Big|_{x_i=0} = 0; i = 1 \ldots d, \end{cases}$$

furthermore it is infinitely differentiable on $\mathbb{R}_+^d$, even with respect to each variable and satisfies the following important result, for all $x, \lambda \in \mathbb{R}^d +$ we have

$$|\psi_{\alpha,d}(\lambda x)| \leq 1. \tag{2.5}$$

We will use this function to define the multidimensional Hankel transform.

2.2. **The multidimensional Hankel transform.**

Definition 2.1. *The multidimensional Hankel transform $\mathscr{H}_\alpha$ is defined on $L_\alpha^1\left(\mathbb{R}_+^d\right)$ by*

$$\mathscr{H}_\alpha(f)(\lambda) = \int_{\mathbb{R}_+^d} f(x)\psi_{\alpha,d}(\lambda x)\mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}_+^d, \tag{2.6}$$

where μ_α is the measure on $\mathbb{R}_+^d$ given by the relation (2.1). Some basic properties of this transform are as follows, for the proofs, we refer the reader to [2,7].

Proposition 2.2. *(1) For all $f \in L_\alpha^1\left(\mathbb{R}_+^d\right)$, we have*

$$\|\mathscr{H}_\alpha(f)\|_{\infty,\mu_\alpha} \leq \|f\|_{1,\mu_\alpha}. \tag{2.7}$$

(2) (Parseval's formula) For all $f, g \in L_\alpha^2\left(\mathbb{R}_+^d\right)$, we have

$$\int_{\mathbb{R}_+^d} f(x)\overline{g(x)}\mathrm{d}\mu_\alpha(x) = \int_{\mathbb{R}_+^d} \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}\mathrm{d}\mu_\alpha(\lambda) \tag{2.8}$$

(3) (Plancherel's theorem) The multidimensional Hankel transform $\mathscr{H}_\alpha$ extends uniquely to an isometric isomorphism on $L_\alpha^2\left(\mathbb{R}_+^d\right)$ and we have

$$\|\mathscr{H}_\alpha(f)\|_{2,\mu_\alpha} = \|f\|_{2,\mu_\alpha}, \tag{2.9}$$

for all $L_\alpha^2\left(\mathbb{R}_+^d\right)$.
(4) (Inversion formula) Let $f \in L_\alpha^1(\mathbb{R}_+^d)$ such that $\mathscr{H}_\alpha(f) \in L_\alpha^1(\mathbb{R}_+^d)$, then we have

$$f(x) = \int_{\mathbb{R}_+^d} \mathscr{H}_\alpha(f)(\lambda)\psi_{\alpha,d}(\lambda x)\mathrm{d}\mu_\alpha(\lambda), \quad a.e.\ \lambda \in \mathbb{R}_+^d. \tag{2.10}$$

2.3. **Generalized Translation Operator Associated With the Multidimensional Hankel Transform.**

Definition 2.2. *The translation operator $\tau_\alpha^x, x \in \mathbb{R}_+^d$ associated with the multidimensional Bessel operator $\Delta_{\alpha,d}$, is defined for a suitable function f by*

$$\tau_\alpha^x f(y) = c_\alpha' \int_{[0,\pi]^d} f\left(X_1, \ldots, X_n\right) \prod_{i=1}^{d} \left(\sin\theta_i\right)^{2\alpha_i} d\theta_1 \ldots d\theta_d,$$

with $c'_\alpha = \prod_{i=1}^d \frac{\Gamma(\alpha_i+1)}{\pi^{d/2}\Gamma(\alpha_i+1/2)}$ *and* $X_i = \sqrt{x_i^2+y_i^2-2y_ix_i\cos\theta_i}$*, for* $i=1,\dots,d$*. The following proposition summarizes some properties of the generalized translation operator see [2,7].*

Proposition 2.3. *. (1) For all* $x, y \in \mathbb{R}_+^{d+1}$*, we have*

(1)

$$\tau_\alpha^x f(y) = \tau_\alpha^y f(x) \tag{2.11}$$

(2)

$$\int_{\mathbb{R}_+^d} \tau_\alpha^x(f)(y)\mathrm{d}\mu_\alpha(y) = \int_{\mathbb{R}_+^d} f(y)\mathrm{d}\mu_\alpha(y). \tag{2.12}$$

(3) for $f \in L_\alpha^p\left(\mathbb{R}_+^d\right)$ *with* $p \in [1;+\infty]$ $\tau_\alpha^x(f) \in L_\alpha^p\left(\mathbb{R}_+^d\right)$ *and we have*

$$\|\tau_\alpha^x(f)\|_{p,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha} \tag{2.13}$$

(4)

$$\mathscr{H}_\alpha(\tau_\alpha^x(f))(\lambda) = \psi_{\alpha,d}(\lambda x)\mathscr{H}_\alpha(f)(\lambda) \tag{2.14}$$

By using the generalized translation, we define the generalized convolution product of f, g *by*

$$(f *_\alpha g)(x) = \int_{\mathbb{R}_+^d} \tau_\alpha^x(f)(y)g(y)\mathrm{d}\mu_\alpha(y).$$

This convolution is commutative, associative and its satisfies the following properties.

Proposition 2.4. *For* $f, g \in L_\alpha^2\left(\mathbb{R}_+^d\right)$ *the function* $f *_\alpha g$ *belongs to* $L_\alpha^2\left(\mathbb{R}_+^d\right)$ *if and only if the function* $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)$ *belongs to* $L_\alpha^2\left(\mathbb{R}_+^d\right)$ *and in this case we have*

$$\mathscr{H}_\alpha(f *_\alpha g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g) \tag{2.15}$$

and we have

$$\int_{\mathbb{R}_+^d} |f *_\alpha g(x)|^2 \, \mathrm{d}\mu_\alpha(x) = \int_{\mathbb{R}_+^d} |\mathscr{H}_\alpha(f)(\lambda)|^2 |\mathscr{H}_\alpha(g)(\lambda)|^2 \, \mathrm{d}\mu_\alpha(\lambda), \tag{2.16}$$

where both integrals are simultaneously finite or infinite.

3. Wavelet transform Associated with the Multidimensional Hankel transform

The main purpose of this section is to define the wavelet transform associated with the multidimensional Bessel operator $\Delta_{\alpha,d}$ *given by the relation (1.1).*

Notation : *we denote by*

•$L_\alpha^p(\mathbb{R}_+ \times \mathbb{R}_+^d)$*,*$1 \leq p \leq +\infty$ *the space of measurable functions on* $\mathbb{R}_+ \times \mathbb{R}_+^d$ *satisfying*

$$\|f\|_{p,\theta_\alpha} := \begin{cases} \left(\int_{\mathbb{R}_+\times\mathbb{R}_+^d} |f(a,x)|^p \mathrm{d}\theta_\alpha(a,x)\right)^{\frac{1}{p}} < \infty, & \text{if } p \in [1,+\infty[, \\ \operatorname*{ess\,sup}_{(a,x)\in\mathbb{R}_+\times\mathbb{R}_+^d} |f(a,x)| < \infty, & \text{if } p = +\infty. \end{cases}$$

where θ_α *is the measure defined on* $\mathbb{R}_+ \times \mathbb{R}_+^d$ *by*

$$\mathrm{d}\theta_\alpha(a,x) := \frac{\mathrm{d}a \otimes \mathrm{d}\mu_\alpha(x)}{a}.$$

Definition 3.1. *Let* u, v *be two functions in* $L_\alpha^2(\mathbb{R}_+^d)$ *we say that the pair* (u,v) *is a multidimensional Hankel two-wavelet on* $\mathbb{R}_+^d$ *if the following integral*

$$C_{u,v} := \int_0^{+\infty} \overline{\mathscr{H}_\alpha(u)(a\lambda)}\mathscr{H}_\alpha(v)(a\lambda)\frac{\mathrm{d}a}{a} \tag{3.1}$$

is finite for almost all $\lambda \in \mathbb{R}_+^d$ *and we call the number* $C_{u,v}$ *the multidimensional Hankel two-wavelet constant associated to the functions* u, v.

Remark 3.1. *It is clear that if* $u = v$*, we have*

$$0 < C_{u,u} = C_u := \int_0^{+\infty} |\mathscr{H}_\alpha(u)(a\lambda)|^2 \frac{\mathrm{d}a}{a} < \infty. \tag{3.2}$$

In this case, we say that u *is a multidimensional Hankel-wavelet in* $L^2_\alpha(\mathbb{R}_+^d)$.

Let $a > 0$*, we define the dilatation operator* D_a *of a measurable function* u *on* $\mathbb{R}_+^d$ *by*

$$D_a(u)(x) := \frac{1}{a^{2|\alpha|+2d}} u\left(\frac{x}{a}\right). \tag{3.3}$$

The dilatation operator D_a *satisfies the following properties*

- *For all* $u \in L^2_\alpha(\mathbb{R}_+^d)$ *we have* $D_a(u) \in L^2_\alpha(\mathbb{R}_+^d)$ *and*

$$\|D_a(u)\|_{2,\mu_\alpha} = \frac{1}{a^{|\alpha|+d}} \|u\|_{2,\mu_\alpha}. \tag{3.4}$$

- *For all* $u \in L^2(\mathbb{R}_+^d)$ *we have*

$$\mathscr{H}_\alpha(D_a(u))(\lambda) = \mathscr{H}_\alpha(u)(a\lambda). \tag{3.5}$$

Let u *be a multidimensional Hankel-wavelet on* $\mathbb{R}_+^d$ *in* $L^p_\alpha(\mathbb{R}_+^d)$ *with* $1 \le p \le \infty$*, for all* $a > 0$*,* $x \in \mathbb{R}$ *we define the function*

$$u_{a,x}(y) = \tau_\alpha^x(D_a(u))(y). \tag{3.6}$$

Definition 3.2. *([18]) Let* u *be a multidimensional Hankel-wavelet on* $\mathbb{R}_+^d$ *in* $L^2_\alpha(\mathbb{R}_+^d)$ *the continuous wavelet transform associated with the multidimensional Bessel operator (1.1) is defined for a function* $f \in L^2_\alpha(\mathbb{R}_+^d)$ *and* $(a, x) \in \mathbb{R}_+ \times \mathbb{R}_+^d$ *by*

$$\mathscr{W}_u^\alpha(f)(a,x) := \int_{\mathbb{R}_+^d} f(y)\overline{u_{a,x}(y)} d\mu_\alpha(y). \tag{3.7}$$

Remark 3.2. *The multidimensional Hankel-wavelet transform can be written as*

$$\mathscr{W}_u^\alpha(f)(a,x) = (\overline{D_a(u)} *_\alpha f)(x). \tag{3.8}$$

We have the following Parseval's formula for $\mathscr{W}_u^\alpha$ *which generalize the Parseval's formula proved in [18].*

Theorem 3.1. *Let* u, v *be two multidimensional Hankel-wavelet functions and* $f, g \in L^2_\alpha(\mathbb{R}_+^d)$ *we have*

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+^d} \mathscr{W}_u^\alpha(f)(a,x)\overline{\mathscr{W}_v^\alpha(g)(a,x)} \mathrm{d}\theta_\alpha(a,x) = C_{u,v} \int_{\mathbb{R}_+^d} f(y)\overline{g(y)} d\mu_\alpha(y), \tag{3.9}$$

where $C_{u,v}$ *is given by the relation (3.1).*

Proof. Using the relations (3.5),(3.8) and Fubini's theorem we get

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+^d} \mathscr{W}_u^\alpha(f)(a,x)\overline{\mathscr{W}_v^\alpha(g)(a,x)} \mathrm{d}\theta_\alpha(a,x) = \int_0^{+\infty} \overline{\mathscr{H}_\alpha(u)(a\lambda)}\mathscr{H}_\alpha(v)(a\lambda)[\int_{\mathbb{R}_+^d} \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)} d\mu_\alpha(\lambda)] \frac{\mathrm{d}a}{a},$$

by using Parseval's formula for the multidimensional Hankel transform (2.8) we find the disered result. □

corollary 3.1. *For all* $f \in L^2_\alpha(\mathbb{R}_+^d)$ *and* u *a multidimensional Hankel-wavelet on* $\mathbb{R}_+^d$ *we have the following Plancherel's formula for* $\mathscr{W}_u^\alpha$

$$\|\mathscr{W}_u^\alpha(f)\|_{2,\theta\alpha} = \sqrt{C_u} \|f\|_{2,\mu_\alpha}. \tag{3.10}$$

Where C_u *is given by the relation (3.2).*

In the following we establish an inversion formula for the multidimensional Hankel-wavelet transform $\mathscr{W}_u^\alpha$ which more general than that proved in [18].

Theorem 3.2. *Let (u,v) be a multidimensional Hankel two-wavelet such that $C_{u,v} \neq 0$ for all $f \in L^1_\alpha(\mathbb{R}^d_+)$ such that $\mathscr{H}_\alpha(f) \in L^1_\alpha(\mathbb{R}^d_+) \cap L^\infty_\alpha(\mathbb{R}^d_+)$ we have*

$$f(\cdot) = \frac{1}{C_{u,v}} \int_0^\infty \left(\int_{\mathbb{R}^d_+} \mathscr{W}_u^\alpha(f)(a,x) v_{a,x}(\cdot) d\mu_\alpha(x) \right) \frac{da}{a}.$$

Proof. Let $f, g \in L^2_\alpha(\mathbb{R}^d_+)$, by using the relation (3.9)and Fubini's theorem we find that

$$\begin{aligned}\int_{\mathbb{R}^d_+} f(y)\overline{g(y)} d\mu_\alpha(y) &= \frac{1}{C_{u,v}} \int_{\mathbb{R}_+ \times \mathbb{R}^d_+} \mathscr{W}_u^\alpha(f)(a,x) [\int_{\mathbb{R}^d_+} \overline{g(y)} v_{a,x}(y) d\mu_\alpha(y)] \mathrm{d}\theta_\alpha(a,x) \\ &= \frac{1}{C_{u,v}} \int_{\mathbb{R}^d_+} [\int_{\mathbb{R}_+ \times \mathbb{R}^d_+} \mathscr{W}_u^\alpha(f)(a,x) v_{a,x}(y) \mathrm{d}\theta_\alpha(a,x)] \overline{g(y)} d\mu_\alpha(y),\end{aligned}$$

which gives the result. □

The reproducing kernels for Hilbert space play an important role in harmonic analysis [16,17]. In this context, we have the following result.

Theorem 3.3. *Let u be a multidimensional Hankel wavelet, then the space $\mathscr{W}_u^\alpha\left(L^2_\alpha\left(\mathbb{R}^d_+\right)\right)$ is a reproducing kernel Hilbert space in $L^2_\alpha\left(\mathbb{R}^{2d}_+\right)$ with kernel function $\mathcal{K}_u$ defined by*

$$\mathcal{K}_u\left((a',x');(a,x)\right) = \frac{1}{C_u}\left(D_{a'}(u) *_\alpha u_{a,x}\right)(x'), \tag{3.11}$$

where C_u is given by the relation (3.2).
Furthermore, the kernel is pointwise bounded and we have

$$|\mathcal{K}_u\left((a',x');(a,x)\right)| \leq \|u\|^2_{2,\mu_\alpha}, \quad \forall (a,x);(a',x') \in \mathbb{R}^{2d}_+. \tag{3.12}$$

Proof. By using the relations (3.7) and (3.9) we find that

$$\begin{aligned}\mathscr{W}_u^\alpha(a,x) &= \frac{1}{C_u} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d_+} \mathscr{W}_u^\alpha(f)\left(a',x'\right) \overline{\mathscr{W}_u^\alpha\left(u_{a,x}\right)\left(a',x'\right)} \mathrm{d}\theta_\alpha(a',x') \\ &= \langle \mathscr{W}_u^\alpha(f) \mid \mathcal{K}_u((.);(a,x)) \rangle_{\theta_\alpha},\end{aligned}$$

where

$$\mathcal{K}_u\left((a',x');(a,x)\right) = \frac{1}{C_u}\left(D_{a'}(u) *_\alpha u_{a,x}\right)(x'),$$

Finally by using the Cauchy-Schwarz inequality, we get

$$|\mathcal{K}_u\left((a',x');(a,x)\right)| \leq \|u\|^2_{2,\mu_\alpha}, \quad \forall (a,x);(a',x') \in \mathbb{R}^{2d}_+.$$

□

The rest of this subsection is devoted to give a generalized version of Calderón's reproducing formula for the multidimensional Hankel two-wavelet (u,v) under the following condition

$$C_{u,v} \neq 0 \quad \textit{and} \quad \mathscr{H}_\alpha(u), \mathscr{H}_\alpha(v) \in L^\infty_\alpha(\mathbb{R}^d_+). \tag{3.13}$$

We begin by the following result

Proposition 3.1. *For $0 < \varepsilon < \delta < \infty$, we put*

$$G_{\varepsilon,\delta}(x) := \frac{1}{C_{u,v}} \int_\varepsilon^\delta \left(\overline{D_a(u)} *_\alpha D_a(v)\right)(x) \frac{da}{a}$$

and

$$K_{\varepsilon,\delta}(\lambda) := \frac{1}{C_{u,v}} \int_{\varepsilon}^{\delta} \overline{\mathscr{H}_\alpha(u)(a\lambda)} \mathscr{H}_\alpha(v)(a\lambda) \frac{\mathrm{d}a}{a}.$$

Under the condition (3.13) we have

$$G_{\varepsilon,\delta} \in L^2_\alpha(\mathbb{R}^d_+), K_{\varepsilon,\delta} \in \ L^1_\alpha(\mathbb{R}^d_+) \cap L^2_\alpha(\mathbb{R}^d_+)$$

and

$$\mathscr{H}_\alpha(G_{\varepsilon,\delta})(\lambda) = K_{\varepsilon,\delta}(\lambda) \tag{3.14}$$

Proof. by using Hölder's inequality for the measure $\frac{da}{a}$ we find that

$$\|G_{\varepsilon,\delta}\|^2_{2,\mu_\alpha} \leq \frac{\log(\delta/\varepsilon)}{C^2_{u,v}} \|\mathscr{H}_\alpha(u)\|^2_{\infty,\mu_\alpha} \|v\|^2_{2,\mu_\alpha} \int_\varepsilon^\delta \frac{da}{a^{2\alpha+2}} < \infty.$$

Which prove that $G_{\varepsilon,\delta} \in L^2_\alpha(\mathbb{R}^d_+)$, the result $K_{\varepsilon,\delta} \in \ L^1_\alpha(\mathbb{R}^d_+) \cap L^\infty_\alpha(\mathbb{R}^d_+)$ can be easily checked, on the other hand we have

$$\left(\overline{D_a(u)} *_\alpha D_a(v)\right)(x) = \int_{\mathbb{R}^d_+} \mathscr{H}_\alpha(v)(a\lambda)\overline{\mathscr{H}_\alpha(u)(a\lambda)} B_\alpha(\lambda x) d\mu_\alpha(\lambda),$$

hence applying Fubini's theorem we get

$$G_{\varepsilon,\delta}(x) = \int_{\mathbb{R}^d_+} \psi_{\alpha,d}(\lambda x) K_{\varepsilon,\delta}(\lambda) d\mu_\alpha(\lambda),$$

inversion formula (2.10) gives the relation (3.14). □

We can now state the main result of this section

Theorem 3.4. *(Calderón's reproducing formula) Let (u,v) be a multidimensional Hankel two-wavelet satisfying the condition (3.11) and let $0 < \varepsilon < \delta < \infty$, then for all $f \in L^2_\alpha(\mathbb{R}^d_+)$, the function $f_{\varepsilon,\delta}$ given by*

$$f_{\varepsilon,\delta}(x) = \frac{1}{C_{u,v}} \int_\varepsilon^\delta \left(\int_{\mathbb{R}^d_+} \mathscr{W}^\alpha_u(f)(a,y) v_{a,x}(y) d\mu_\alpha(y) \right) \frac{da}{a}, x \in \mathbb{R}^d_+,$$

belongs to $L^2_\alpha(\mathbb{R})$ and satisfies

$$\lim_{\varepsilon\to 0, \delta\to\infty} \|f_{\varepsilon,\delta} - f\|_{2,\mu_\alpha} = 0. \tag{3.15}$$

Proof. It is easy to see that

$$f_{\varepsilon,\delta} = f *_\alpha G_{\varepsilon,\delta}$$

then by using the relations (2.9) and (3.12) we find that

$$\|f_{\varepsilon,\delta} - f\|^2_{2,\mu_\alpha} = \int_{\mathbb{R}^d_+} |\mathscr{H}_\alpha(f)(\lambda)|^2 (1 - K_{\varepsilon,\delta}(\lambda))^2 d\mu_\alpha(\lambda),$$

by using the relation (3.1), the relation (3.15) follows from the dominated convergence theorem. □

4. Uncertainty Principles Associated With the Multidimensional Hankel-Wavelet Transform

In this section, we estimate the concentration of $\mathscr{W}^\alpha_u(f)$ on subset of $\mathbb{R}^d_+ \times \mathbb{R}^d_+$ of finite measure, similar results have been checked in [1, 6–8, 10, 20] and we establish the uncertainty principle for orthonormal sequences associated with the multidimensional Hankel-wavelet transform, first we consider the following orthogonal projections

(1) Let P_u be the orthogonal projection from $L^2_\alpha\left(\mathbb{R}^{2d}_+\right)$onto $\mathscr{W}^\alpha_u\left(L^2_\alpha\left(\mathbb{R}^d_+\right)\right)$ and $\operatorname{Im} P_u$ denotes the range of P_u.
(2) Let P_E be the orthogonal projection on $L^2_\alpha\left(\mathbb{R}^{2d}_+\right)$ defined by

$$P_E F = \chi_E F, \quad F \in L^2_\alpha\left(\mathbb{R}^{2d}_+\right), \tag{4.1}$$

where $E \subset \mathbb{R}^d_+ \times \mathbb{R}^d_+$ and $\operatorname{Im} P_E$ is the range of P_E. Also, we define

$$\|P_E P_u\| = \sup\left\{\|P_E P_u(F)\|_{2,\mu_\alpha\otimes\mu_\alpha} : F \in L^2_\alpha\left(\mathbb{R}^{2d}_+\right), \|F\|_{2,\mu_\alpha\otimes\mu_\alpha} = 1\right\}.$$

We first need the following result.

Theorem 4.1. *Let u be a multidimensional Hankel-wavelet in $L^2_\alpha(\mathbb{R}^d_+)$. Then for any $E \subset \mathbb{R}^d_+ \times \mathbb{R}^d_+$ of finite measure $\mu_\alpha \otimes \mu_\alpha(E) < \infty$, the operator $P_E P_u$ is a Hilbert-Schmidt operator. Moreover, we have the following estimation*

$$\|P_E P_u\| \leq \|u\|^2_{2,\alpha}\sqrt{\mu_\alpha \otimes \mu_\alpha(E)}.$$

Proof. since P_u is a projection onto a reproducing karnel Hilbert space, for any function $F \in L^2_\alpha(\mathbb{R}^2)$, the orthogonal projection P_u can be expressed as

$$P_u(F)(x,\xi) = \iint_{\mathbb{R}^{2d}_+} F(x',\xi')\, K_u((x',\xi');(x,\xi))\, d\mu_\alpha(x') \otimes d\mu_\alpha(\xi'),$$

where $\mathcal{K}_u((x',\xi');(x,\xi))$ is given by the relation (3.11), using the relation (4.1), we find that

$$P_E P_u(F)(x,\xi) = \iint_{\mathbb{R}^{2d}_+} \chi_E(x,\xi) F(x',\xi')\, \mathcal{K}_u((x',\xi');(x,\xi))\, d\mu_\alpha(x') \otimes d\mu_\alpha(\xi').$$

This shows that the operator $P_E P_u$ is an integral operator with kernel $K((x',\xi');(x,\xi)) = \chi_E(x,\xi)\mathcal{K}_\psi((x',\xi');(x,\xi))$. Using the relation (3.6) and Fubini's theorem, we find that

$$\|P_\Sigma P_u\|^2_{HS} = \iint_{\mathbb{R}^{2d}_+}\iint_{\mathbb{R}^{2d}_+} |\chi_\Sigma(x,\xi)|^2 \left|\mathcal{K}_u((x',\xi');(x,\xi))\right|^2 d\mu_\alpha(x') \otimes d\mu_\alpha(\xi') d\mu_\alpha(x) \otimes d\mu_\alpha(\xi)$$

$$\leq \|u\|^2_{2,\alpha}\sqrt{\mu_\alpha \otimes \mu_\alpha(E)} < \infty. \tag{4.2}$$

Thus, the operator $P_E P_u$ is a Hilbert-Schmidt operator. Now, the proof follows from the fact that $\|P_E P_u\| \leq \|P_E P_u\|_{HS}$. □

In the following, we obtain the uncertainty principle for orthonormal sequences associated with the multidimensional Hankel-wavelet transform.

Theorem 4.2. *Let u be a multidimensional Hankel-wavelet in $L^2_\alpha(\mathbb{R}^d_+)$ and $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}^d_+)$. Then for any subset $E \subset \mathbb{R}^d_+ \times \mathbb{R}^d_+$ of finite measure $\mu_\alpha \otimes \mu_\alpha(E) < \infty$, we have*

$$\sum_{n=1}^{N}\left(1 - \|\chi_{E^c}\mathscr{W}^\alpha_u(\phi_n)\|_{2,\mu_\alpha\otimes\mu_\alpha}\right) \leq \|u\|^2_{2,\alpha}\sqrt{\mu_\alpha \otimes \mu_\alpha(E)},$$

for every $N \in \mathbb{N}$.

Proof. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2_\alpha\left(\mathbb{R}^{2d}_+\right)$. Since $P_E P_u$ is a Hilbert-Schmidt operator, and satisfied the relation (4.2) and we have

$$\sum_{n\in\mathbb{N}} \langle P_\psi P_E P_u \phi_n, \phi_n\rangle_{\mu_\alpha\otimes\mu_\alpha} = \|P_E P_u\|^2_{HS} \leq \|u\|^2_{2,\alpha}\sqrt{\mu_\alpha \otimes \mu_\alpha(E)} < \infty..$$

According to the paper [11], the positive operator $P_u P_E P_u$ is a trace class operator and we have

$$\operatorname{tr}(P_u P_E P_u) = \|P_E P_u\|^2_{HS} \leq \|u\|^2_{2,\alpha}\sqrt{\mu_\alpha \otimes \mu_\alpha(E)} < \infty.$$

where $\operatorname{tr}(P_u P_E P_u)$ denotes the trace of the operator $P_u P_E P_u$. Since $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}^d_+)$, from the orthogonality relation (3.10), we obtain that $\{\mathscr{W}^\alpha_u(\phi_n)\}_{n\in\mathbb{N}}$ is also an orthonormal sequence in $L^2_\alpha\left(\mathbb{R}^{2d}_+\right)$ thus

$$\sum_{n=1}^{N} \langle P_E \mathscr{W}^\alpha_u(\phi_n), \mathscr{W}^\alpha_u(\phi_n)\rangle_{\mu_\alpha\otimes\mu_\alpha} = \sum_{n=1}^{N} \langle P_u P_\Sigma P_u \mathscr{W}^\alpha_u(\phi_n), \mathscr{W}^\alpha_u(\phi_n)\rangle_{\mu_\alpha\otimes\mu_\alpha} \leq \operatorname{tr}(P_u P_E P_u)$$

Moreover, for any n with $1 \leq n \leq N$, using the Cauchy-Schwarz inequality, we get

$$\sum_{n=1}^{N}\left(1 - \|\chi_{E^c}\mathscr{W}^\alpha_u(\phi_n)\|_{2,\mu_\alpha\otimes\mu_\alpha}\right) \leq \sum_{n=1}^{N} \langle P_E \mathscr{W}^\alpha_u(\phi_n), \mathscr{W}^\alpha_u(\phi_n)\rangle_{\mu_\alpha\otimes\mu_\alpha} \leq \|u\|^2_{2,\alpha}\sqrt{\mu_\alpha\otimes\mu_\alpha(E)} < \infty.$$

This completes the proof of the theorem. □

5. Localization Operators Associated With the Multidimensional Hankel-Wavelet Transform

tionIntroduction The main purpose of this section is to introduce and to give sufficient conditions for boundedness, compactness, and Schatten class properties of localization operators $\mathscr{L}_{u,v}(\sigma)$ associated with the multidimensional Hankel-wavelet transform.

In the sequel of this section u, v belongs to $L^2_\alpha\left(\mathbb{R}^d_+\right)$ such that $\|u\|_{2,\alpha} = \|v\|_{2,\alpha} = 1$, we note that this hypothesis is not essential and the result still hold true up to some contant depending on $\|u\|_{2,\alpha}$ and $\|v\|_{2,\alpha}$.

Notation: we denote by

- $B\left(L^p_\alpha\left(\mathbb{R}^d_+\right)\right), 1 \leq p \leq \infty$, *the space of bounded operators from $\mathbb{R}^d_+$ into itself.*

In particular for $p = 2$, we define

$$S_\infty := B\left(L^2_\alpha\left(\mathbb{R}^d_+\right)\right),$$

equipped with the norm

$$\|A\|_{S_\infty} := \sup_{v\in L^2_\alpha\left(\mathbb{R}^d_+\right):\|v\|_{L^2_\alpha\left(\mathbb{R}^d_+\right)}\leq 1} \|Av\|_{2,\mu_\alpha}. \tag{5.1}$$

Definition 5.1. *The trace of an operator A in S_1 is defined by*

$$\operatorname{tr}(A) = \sum_{k=1}^{\infty} \langle A\varphi_k, \varphi_k\rangle_\alpha, \tag{5.2}$$

where $(\varphi_k)_k$ is any orthonormal basis of $L^2_\alpha\left(\mathbb{R}^d_+\right)$.

Remark 5.1. *A compact operator A on the Hilbert space $L^2_\alpha\left(\mathbb{R}^d_+\right)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class in this case we have*

$$\|A\|^2_{HS} := \operatorname{tr}(A^*A) = \sum_{k=1}^{\infty} \|A\varphi_k\|^2_{2,\mu_\alpha}, \tag{5.3}$$

for any orthonormal basis $(\varphi_k)_k$ of $L^2_\alpha\left(\mathbb{R}^d_+\right)$. For more information about the Schatten classes one can see [21].

5.1. Boundedness and compactness of localization in S_∞.

In this subsection, we will define and study the boundedness and compactness of localization operator in $L^2_\alpha\left(\mathbb{R}^d_+\right)$ we begin by the following result.

Proposition 5.1. *For every $p \in [1, +\infty]$ and all $f, g \in L^2_\alpha\left(\mathbb{R}^d_+\right)$, the function $\mathscr{W}^\alpha_u(f)\mathscr{W}^\alpha_v(g)$ belongs to $L^p_\alpha\left(\mathbb{R}^{2d}_+\right)$ and*

$$\|\mathscr{W}^\alpha_u(f)\mathscr{W}^\alpha_v(g)\|_{p,\mu_\alpha\otimes\mu_\alpha} \leq \|f\|_{2,\mu_\alpha}\|g\|_{2,\mu_\alpha} \tag{5.4}$$

Proof. From the relation (3.6), the function $\mathscr{W}_u^\alpha(f)\mathscr{W}_v^\alpha(g)$ belongs to $L_\alpha^\infty\left(\mathbb{R}_+^{2d}\right)$ and we have

$$\|\mathscr{W}_u^\alpha(f)\mathscr{W}_v^\alpha(g)\|_{\infty,\mu_\alpha\otimes\mu_\alpha} \leq \|f\|_{2,\mu_\alpha}\|g\|_{2,\mu_\alpha}. \tag{5.5}$$

From the Cauchy-Schwarz inequality and the relation (3.7), we find that the function $\mathscr{W}_u^\alpha(f)\mathscr{W}_v^\alpha(g)$ belongs to $L_\alpha^1\left(\mathbb{R}_+^{2d}\right)$ and

$$\|\mathscr{W}_u^\alpha(f)\mathscr{W}_v^\alpha(h)\|_{1,\mu_\alpha\otimes\mu_\alpha} \leq \|f\|_{2,\alpha}\|h\|_{2,\alpha} \tag{5.6}$$

Let now $p \in]1,+\infty[$, we have

$$\|\mathscr{W}_u^\alpha(f)\mathscr{W}_v^\alpha(g)\|_{p,\mu_\alpha\otimes\mu_\alpha} \leq \|\mathscr{W}_u^\alpha(f)\mathscr{W}_v^\alpha(g)\|_{\infty,\mu_\alpha\otimes\mu_\alpha}^{p-1}\|\mathscr{W}_u^\alpha(f)\mathscr{W}_v^\alpha(h)\|_{1,\mu_\alpha\otimes\mu_\alpha}$$

the relations (5.5) and (5.6) gives the desired result. □

Proposition 5.2. *Let $\sigma \in L_\alpha^p\left(\mathbb{R}_+^{2d}\right), p \in [1,+\infty]$. For every $f \in L_\alpha^2\left(\mathbb{R}_+^d\right)$, there exists a unique function in $L_\alpha^2\left(\mathbb{R}_+^d\right)$, denoted by $\mathscr{L}_{u,v}(\sigma)(f)$ such that for every $g \in L_\alpha^2\left(\mathbb{R}_+^d\right)$ we have*

$$\langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\rangle_\alpha = \int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d} \sigma(x,y)\mathscr{W}_u^\alpha(f)(x,y)\overline{\mathscr{W}_v^\alpha(g)(x,y)}\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y). \tag{5.7}$$

Proof. $\sigma \in L_\alpha^p\left(\mathbb{R}_+^{2d}\right), p \in [1,+\infty]$ and let p' be the conjugate exposant of p by proposition 5.1 we have the function $\mathscr{W}_u^\alpha(f)\mathscr{W}_v^\alpha(g)$ belongs to $L_\alpha^{p'}\left(\mathbb{R}_+^{2d}\right)$, by using Hölder's inequality we find that

$$\left|\int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d} \sigma(x,y)\mathscr{W}_u^\alpha(f)(x,y)\overline{\mathscr{W}_v^\alpha(g)(x,y)}\mathrm{d}\mu_\alpha(x)\mathrm{d}\mu_\alpha(y)\right| \leq \|\sigma\|_{p,\mu_\alpha\otimes\mu_\alpha}\|f\|_{2,\mu_\alpha}\|g\|_{2,\mu_\alpha}. \tag{5.8}$$

The relation (5.8) shows that the mapping

$$g \longmapsto \int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d} \sigma(x,y)\mathscr{W}_u^\alpha(f)(x,y)\overline{\mathscr{W}_v^\alpha(g)(x,y)}\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y),$$

is an antilinear continous form on the Hilbert space $L_\alpha^p\left(\mathbb{R}_+^d\right)$ by the Riesz representation theorem, there exist a unique function $\mathscr{L}_{u,v}(\sigma)(f)$ satisfying the equality (5.7). □

Definition 5.2. *For every $\sigma \in L_\alpha^p\left(\mathbb{R}_+^{2d}\right), p \in [1,+\infty]$, the localization operators associated with the window functions u, v and the symbol $\sigma, \mathscr{L}_{u,v}(\sigma)$ is defined for all $f, g \in L_\alpha^2\left(\mathbb{R}_+^d\right)$ by*

$$\langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\rangle_\alpha = \int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d} \sigma(x,y)\mathscr{W}_u^\alpha(f)(x,y)\overline{\mathscr{W}_v^\alpha(g)(x,y)}\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y).$$

We have the following result

Theorem 5.1. *For every $\sigma \in L_\alpha^p\left(\mathbb{R}_+^{2d}\right), p \in [1,+\infty]$, the localization operator*

$$\mathscr{L}_{u,v}(\sigma) : L_\alpha^2\left(\mathbb{R}_+^d\right) \longrightarrow L_\alpha^2\left(\mathbb{R}_+^d\right)$$

belongs to S_∞ and we have

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{p,\mu_\alpha\otimes\mu_\alpha}. \tag{5.9}$$

Furthermore we have

$$\mathscr{L}_{u,v}^*(\sigma) = \mathscr{L}_{v,u}(\bar{\sigma}) \tag{5.10}$$

where $\mathscr{L}_{u,v}^*(\sigma)$ is the adjoint operator of $\mathscr{L}_{u,v}(\sigma)$.

Proof. For $f \in L_\alpha^2\left(\mathbb{R}_+^d\right)$ by using the relation (5.8) we find that

$$\|\mathscr{L}_{u,v}(\sigma)(f)\|_{2,\mu_\alpha} \leq \|\sigma\|_{p,\mu_\alpha\otimes\mu_\alpha}\|f\|_{2,v_\alpha},$$

this show that the localization operator $\mathscr{L}_{u,v}(\sigma)$ is bounded on $L_\alpha^2\left(\mathbb{R}_+^d\right)$ and by using the relation (5.1) we find that

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} = \sup_{\|f\|_{2,\alpha}\le 1} \|\mathscr{L}_{u,v}(\sigma)(f)\|_{2,\alpha} \le \|\sigma\|_{p,\mu_\alpha\otimes\mu_\alpha} < \infty,$$

since $\mathscr{L}_{u,v}(\sigma) \in S_\infty$. Let us determine his adjoint, for all $f, g \in L^2_\alpha\left(\mathbb{R}^d_+\right)$ we have

$$\begin{aligned}\langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\rangle_\alpha &= \overline{\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} \mathscr{W}^\alpha_v(g)(x,y)\overline{\sigma(x,y)\mathscr{W}^\alpha_u(f)(x,y)}\mathrm{d}\mu_\alpha(x)\mathrm{d}\mu_\alpha(y)} \\ &= \overline{\langle \mathscr{L}_{v,u}((\bar\sigma))(f) \mid g\rangle_\alpha} = \langle g \mid \mathscr{L}_{v,u}((\bar\sigma))(f)\rangle_\alpha ,\end{aligned}$$

which gives the result. □

We have the following result

Theorem 5.2. *For every $\sigma \in L^p_\alpha\left(\mathbb{R}^{2d}_+\right), p \in [1, +\infty]$, the localization operator*

$$\mathscr{L}_{u,v}(\sigma) : L^2_\alpha\left(\mathbb{R}^d_+\right) \longrightarrow L^2_\alpha\left(\mathbb{R}^d_+\right)$$

is compact.

Proof. Let $\sigma \in L^1_\alpha\left(\mathbb{R}^{2d}_+\right)$ and let $(\varphi_k)_k$ be an orthonormal basis of $L^2_\alpha\left(\mathbb{R}^d_+\right)$, for every $k \in \mathbb{N}$ we have

$$\begin{aligned}||\mathscr{L}_{u,v}(\sigma)(\varphi_k)||^2_{2,\mu_\alpha} &= \left|\langle \mathscr{L}_{u,v}(\sigma)(\varphi_k) \mid \mathscr{L}_{u,v}(\sigma)(\varphi_k)\rangle_\alpha\right| \\ \le \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} |\sigma(x,y)| &\left[\left|\langle \varphi_k \mid u^{x,y}\rangle_\alpha\right|^2 + \left|\langle \mathscr{L}_{v,u}((\bar\sigma))(v^{x,y}) \mid \varphi_k\rangle_\alpha\right|^2\right]\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y)\end{aligned}$$

by using Bessel's inequality, Fubini's theorem and and the relations (2.13),(3.3), (5.3),(5.9) we find that

$$\|\mathscr{L}_{u,v}(\sigma)\|^2_{HS} \le \frac{1}{2}\|\sigma\|_{1,\mu_\alpha\otimes\mu_\alpha}\left(1 + \|\sigma\|^2_{1,\mu_\alpha\otimes\mu_\alpha}\right) \le \left(1 + \|\sigma\|^2_{1,\mu_\alpha\otimes\mu_\alpha}\right)^2 < \infty$$

this show that $\mathscr{L}_{u,v}(\sigma)$ is an Hilbert-Schmidt operator in particular $\mathscr{L}_{u,v}(\sigma)$ is compact. Let $\sigma \in L^p_\alpha\left(\mathbb{R}^{2d}_+\right), p \in [1, +\infty]$, Since $L^1_\alpha\left(\mathbb{R}^{2d}_+\right) \cap L^p_\alpha\left(\mathbb{R}^{2d}_+\right)$ is dense in $L^p_\alpha\left(\mathbb{R}^{2d}_+\right)$, there exists $(\sigma_k)_k \subset L^1_\alpha\left(\mathbb{R}^{2d}_+\right)$, such that

$$\lim_{k\to+\infty}\|\sigma_k - \sigma\|_{p,\mu_\alpha\otimes\mu_\alpha} = 0.$$

From the relation (5.9), we have $\|\mathscr{L}_{u,v}(\sigma_k) - \mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \le \|\sigma_k - \sigma\|_{p,\mu_\alpha\otimes\mu_\alpha}$. Consequently, $\lim_{k\to+\infty}\mathscr{L}_{u,v}(\sigma_k) = \mathscr{L}_{u,v}(\sigma)$, in $B\left(L^2_\alpha\left(\mathbb{R}^d_+\right)\right)$. Since the set of compact operators is a closed ideal of $B\left(L^2_\alpha\left(\mathbb{R}^d_+\right)\right)$, we deduce that $\mathscr{L}_{u,v}(\sigma)$ is a compact operator. The proof is complete. □

5.2. **Trace of the localization operators.** *In this subsection, we consider the separable Hilbert space $L^2_\alpha\left(\mathbb{R}^d_+\right)$, we will prove that for every $\sigma \in L^p_\alpha\left(\mathbb{R}^{2d}_+\right), p \in [1, +\infty]$, the localization operator $\mathscr{L}_{u,v}(\sigma)$ belongs to the Schatten-von Neumann classes S^p in particular for $\sigma \in L^1_\alpha\left(\mathbb{R}^{2d}_+\right)$ we have $\mathscr{L}_{u,v}(\sigma)$ belongs to the trace class and we give explicitly its trace.*

Theorem 5.3. *Let σ in $L^1_\alpha\left(\mathbb{R}^{2d}_+\right)$. Then the localization operator*

$$\mathscr{L}_{u,v}(\sigma) : L^2_\alpha\left(\mathbb{R}^d_+\right) \to L^2_\alpha\left(\mathbb{R}^d_+\right)$$

is in S_1 and we have

$$\|\widetilde{\sigma}\|_{1,\mu_\alpha\otimes\mu_\alpha} \le \|\mathscr{L}_{u,v}(\sigma)\|_{S_1} \le 4\|\sigma\|_{1,v_\alpha\otimes v_\alpha}. \tag{5.11}$$

where

$$\widetilde{\sigma}(x,y) = \langle \mathscr{L}_{u,v}(\sigma)(u^{x,y}) \mid v^{x,y}\rangle_\alpha$$

Proof. We begin first by proving the right hand of (5.11), we first assume that σ is non-negative. Let $(\varphi_k)_k$ be an orthonormal basis for $L^2_\alpha\left(\mathbb{R}^d_+\right)$. Then, by using Fubini's theorem and the Parseval's identity, we find that

(5.12)
$$\sum_{k=1}^{+\infty}\left\langle\mathscr{L}_{u,v}(\sigma)\left(\varphi_k\right)\mid\varphi_k\right\rangle_\alpha\leq\|\sigma\|_{1,\mu_\alpha\otimes\mu_\alpha},$$

for all orthonormal basis $\left(\varphi_k\right)_k$. Hence, by [21], the localization operator $\mathscr{L}_{u,v}(\sigma)$ is in S^1.Now to prove the right estimate in (5.11), let σ be a non-negative function in $L_\alpha^1\left(\mathbb{R}_+^{2d}\right)$ then

$$\left(\mathscr{L}_{u,v}^*(\sigma)\mathscr{L}_{u,v}(\sigma)\right)^{1/2}=\mathscr{L}_{u,v}(\sigma).$$

Thus, if $\left(\varphi_k\right)_k$ is an orthonormal basis for $L_\alpha^2\left(\mathbb{R}_+^d\right)$ consisting of eigenvalues of

$$\left(\mathscr{L}_{u,v}^*(\sigma)\mathscr{L}_{u,v}(\sigma)\right)^{1/2}:L_\alpha^2\left(\mathbb{R}_+^d\right)\rightarrow L_\alpha^2\left(\mathbb{R}_+^d\right)$$

by using the relation (5.12), we get

(5.13)
$$\left\|\mathscr{L}_{u,v}(\sigma)\right\|_{S_1}=\sum_{k=1}^{+\infty}\left\langle\left(\mathscr{L}_{u,v}^*(\sigma)\mathscr{L}_{u,v}(\sigma)\right)^{1/2}\left(\varphi_k\right)\mid\varphi_k\right\rangle_\alpha\leq\|\sigma\|_{1,\mu_\alpha\otimes\mu_\alpha}.$$

Now, for an arbitrary real-valued symbol σ in $L_\alpha^1\left(\mathbb{R}_+^{2d}\right)$ we can write $\sigma=\sigma_+-\sigma_-$, where

$$\sigma_+=\max(\sigma,0);\sigma-=-\min(\sigma,0).$$

Then, by using relation (5.13), we find that

(5.14)
$$\left\|\mathscr{L}_{u,v}(\sigma)\right\|_{S_1}\leq\left\|\mathscr{L}_{u,v}\left(\sigma_+\right)\right\|_{S_1}+\left\|\mathscr{L}_{u,v}\left(\sigma_-\right)\right\|_{S_1}\leq\left\|\sigma_+\right\|_{1,\mu_\alpha\otimes\mu_\alpha}+\left\|\sigma_-\right\|_{1,\mu_\alpha\otimes\mu_\alpha}\leq2\|\sigma\|_{1,\mu_\alpha\otimes v_\alpha}.$$

Finally, let σ in $L_\alpha^1\left(\mathbb{R}_+^{2d}\right)$ be a complex-valued function. Then, we can write
$\sigma=\sigma_1+i\sigma_2$, where σ_1 and σ_2 are the real and imaginary parts of σ respectively.From inequality (5.14), we get

$$\begin{aligned}\left\|\mathscr{L}_{u,v}(\sigma)\right\|_{S_1}&=\left\|\mathscr{L}_{u,v}(\sigma)\left(\sigma_1\right)+i\mathscr{L}_{u,v}(\sigma)\left(\sigma_2\right)\right\|_{S_1}\leq\left\|\mathscr{L}_{u,v}(\sigma)\left(\sigma_1\right)\right\|_{S_1}+\left\|\mathscr{L}_{u,v}(\sigma)\left(\sigma_2\right)\right\|_{S_1}\\&\leq2\left(\left\|\sigma_1\right\|_{1,v_\alpha\otimes v_\alpha}+\left\|\sigma_2\right\|_{1,v_\alpha\otimes v_\alpha}\right)\leq4\|\sigma\|_{1,\mu_\alpha\otimes\mu_\alpha}.\end{aligned}$$

Which proves the right hand of (5.11). For the left hand of (5.11), by using theorem 5.2 $\mathscr{L}_{u,v}(\sigma)$ is a compact operator from [21], $\mathscr{L}_{u,v}(\sigma)$ can be diagonalized as follows

(5.15)
$$\mathscr{L}_{u,v}(\sigma)(f)=\sum_{k=1}^{+\infty}s_k\left\langle f\mid\varphi_k\right\rangle_\alpha\psi_k$$

where $\left(s_k\right)_k$ are the positive singular values of $\mathscr{L}_{u,v}(\sigma),\left(\varphi_k\right)_k$ is an orthonormal basis for the orthogonal complement of the null space of $\mathscr{L}_{u,v}(\sigma)$ and $\left(\psi_k\right)_k$ is an orthonormal set in $L_\alpha^2\left(\mathbb{R}_+^d\right)$. Then

$$\sum_{k=1}^{+\infty}\left\langle\mathscr{L}_{u,v}(\sigma)\left(\varphi_k\right)\mid\psi_k\right\rangle_{\mu_\alpha}=\sum_{k=1}^{+\infty}s_k$$

Now it is easy to see that $\widetilde{\sigma}\in L_\alpha^1\left(\mathbb{R}_+^{2d}\right)$ and by using the canonical form of $\mathscr{L}_{u,v}(\sigma)$ (5.15) we find that

$$\int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d}|\tilde{\sigma}(x,y)|\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y)\leq\sum_{k=1}^{+\infty}s_k=\left\|\mathscr{L}_{u,v}(\sigma)\right\|_{S_1}.$$

The proof is complete.

□

In the following we give a trace formula for the localization operators $\mathscr{L}_{u,v}(\sigma)$.

Theorem 5.4. *For all $\sigma \in L^1_\alpha\left(\mathbb{R}^{2d}_+\right)$ we have the following trace formula*

$$\operatorname{Tr}\left(\mathscr{L}_{u,v}(\sigma)\right)=\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\sigma(x,y)\left\langle v^{x,y}\mid u^{x,y}\right\rangle_\alpha \mathrm{d}\mu_\alpha(x)\otimes \mathrm{d}\mu_\alpha(y).$$

Proof. let $(\varphi_k)_k$ an orthonormal basis in $L^2_\alpha\left(\mathbb{R}^d_+\right)$ by using theorem 5.3 the localization operators $\mathscr{L}_{u,v}(\sigma)$ belongs to S^1 and by using Parseval's identity and Fubini's theorem we get

$$\begin{aligned}\operatorname{Tr}\left(\mathscr{L}_{u,v}(\sigma)\right)&=\sum_{k=1}^{\infty}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\sigma(x,y)\left\langle \varphi_k\mid u^{x,y}\right\rangle_\alpha \overline{\left\langle \varphi_k\mid v^{x,y}\right\rangle_{\mu_\alpha}}\mathrm{d}\mu_\alpha(x)\otimes \mathrm{d}\mu_\alpha(y)\\&=\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\sigma(x,y)\left\langle v^{x,y}\mid u^{x,y}\right\rangle_\alpha \mathrm{d}\mu_\alpha(x)\otimes \mathrm{d}\mu_\alpha(y).\end{aligned}$$

□

We have the following result

Theorem 5.5. *Let $\sigma \in L^p_\alpha\left(\mathbb{R}^{2d}_+\right), p\in[1,+\infty]$ then the Gabor multipliers*

$$\mathscr{L}_{u,v}(\sigma): L^2_\alpha\left(\mathbb{R}^d_+\right)\to L^2_\alpha\left(\mathbb{R}^d_+\right)$$

is in S_p and we have

$$\left\|\mathscr{L}_{u,v}(\sigma)\right\|_{S_p}\leq 4^{\frac{1}{p}}\|\sigma\|_{p,\mu_\alpha\otimes\mu_\alpha}.$$

Proof. The result follows from the relations (5.9),(5.11) and by interpolation theorems [21]. □

6. Conclusion

The main purpose of this paper is to define a new time-frequency analysis called multidimensional Hankel-wavelet transform and to give some new results associated with this new integral transform as Parseval's, Plancherel's and Calderón's reproducing formulas. As application of these results we analyse the concentration of this transform on sets of finite measure and we give uncertainty principle for orthonormal sequences and Donoho-Stark's type uncertainty principle, also we introduce a new class of pseudo-differential operator $\mathscr{L}_{u,v}(\sigma)$ called localization operator which depend on a symbol σ and two functions u and v, we give a criteria in terms of the symbol σ for its boundedness and compactness, we also show that this operator belongs to the Schatten-Von Neumann classes S^p for all $p\in[1;+\infty]$ and we give a trace formula.

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