ISSN: 2832-4293

Pan-American Journal of Mathematics 4 (2025), 13
https://doi.org/10.28919/cpr-pajm/4-13

STUDY OF SEVERAL NEW VARIATIONS OF THE HARDY-HILBERT INTEGRAL INEQUALITY

CHRISTOPHE CHESNEAU

ABSTRACT. In this article, we introduce new variations of the Hardy-Hilbert integral inequality, incorporating weight functions that depend on sums, products, absolute differences of variables, and a power parameter. Some of these inequalities involve the primitives of the main functions, as well as original weight functions of logarithmic and exponential types, along with several adjustable parameters. Additionally, we extend our analysis to a three-dimensional setting, deriving related results of potential significance. The optimality of certain constants in the obtained inequalities is also established. By formulating these new variations, we expand the family of Hardy-Hilbert-type integral inequalities, with possible applications in various branches of mathematical analysis.

1. INTRODUCTION

The classical Hardy-Hilbert integral inequality, introduced by Hardy in [13], provides an upper bound on the weighted double integral of the product of two functions. A notable characteristic of this bound is that it depends on the unweighted integral norms of the functions. This result has many applications in analysis, particularly in operator theory. For a rigorous formulation, let $p > 1$, $q = p/(p-1)$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two (non-negative) functions. Then the Hardy-Hilbert integral inequality states that

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}, \tag{1.1}$$

provided that the two integrals on the right-hand side converge. The weight function is thus defined by $w(x, y) = 1/(x + y)$ and the constant in the factor by $\pi/\sin(\pi/p)$. The case $p = 2$ corresponds to the classical Hilbert integral inequality, with the constant in the factor reduced to π. The Hardy-Hilbert integral inequality has been studied extensively and has inspired numerous refinements, generalizations, and variations. Key works can be found in [2–4, 7–12, 14, 18, 21, 26–28, 31–33], complemented by the book [34]. A particularly important generalization was made in [29] with the introduction of a power parameter. Denoting this parameter as α, it takes into account the weight function defined by $w(x, y) = 1/(x + y)^\alpha$. A rigorous formulation of this generalization is presented below. Let $p > 1$, $q = p/(p-1)$, $\alpha > 2 - \min(p, q)$ and

LMNO, UNIVERSITÉ DE CAEN-NORMANDIE, CAEN 14032, FRANCE
E-mail address: christophe.chesneau@unicaen.fr.
Submitted on March 05, 2025.
2020 *Mathematics Subject Classification.* Primary 26D15; Secondary 33E20.
Key words and phrases. Hardy-Hilbert integral inequality; Hardy integral inequality; generalized Hölder integral inequality; Fubini-Tonelli integral theorem; optimal constants.

$f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\alpha} dxdy \leq B\left(1 - \frac{2-\alpha}{p}, 1 - \frac{2-\alpha}{p}\right) \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}, \tag{1.2}$$

where $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ with $a, b > 0$ is the standard beta function, provided that the two integrals on the right-hand side converge. Obviously, if we take $\alpha = 1$ and use the basic properties of the beta function, this inequality reduces to the classical Hardy-Hilbert integral inequality as given in Equation (1.1). Beyond this generalization, multidimensional modifications of the Hardy-Hilbert integral inequality have also been examined. For more information, we refer to [1, 6, 15–17, 19, 20, 22–24, 30, 35, 36]. These studies extend classical Hardy-Hilbert-type integral inequalities to multiple variables. The Hardy-Hilbert-type integral inequalities therefore remain an active area of research. In particular, new variations need to be developed to accommodate different function spaces and weighted versions, as well as to address applications in modern analysis.

In this article, we contribute to the study of Hardy-Hilbert integral inequalities by establishing new variations that incorporate weight functions depending on sums, products, absolute differences of variables, and a power parameter. To provide a precise formulation, our first main variations is presented below. Let $p > 1$, $q = p/(p-1)$, $\beta > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x)g(y)dxdy \leq \frac{1}{\beta} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}, \tag{1.3}$$

provided that the two integrals on the right-hand side converge. Two notable facts on the upper bounds are (i) the contrast between the complexity of the weight function, i.e., $w(x, y) = x^{1/p}y^{1/q}|x-y|^{\beta-1}/(x+y)^{\beta+1}$, and the simplicity of the constant in the factor, i.e., $1/\beta$, with no particular constraint on β other than $\beta > 0$ and no dependence on p, and (ii) the unweighted norms of f and g. Furthermore, we will prove that the constant $1/\beta$ is optimal; it cannot be improved in such a general setting. In a sense, this result is more tractable than some variations of the Hardy-Hilbert integral inequality with constants in the factor involving special functions, such as that in Equation (1.2). To support this claim, we use our first result to derive several new integral inequalities. Some of these involve the primitives of the main functions, in the same spirit as the classical Hardy integral inequality (see [14]), while others involve weight functions of the power, logarithmic and exponential forms. They also have the property of depending on one or two parameters, which gives them an interesting degree of flexibility. The originality of these inequalities opens up new perspectives for possible applications.

We then develop another variation based on the same mathematical framework, incorporating the weight function defined by $w(x, y) = x^{1/p}y^{1/q}|1-xy|^{\gamma-1}/(1+xy)^{\gamma+1}$ with $\gamma > 0$. This can be seen as a variable product (or multiplicative) modification of the first proposed variation. Due to its flexibility, several additional integral inequalities are established, analogous to those derived for the first variation. The key innovations lie in the originality and adaptability of the resulting inequalities.

In the final part, we explore a technical three-dimensional extension of this variation, further extending the scope of Hardy-Hilbert-type integral inequalities to higher-dimensional settings. This leads to the derivation of new integral inequalities, including those with original power, logarithmic and exponential weight functions, marking a significant advance in the three-dimensional case.

The rest of the article consists of the following sections: Section 2 presents the first variation of the Hardy-Hilbert integral inequality. The other two variations are studied in Sections 3 and 4, respectively. The proofs of all results are contained in Section 5. Section 6 concludes the article with some comments and perspectives.

2. First variation

This section deals with the first proposed variation of the Hardy-Hilbert integral inequality, as described in Equation (1.3).

2.1. Preliminary result.

We start with a preliminary result that relates our first variation to the inequality in Equation (1.2).

Proposition 2.1. *Let $p > 1$, $q = p/(p-1)$, $\beta \geq 1$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) g(y) dx dy \leq \left[\int_0^{+\infty} x f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y g^q(y) dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

The proof of this statement, as well as that of all the results, is given in Section 5. We now claim that this result can be refined, with a sharper constant in the factor and unweighted integral norms of the main functions. This is developed in more detail in the subsection below.

2.2. Main results.

We are now in the position to formulate the main result.

Theorem 2.2. *Let $p > 1$, $q = p/(p-1)$, $\beta > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) g(y) dx dy \leq \frac{1}{\beta} \left[\int_0^{+\infty} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y) dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

Remark 2.3. *In the framework of Theorem 2.2, the inequality can also be expressed as follows:*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f_\Upsilon(x) g_\Upsilon(y) dx dy \leq \frac{1}{\beta} \left[\int_0^{+\infty} \frac{1}{x} f_\Upsilon^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y} g_\Upsilon^q(y) dy\right]^{1/q},$$

where $f_\Upsilon(x) = x^{1/p} f(x)$ and $g_\Upsilon(y) = y^{1/q} g(y)$ are adjustable functions, provided that the two integrals on the right-hand side converge.

Remark 2.4. *If we take $p = 2$, the inequality in Theorem 2.2 becomes*

$$\int_0^{+\infty} \int_0^{+\infty} \sqrt{xy} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) g(y) dx dy \leq \frac{1}{\beta} \sqrt{\int_0^{+\infty} f^2(x) dx} \sqrt{\int_0^{+\infty} g^2(y) dy}.$$

To the best of our knowledge, Theorem 2.2 introduces a new variation of the Hardy-Hilbert integral inequality. Note that the case $\beta \in (0,1)$ is covered contrary to the inequality in Proposition 2.1. As an example, if we take, $\beta = 1/2$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/p}y^{1/q}}{(x+y)^{3/2}\sqrt{|x-y|}} f(x)g(y)dxdy \leq 2\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

This theorem is also remarkable for the simplicity of the constant in the factor, i.e., $1/\beta$, and the fact that it does not depend on p. This independence is significant, as many classical results involve constants that explicitly depend on p. Furthermore, this constant cannot be improved, as formulated in the proposition below.

Proposition 2.5. *In the framework of Theorem 2.2, the constant in the factor, i.e., $1/\beta$, is optimal.*

We will exploit the simplicity, optimality and originality of Theorem 2.2 to derive new integral inequalities with a certain potential of application in analysis, starting with the proposition below.

Proposition 2.6. *Let $p > 1$, $q = p/(p-1)$, $\beta > 0$ and $f : [0,+\infty) \mapsto [0,+\infty)$ be a function. Then the following inequality holds:*

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)dx\right]^p dy \leq \frac{1}{\beta^p}\int_0^{+\infty} f^p(x)dx,$$

provided that the integral on the right-hand side converges.

By defining $L_p((0+\infty)) = \{f : [0,+\infty) \mapsto [0,+\infty); \|f\|_p^p = \int_0^{+\infty} f^p(x)dx < +\infty\}$, this result thus shows the boundedness of the linear operator $R : L_p([0+\infty)) \mapsto L_p([0+\infty))$ defined by

$$R(f)(y) = \int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)dx,$$

i.e., $\|R(f)\|_p \leq (1/\beta)\|f\|_p$. As far as we know, this is a new result in operator theory.

2.3. **Complementary results involving primitives.** In the spirit of the classical Hardy integral inequality, we can use Theorem 2.2 to derive variations of the Hardy-Hilbert integral inequality that are defined with primitives of the main functions. The first result of this kind is presented below.

Proposition 2.7. *Let $p > 1$, $q = p/(p-1)$, $\beta > 0$, $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two functions and $F, G : [0,+\infty) \mapsto [0,+\infty)$ be defined by*

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt,$$

provided that the integrals converge. Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}F(x)G(y)dxdy \leq \frac{1}{\beta}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

In the proof, the constant in the factor is based on two optimal constants: that of Theorem 2.2, i.e., $1/\beta$, and that of the Hardy integral inequality, i.e., $[p/(p-1)]^p$ when defined with the parameter p. We therefore claim that it is sharp by construction. Using the relation $1/p+1/q=1$, we can also express this constant as follows:

$$\frac{1}{\beta}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)=\frac{p^2}{\beta(p-1)}.$$

This form is more concise, but we lose some of the understanding of the underlying use of the Hardy integral inequality. We also emphasize the unweighted integral norms of f and g in the upper bound.

A primitive variation of Proposition 2.6 is developed below.

Proposition 2.8. *Let $p>1$, $q=p/(p-1)$, $\beta>0$, $f:[0,+\infty)\mapsto[0,+\infty)$ be a function and $F:[0,+\infty)\mapsto[0,+\infty)$ be defined by*

$$F(x)=\int_0^x f(t)dt,$$

provided that the integral converges. Then the following inequality holds:

$$\int_0^{+\infty}\left[\int_0^{+\infty}x^{1/p-1}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}F(x)dx\right]^p dy\leq\frac{1}{\beta^p}\left(\frac{p}{p-1}\right)^p\int_0^{+\infty}f^p(x)dx,$$

provided that the integral on the right-hand side converges.

This proposition ensures the boundedness of the linear operator $S:L_p([0+\infty))\mapsto L_p([0+\infty))$ defined by

$$\begin{aligned}S(f)(y)&=\int_0^{+\infty}x^{1/p-1}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}F(x)dx\\&=\int_0^{+\infty}x^{1/p-1}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}\left[\int_0^x f(t)dt\right]dx,\end{aligned}$$

i.e., $\|S(f)\|_p\leq(1/\beta)[p/(p-1)]\|f\|_p$. As far as we know, this is a new result in operator theory.

The rest of this section deals with complementary integral inequalities of the Hardy-Hilbert type involving different parametric weight functions.

2.4. **Complementary results involving different weight functions.** A variation of the Hardy-Hilbert integral inequality using a one-parameter logarithmic power weight is given below.

Proposition 2.9. *Let $p>1$, $q=p/(p-1)$, $\delta>1$ and $f,g:[0,+\infty)\mapsto[0,+\infty)$ be two functions. Then the following inequality holds:*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\frac{1}{\log[(x+y)/|x-y|]}\left[1-\left(\frac{|x-y|}{x+y}\right)^{\delta-1}\right]f(x)g(y)dxdy\\&\leq\log(\delta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},\end{aligned}$$

provided that the two integrals on the right-hand side converge.

The proof is based on integrating both sides of the inequality in Theorem 2.2 with respect to β. To the best of our knowledge, this inequality is not referenced elsewhere. We emphasize the exact logarithmic constant $\log(\delta)$, as well as its simplicity and independence on p. In addition to being original, the parameter δ offers a degree of flexibility that can be adapted to different mathematical situations.

A two-parameter power variation of Theorem 2.2 is described below.

Proposition 2.10. *Let $p > 1$, $q = p/(p-1)$, $\eta \in (0,1)$, $\varepsilon > 0$ and $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\varepsilon-1}(x+y)^{\varepsilon-1}}{[(x+y)^{\varepsilon}-\eta|x-y|^{\varepsilon}]^2}f(x)g(y)dxdy$$
$$\leq \frac{1}{\varepsilon(1-\eta)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

The proof is based on Theorem 2.2 and geometric series formulas. Again, we point out the simplicity of the constant in the factor.

The proposition below offers a one-parameter logarithmic variation of the inequality Theorem 2.2.

Proposition 2.11. *Let $p > 1$, $q = p/(p-1)$, $\xi > 0$ and $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\log\left[\frac{(x+y)^{\xi}}{(x+y)^{\xi}-|x-y|^{\xi}}\right]f(x)g(y)dxdy$$
$$\leq \frac{\pi^2}{6\xi}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

The proof uses Theorem 2.2, the classical formula $\sum_{i=1}^{+\infty} 1/i^2 = \pi^2/6$ explaining the presence of π and a logarithmic series formula. Note that, if we take $\xi = 1$ and use $\min(x,y) = (1/2)(x+y-|x-y|)$ for $x, y \in \mathbb{R}$, the inequality is reduced to

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\log\left[\frac{x+y}{2\min(x,y)}\right]f(x)g(y)dxdy$$
$$\leq \frac{\pi^2}{6}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The significance of this result lies in the presence of the minimum variable, the complexity of the weight function, and the simplicity of the constant in the factor.

A two-parameter exponential power variation of Theorem 2.2 is given below.

Proposition 2.12. *Let $p > 1$, $q = p/(p-1)$, $\upsilon > 0$, $\zeta > 0$ and $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left(\frac{|x-y|}{x+y}\right)^{\zeta}\exp\left[\upsilon\left(\frac{|x-y|}{x+y}\right)^{\zeta}\right]f(x)g(y)dxdy$$
$$\leq \frac{\exp(\upsilon)-1}{\zeta\upsilon}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

The proof relies on Theorem 2.2 and exponential series formulas.

All these integral inequalities illustrate the possible applications of Theorem 2.2. Furthermore, the techniques used in the proofs also contain some ideas that may inspire purposes beyond this article.

3. Second variation

In this section, we emphasize another variation that introduces a product variable modification, and several applications.

3.1. **Main results.** A product variation of Theorem 2.2 is proposed in the statement below.

Theorem 3.1. *Let $p > 1$, $q = p/(p-1)$, $\gamma > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)g(y)dxdy$$
$$\leq \frac{1}{\gamma}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

Remark 3.2. *In the framework of Theorem 3.1, the inequality can also be expressed as follows:*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f_\curlyvee(x)g_\curlyvee(y)dxdy$$
$$\leq \frac{1}{\gamma}\left[\int_0^{+\infty} \frac{1}{x}f_\curlyvee^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} \frac{1}{y}g_\curlyvee^q(y)dy\right]^{1/q},$$

where $f_\curlyvee(x) = x^{1/p}f(x)$ and $g_\curlyvee(y) = y^{1/q}g(y)$ are adjustable functions, provided that the two integrals on the right-hand side converge.

Remark 3.3. *If we take $p = 2$, the inequality in Theorem 3.1 is simplified as follows:*

$$\int_0^{+\infty}\int_0^{+\infty} \sqrt{xy}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)g(y)dxdy \leq \frac{1}{\gamma}\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy}.$$

Compared to Theorem 2.2, the weight function $w(x,y) = x^{1/p}y^{1/q}|x-y|^{\beta-1}/(x+y)^{\beta+1}$ has been replaced by $w(x,y) = x^{1/p}y^{1/q}|1-xy|^{\gamma-1}/(1+xy)^{\gamma+1}$, which is a product variable modification. The upper bound is still defined with the unweighted norms of f and g. The constant in the factor remains simple and independent of p. Furthermore, it cannot be improved, as formulated in the following proposition.

Proposition 3.4. *In the framework of Theorem 3.1, the constant in the factor, i.e., $1/\gamma$, is optimal.*

Thanks to the tractability of Theorem 3.1, it can be used in various mathematical scenarios. As an example, the result below suggests an integral inequality from which we can give an operator interpretation.

Proposition 3.5. *Let $p > 1$, $q = p/(p-1)$, $\gamma > 0$ and $f : [0, +\infty) \mapsto [0, +\infty)$ be a function. Then the following inequality holds:*

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)dx\right]^p dy \leq \frac{1}{\gamma^p}\int_0^{+\infty} f^p(x)dx,$$

provided that the integral on the right-hand side converges.

If we consider the linear operator $T : L_p([0+\infty)) \mapsto L_p([0+\infty))$ defined by

$$T(f)(y) = \int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)dx,$$

this proposition ensures that $\|T(f)\|_p \leq (1/\gamma)\|f\|_p$. It is thus bounded, which opens up some possibilities for defining new function spaces and further operator analysis.

3.2. **Complementary results involving primitives.** This subsection is devoted to the derivation of Hardy-Hilbert type integral inequalities using primitives of the main functions. The first such result is presented below.

Proposition 3.6. *Let $p > 1$, $q = p/(p-1)$, $\gamma > 0$, $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two functions and $F, G : [0,+\infty) \mapsto [0,+\infty)$ be defined by*

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt,$$

provided that the integrals converge. Then the following inequality holds:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}F(x)G(y)dxdy \\ &\leq \frac{1}{\gamma}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},\end{aligned}$$

provided that the two integrals on the right-hand side converge.

This proposition can be seen as a product variable variation of Proposition 2.7. The expression of the constant in the factor uses that of Theorem 3.1, i.e., $1/\gamma$, and that of the Hardy integral inequality. Since both are optimal in their respective context, the product constant obtained can be claimed to be sharp.

A primitive variation of Proposition 3.6 is developed below.

Proposition 3.7. *Let $p > 1$, $q = p/(p-1)$, $\gamma > 0$, $f : [0,+\infty) \mapsto [0,+\infty)$ be a function and $F : [0,+\infty) \mapsto [0,+\infty)$ be defined by*

$$F(x) = \int_0^x f(t)dt,$$

provided that the integral converges. Then the following inequality holds:

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}F(x)dx\right]^p dy \leq \frac{1}{\gamma^p}\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx,$$

provided that the integral on the right-hand side converges.

As a consequence of this proposition, let us consider the linear operator $U : L_p([0+\infty)) \mapsto L_p([0+\infty))$ defined by

$$\begin{aligned}U(f)(y) &= \int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}F(x)dx \\ &= \int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}\left[\int_0^x f(t)dt\right]dx.\end{aligned}$$

We then have $\|U(f)\|_p \leq (1/\gamma)[p/(p-1)]\|f\|_p$. This completes the operator theory with a one-parameter operator, which is innovative by its functional structure.

The rest of this section deals with complementary integral inequalities of the Hardy-Hilbert type with different parametric weight functions.

3.3. **Complementary results involving different weight functions.** A variation of Theorem 3.1 using a one-parameter logarithmic power weight is given below.

Proposition 3.8. *Let $p > 1$, $q = p/(p-1)$, $\nu > 1$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\frac{1}{\log[(1+xy)/|1-xy|]}\left[1-\left(\frac{|1-xy|}{1+xy}\right)^{\nu-1}\right]f(x)g(y)dxdy$$
$$\leq \log(\nu)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

This result can be seen as a product variable modification of Proposition 2.9, with the same mathematical foundations for the proofs.

A two-parameter power variation of Theorem 3.1 is described below.

Proposition 3.9. *Let $p > 1$, $q = p/(p-1)$, $\phi \in (0, 1)$, $\psi > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\psi-1}(1+xy)^{\psi-1}}{[(1+xy)^{\psi}-\phi|1-xy|^{\psi}]^2}f(x)g(y)dxdy$$
$$\leq \frac{1}{\psi(1-\phi)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

This result can be seen as a product variable modification of Proposition 2.10. The proof is based on Theorem 3.1 and geometric series formulas. Again, we point out the simplicity of the constant in the factor.

The proposition below offers a one-parameter logarithmic variation of the inequality Theorem 3.1.

Proposition 3.10. *Let $p > 1$, $q = p/(p-1)$, $\varphi > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{(1+xy)^{\varphi}}{(1+xy)^{\varphi}-|1-xy|^{\varphi}}\right]f(x)g(y)dxdy$$
$$\leq \frac{\pi^2}{6\varphi}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

This result can be interpreted as a product variable modification of Proposition 2.11. The proof is based on Theorem 3.1, the classical formula $\sum_{i=1}^{+\infty} 1/i^2 = \pi^2/6$ and a logarithmic series formula. In particular, if we take $\varphi = 1$, we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{1+xy}{2\min(1,xy)}\right]f(x)g(y)dxdy$$
$$\leq \frac{\pi^2}{6}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The minimum variable adds a degree of originality. We also emphasize the contrast between the complexity of the weight function and the simplicity of the constant in the factor.

A two-parameter exponential power variation of Theorem 3.1 is given below.

Proposition 3.11. *Let $p > 1$, $q = p/(p-1)$, $\varpi > 0$, $\chi > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{1}{(1+xy)|1-xy|} \left(\frac{|1-xy|}{1+xy}\right)^{\chi} \exp\left[\varpi \left(\frac{|1-xy|}{1+xy}\right)^{\chi}\right] f(x)g(y)dxdy$$
$$\leq \frac{\exp(\varpi) - 1}{\chi\varpi} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals on the right-hand side converge.

This proposition can be seen as a product variable modification of Proposition 2.12. The proof relies on Theorem 3.1 and exponential series formulas.

All the integral inequalities presented in this section are thus derived from Theorem 3.1. This illustrates the versatility of the theorem and its power to produce innovative results in this area.

In the next section, a three-dimensional framework is examined, adapting the weight function of the second variation for this purpose.

4. Third variation

This section focuses on a three-dimensional variation of Theorem 3.1.

4.1. **Main result.**

The main three-dimensional integral inequality is given below.

Theorem 4.1. *Let $p, q > 1$, $r = pq/(pq-p-q)$, $\sigma > 0$, $\theta > \max(1/p, 1/q, 1/r)$ and $f, g, h : [0, +\infty) \mapsto [0, +\infty)$ be three functions. Then the following inequality holds:*

$$\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r} \frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}} f(x)g(y)h(z)dxdydz$$
$$\leq \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \times$$
$$\left[\int_0^{+\infty} (1+x)^{\theta p} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} (1+y)^{\theta q} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty} (1+z)^{\theta r} h^r(z)dz\right]^{1/r},$$

provided that the three integrals on the right-hand side converge.

The proof is based on a suitable decomposition of the integrand using the intermediate parameter θ, the generalized Hölder integral inequality (see [5, 25]) and sophisticated integral developments. To the best of our knowledge, it is new in the literature and thus offers a new perspective for various three-dimensional analysis frameworks. The parameter θ can be chosen to make the integrals convergent in the upper bound or to adapt to a particular situation.

4.2. **Complementary result involving primitives.**

The result below can be seen as a three-dimensional variation of Proposition 3.6. It has the property of including the primitives of the main functions.

Proposition 4.2. *Let $p, q > 1$, $r = pq/(pq-p-q)$, $\sigma > 0$, $\theta > \max(1/p, 1/q, 1/r)$, $f, g, h : [0, +\infty) \mapsto [0, +\infty)$ be three functions and $F, G, H : [0, +\infty) \mapsto [0, +\infty)$ be defined by*

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt, \quad H(z) = \int_0^z h(t)dt,$$

provided that the integrals converge. Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} x^{-1/q}y^{-1/r}z^{-1/p}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}\frac{F(x)G(y)H(z)}{(1+x)^{\theta}(1+y)^{\theta}(1+z)^{\theta}}dxdydz$$
$$\leq \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left(\frac{r}{r-1}\right)\times$$
$$\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty} h^r(z)dz\right]^{1/r},$$

provided that the three integrals on the right-hand side converge.

The proof is based on a special configuration of Theorem 4.1 and the classical Hardy integral inequality. The constant in the factor and the unweighted integral norms of the main functions are advantages of this result. This allows it to be adapted to a wide range of situations.

Using the relation $1/p+1/q+1/r=1$, the constant the factor can also be expressed as follows:

$$\frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left(\frac{r}{r-1}\right)$$
$$=\frac{p^2q^2}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}(p-1)(q-1)(p+q)}.$$

This form is more concise, but we lose some of the information about the use of the Hardy integral inequality in the corresponding proof.

4.3. **Complementary results involving different weight functions.** Theorem 4.1 is now completed by several three-dimensional integral inequalities involving original weight functions, starting with a one-parameter logarithmic power weight function in the result below.

Proposition 4.3. *Let $p,q>1$, $r=pq/(pq-p-q)$, $\omega>1$, $\theta>\max(1/p,1/q,1/r)$ and $f,g,h:[0,+\infty)\mapsto[0,+\infty)$ be three functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)^2}\times$$
$$\frac{1}{\log[(1+xyz)/|1-xyz|]}\left[1-\left(\frac{|1-xyz|}{1+xyz}\right)^{\omega-1}\right]f(x)g(y)h(z)dxdydz$$
$$\leq\frac{\log(\omega)}{(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r},$$

provided that the three integrals on the right-hand side converge.

The result below can be presented as a three-dimensional variation of Proposition 3.9.

Proposition 4.4. *Let $p,q>1$, $r=pq/(pq-p-q)$, $\theta>\max(1/p,1/q,1/r)$, $\iota\in(0,1)$, $\varsigma>0$, and $f,g,h:[0,+\infty)\mapsto[0,+\infty)$ be three functions. Then the following inequality holds: Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\varsigma-1}(1+xyz)^{\varsigma-1}}{[(1+xyz)^{\varsigma}-\iota|1-xyz|^{\varsigma}]^2}f(x)g(y)h(z)dxdydz$$
$$\leq\frac{1}{\varsigma(1-\iota)(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r},$$

provided that the three integrals on the right-hand side converge.

The proof is based on Theorem 4.1 and geometric series formulas. Again, we point out the simplicity of the constant in the factor and the presence of θ that allows another degree of flexibility.

The proposition below offers a two-parameter logarithmic variation of Theorem 4.1.

Proposition 4.5. *Let $p, q > 1$, $r = pq/(pq-p-q)$, $\theta > \max(1/p, 1/q, 1/r)$, $\tau > 0$ and $f, g, h : [0, +\infty) \mapsto [0, +\infty)$ be three functions. Then the following inequality holds: Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\times$$
$$\log\left[\frac{(1+xyz)^\tau}{(1+xyz)^\tau-|1-xyz|^\tau}\right]f(x)g(y)h(z)dxdydz$$
$$\leq \frac{\pi^2}{6\tau(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r},$$

provided that the three integrals on the right-hand side converge.

The proof relies on the use of Theorem 4.1, the formula $\sum_{i=1}^{+\infty} 1/i^2 = \pi^2/6$ and a logarithmic series formula. In particular, if we take $\tau = 1$, we find that

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\log\left[\frac{1+xyz}{2\min(1,xyz)}\right]\times$$
$$f(x)g(y)h(z)dxdydz$$
$$\leq \frac{\pi^2}{6(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}.$$

We end this section with a three-parameter exponential power variation of Theorem 4.1.

Proposition 4.6. *Let $p, q > 1$, $r = pq/(pq-p-q)$, $\theta > \max(1/p, 1/q, 1/r)$, $\kappa > 0$, $\zeta > 0$, and $f, g, h : [0, +\infty) \mapsto [0, +\infty)$ be three functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\left(\frac{|1-xyz|}{1+xyz}\right)^\zeta\times$$
$$\exp\left[\kappa\left(\frac{|1-xyz|}{1+xyz}\right)^\zeta\right]f(x)g(y)h(z)dxdydz$$
$$\leq \frac{\exp(\kappa)-1}{\kappa\zeta(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r},$$

provided that the three integrals on the right-hand side converge.

This proposition can be seen as a three-dimensional extension of Proposition 3.11. The proof is based on Theorem 4.1 and exponential series formulas.

The detailed proofs of all results are given in the next section.

5. PROOFS

This section contains four subsections. The first is devoted to an intermediate result, while the remaining three are devoted to results related to the first, second and third variations, respectively.

5.1. An intermediate lemma.

The lemma below is needed to prove our main results.

Lemma 5.1. *Let $\epsilon > 0$. Then we have*

$$\int_0^{+\infty} \frac{|1-x|^{\epsilon-1}}{(1+x)^{\epsilon+1}} dx = \frac{1}{\epsilon}.$$

Proof of Lemma 5.1. Using the Chasles integral relation at the cutoff value $x = 1$, we have

$$\int_0^{+\infty} \frac{|1-x|^{\epsilon-1}}{(1+x)^{\epsilon+1}} dx = \int_0^{1} \frac{(1-x)^{\epsilon-1}}{(1+x)^{\epsilon+1}} dx + \int_1^{+\infty} \frac{(x-1)^{\epsilon-1}}{(1+x)^{\epsilon+1}} dx.$$

Making the change of variables $x = 1/y$, the second integral term of the sum can be expressed as follows:

$$\int_1^{+\infty} \frac{(x-1)^{\epsilon-1}}{(1+x)^{\epsilon+1}} dx = \int_1^{0} \frac{(1/y-1)^{\epsilon-1}}{(1+1/y)^{\epsilon+1}} \left(-\frac{1}{y^2}dy\right) = \int_0^{1} \frac{(1-y)^{\epsilon-1}}{(1+y)^{\epsilon+1}} dy.$$

This corresponds to the first integral term of the sum.

We therefore have

$$\int_0^{+\infty} \frac{|1-x|^{\epsilon-1}}{(1+x)^{\epsilon+1}} dx = 2\int_0^{1} \frac{(1-x)^{\epsilon-1}}{(1+x)^{\epsilon+1}} dx. \tag{5.1}$$

Making the change of variables $z = (1-x)/(1+x)$, with $dz = [-2/(1+x)^2]dx$, we obtain

$$2\int_0^{1} \frac{(1-x)^{\epsilon-1}}{(1+x)^{\epsilon+1}} dx = \int_0^{1} \left(\frac{1-x}{1+x}\right)^{\epsilon-1} \frac{2}{(1+x)^2} dx = \int_1^{0} z^{\epsilon-1}(-dz)$$

$$= \int_0^{1} z^{\epsilon-1} dz = \left[\frac{1}{\epsilon} z^{\epsilon}\right]_{z=0}^{z=1} = \frac{1}{\epsilon}. \tag{5.2}$$

Putting Equations (5.1) and (5.2) together, we get

$$\int_0^{+\infty} \frac{|1-x|^{\epsilon-1}}{(1+x)^{\epsilon+1}} dx = \frac{1}{\epsilon}.$$

This completes the proof of Lemma 5.1. □

5.2. Proofs related to the first variation.

Proof of Proposition 2.1. The triangle inequality and the non-decreasing property of the function $t^{\beta-1}$, $t \in (0, +\infty)$, with $\beta \geq 1$, give, for any $x, y \in [0, +\infty)$,

$$|x-y|^{\beta-1} \leq (|x|+|y|)^{\beta-1} = (x+y)^{\beta-1}.$$

Since f and g are non-negative, we get

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x)g(y)dxdy$$

$$\leq \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q} \frac{(x+y)^{\beta-1}}{(x+y)^{\beta+1}} f(x)g(y)dxdy$$

$$= \int_0^{+\infty}\int_0^{+\infty} \frac{f_\star(x)g_\star(y)}{(x+y)^2} dxdy, \tag{5.3}$$

where $f_\star(x) = x^{1/p} f(x)$ and $g_\star(x) = x^{1/q} g(x)$. It follows from Equation (1.2) applied with these two functions and $\alpha = 2$, and $B(1,1) = 1$, that

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f_\star(x) g_\star(y)}{(x+y)^\alpha} dxdy \leq B(1,1) \left[\int_0^{+\infty} f_\star^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g_\star^q(y)dy\right]^{1/q}$$

$$= \left[\int_0^{+\infty} x f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y g^q(y)dy\right]^{1/q}. \tag{5.4}$$

Putting Equations (5.3) and (5.4) together, we obtain

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x)g(y)dxdy$$
$$\leq \left[\int_0^{+\infty} x f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y g^q(y)dy\right]^{1/q}.$$

This concludes the proof of Proposition 2.1. □

Proof of Theorem 2.2. Decomposing the integrand appropriately, using the relation $1/p + 1/q = 1$ and applying the Hölder integral inequality, we have

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x)g(y)dxdy$$
$$= \int_0^{+\infty} \int_0^{+\infty} x^{1/p} \frac{|x-y|^{(\beta-1)/p}}{(x+y)^{(\beta+1)/p}} f(x) \times y^{1/q} \frac{|x-y|^{(\beta-1)/q}}{(x+y)^{(\beta+1)/q}} g(y)dxdy$$
$$\leq \left[\int_0^{+\infty} \int_0^{+\infty} x \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f^p(x)dxdy\right]^{1/p} \left[\int_0^{+\infty} \int_0^{+\infty} y \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} g^q(y)dxdy\right]^{1/q}. \tag{5.5}$$

Let us now determine the two double integral terms of this bound. For the first double integral term, using the Fubini-Tonelli integral theorem to exchange the order of integration, the change of variables $u = y/x$ and Lemma 5.1 with $\epsilon = \beta$, we obtain

$$\int_0^{+\infty} \int_0^{+\infty} x \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f^p(x)dxdy = \int_0^{+\infty} f^p(x) \left[\int_0^{+\infty} \frac{|1-y/x|^{\beta-1}}{(1+y/x)^{\beta+1}} \frac{1}{x} dy\right] dx$$
$$= \int_0^{+\infty} f^p(x) \left[\int_0^{+\infty} \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} du\right] dx = \frac{1}{\beta} \int_0^{+\infty} f^p(x)dx. \tag{5.6}$$

For the second double integral term, similarly but with the change of variables $v = x/y$, we obtain

$$\int_0^{+\infty} \int_0^{+\infty} y \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} g^q(y)dxdy = \int_0^{+\infty} g^q(y) \left[\int_0^{+\infty} \frac{|1-x/y|^{\beta-1}}{(1+x/y)^{\beta+1}} \frac{1}{y} dx\right] dy$$
$$= \int_0^{+\infty} g^q(y) \left[\int_0^{+\infty} \frac{|1-v|^{\beta-1}}{(1+v)^{\beta+1}} dv\right] dy = \frac{1}{\beta} \int_0^{+\infty} g^p(y)dy. \tag{5.7}$$

Putting Equations (5.5), (5.6) and (5.7) together and using the relation $1/p + 1/q = 1$, we get

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x)g(y)dxdy$$
$$\leq \left[\frac{1}{\beta} \int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\frac{1}{\beta} \int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\beta} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

This concludes the proof of Theorem 2.2. □

Proof of Proposition 2.5. We proceed by contradiction. Based on the statement of Theorem 2.2, let us suppose the existence of a constant $\kappa \in (0, 1/\beta)$ such that, for any $f, g : [0, +\infty) \mapsto [0, +\infty)$,

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)g(y)dxdy\\ &\leq \kappa\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},\end{aligned} \tag{5.8}$$

provided that the two integrals on the right-hand side converge. For any $n \in \mathbb{N}\backslash\{0\}$, we consider the function $f_n : [0, +\infty) \mapsto [0, +\infty)$ defined by $f_n(x) = 0$ for any $x \in [0, 1)$, and $f_n(x) = x^{-(1+1/n)/p}$ for any $x \in [1, +\infty)$, and the function $g_n : [0, +\infty) \mapsto [0, +\infty)$ defined by $g_n(y) = 0$ for any $y \in [0, 1)$, and $g_n(y) = y^{-(1+1/n)/q}$ for any $y \in [1, +\infty)$. We then have

$$\int_0^{+\infty} f_n^p(x)dx = \int_1^{+\infty}(x^{-(1+1/n)/p})^p dx = \left[-nx^{-1/n}\right]_{x=1}^{x\to+\infty} = n$$

and, similarly,

$$\int_0^{+\infty} g_n^q(y)dy = \int_1^{+\infty}(y^{-(1+1/n)/q})^q dy = \left[-ny^{-1/n}\right]_{y=1}^{y\to+\infty} = n.$$

By these integral values combined with the relation $1/p + 1/q = 1$ and Equation (5.8), we obtain

$$\begin{aligned}\kappa &= \kappa\frac{1}{n}n^{1/p}n^{1/q} = \frac{1}{n}\left\{\kappa\left[\int_0^{+\infty} f_n^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_n^q(y)dy\right]^{1/q}\right\}\\ &\geq \frac{1}{n}\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f_n(x)g_n(y)dxdy.\end{aligned} \tag{5.9}$$

We now aim to determine the double integral term. Using the definitions of f_n and g_n, the change of variables $x = uy$, the Fubini-Tonelli integral theorem and the relation $1/p + 1/q = 1$, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f_n(x)g_n(y)dxdy\\ &= \int_1^{+\infty}\int_1^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}x^{-(1+1/n)/p}y^{-(1+1/n)/q}dxdy\\ &= \int_1^{+\infty}\left[\int_1^{+\infty}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}x^{-1/(np)}dx\right]y^{-1/(nq)}dy\\ &= \int_1^{+\infty}\left[\int_{1/y}^{+\infty}\frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}}\frac{1}{y^2}u^{-1/(np)}y^{-1/(np)}ydu\right]y^{-1/(nq)}dy\\ &= \int_1^{+\infty}\left[\int_{1/y}^{+\infty}\frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}}u^{-1/(np)}du\right]y^{-(1+1/n)}dy.\end{aligned} \tag{5.10}$$

It follows from the Chasles integral relation at the cutoff value $u = 1$, the Fubini-Tonelli integral theorem, simple integral calculus and the relation $1/p + 1/q = 1$ that

$$\begin{aligned}&\int_1^{+\infty}\left[\int_{1/y}^{+\infty}\frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}}u^{-1/(np)}du\right]y^{-(1+1/n)}dy\\ &= \int_1^{+\infty}\left[\int_{1/y}^{1}\frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}}u^{-1/(np)}du\right]y^{-(1+1/n)}dy\\ &+ \int_1^{+\infty}\left[\int_1^{+\infty}\frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}}u^{-1/(np)}du\right]y^{-(1+1/n)}dy\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left[\int_{1/u}^{+\infty} y^{-(1+1/n)} dy\right] \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{-1/(np)} du \\
&+ \left[\int_1^{+\infty} \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{-1/(np)} du\right] \left[\int_1^{+\infty} y^{-(1+1/n)} dy\right] \\
&= \int_0^1 (nu^{1/n}) \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{-1/(np)} du + n\left[\int_1^{+\infty} \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{-1/(np)} du\right] \\
&= n\left[\int_0^1 \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{1/(nq)} du + \int_1^{+\infty} \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{-1/(np)} du\right]. \qquad (5.11)
\end{aligned}
$$

Putting Equations (5.9), (5.10) and (5.11) together, we get

$$
\kappa \geq \int_0^1 \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{1/(nq)} du + \int_1^{+\infty} \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{-1/(np)} du.
$$

This is valid for any $n \in \mathbb{N}\backslash\{0\}$. Applying the inferior limit with respect to n, the Fatou integral lemma, which is possible because the integrand is non-negative, $\liminf_{n\to+\infty} u^{1/(nq)} = 1$ for $u \in (0,1)$, $\liminf_{n\to+\infty} u^{-1/(np)} = 1$ for $u \in [1,+\infty)$, the Chasles integral relation and Lemma 5.1 with $\epsilon = \beta$, we obtain

$$
\begin{aligned}
\kappa &\geq \liminf_{n\to+\infty} \int_0^1 \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{1/(nq)} du + \liminf_{n\to+\infty} \int_1^{+\infty} \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} u^{-1/(np)} du \\
&\geq \int_0^1 \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} \left[\liminf_{n\to+\infty} u^{1/(nq)}\right] du + \int_1^{+\infty} \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} \left[\liminf_{n\to+\infty} u^{-1/(np)}\right] du \\
&= \int_0^1 \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} du + \int_1^{+\infty} \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} du = \int_0^{+\infty} \frac{|1-u|^{\beta-1}}{(1+u)^{\beta+1}} du = \frac{1}{\beta}.
\end{aligned}
$$

This contradicts the assumption $\kappa \in (0, 1/\beta)$. As a result, in the framework of Theorem 2.2, the constant $1/\beta$ can not be improved; it is optimal. This completes the proof of Proposition 2.5. □

Proof of Proposition 2.6. The Fubini-Tonelli integral theorem and a suitable decomposition of the integrand give

$$
\begin{aligned}
&\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) dx\right]^p dy \\
&= \int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) dx\right] \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) dx\right]^{p-1} dy \\
&= \int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) dx\right]^{p-1} dx dy \\
&= \int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) g_\dagger(y) dx dy, \qquad (5.12)
\end{aligned}
$$

where

$$
g_\dagger(y) = \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) dx\right]^{p-1}.
$$

Applying Theorem 2.2 to the functions f and $g_\dagger$, we obtain

$$
\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x) g_\dagger(y) dx dy \\
&\leq \frac{1}{\beta} \left[\int_0^{+\infty} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} g_\dagger^q(y) dy\right]^{1/q}. \qquad (5.13)
\end{aligned}
$$

Let us now determine the second integral. Since $q(p-1)=p$, we have

$$\begin{aligned}\int_0^{+\infty} g_\dagger^q(y)dy &= \int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)dx\right]^{q(p-1)}dy\\ &= \int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)dx\right]^{p}dy. \end{aligned}\tag{5.14}$$

Putting Equations (5.12), (5.13) and (5.14) together, we get

$$\begin{aligned}&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)dx\right]^{p}dy\\ &\leq \frac{1}{\beta}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left\{\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)dx\right]^{p}dy\right\}^{1/q}.\end{aligned}$$

We thus have

$$\left\{\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)dx\right]^{p}dy\right\}^{1-1/q} \leq \frac{1}{\beta}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}.$$

Using the relation $1/p+1/q=1$, this is equivalent to

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)dx\right]^{p}dy \leq \frac{1}{\beta^p}\int_0^{+\infty} f^p(x)dx.$$

The proof of Proposition 2.6 ends. □

Proof of Proposition 2.7. Let us notice that

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}F(x)G(y)dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f_\diamond(x)g_\diamond(y)dxdy,\end{aligned}\tag{5.15}$$

where $f_\diamond(x)=F(x)/x$ and $g_\diamond(y)=G(y)/y$. Applying Theorem 2.2 to these two functions, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f_\diamond(x)g_\diamond(y)dxdy\\ &\leq \frac{1}{\beta}\left[\int_0^{+\infty} f_\diamond^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_\diamond^q(y)dy\right]^{1/q}\\ &= \frac{1}{\beta}\left[\int_0^{+\infty} \frac{F^p(x)}{x^p}dx\right]^{1/p}\left[\int_0^{+\infty} \frac{G^q(y)}{y^q}dy\right]^{1/q}.\end{aligned}\tag{5.16}$$

Upper bounds for these integrals can be derived from an existing result: the classical Hardy integral inequality. Specifically, this gives

$$\int_0^{+\infty}\frac{F^p(x)}{x^p}dx \leq \left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx\tag{5.17}$$

and

$$\int_0^{+\infty}\frac{G^q(y)}{y^q}dy \leq \left(\frac{q}{q-1}\right)^q\int_0^{+\infty} g^q(y)dy.\tag{5.18}$$

Putting Equations (5.15), (5.16), (5.17) and (5.18) together, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}F(x)G(y)dxdy\\&\leq \frac{1}{\beta}\left[\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\left(\frac{q}{q-1}\right)^q\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\&=\frac{1}{\beta}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.\end{aligned}$$

This concludes the proof of Proposition 2.7. □

Proof of Proposition 2.8. Let us notice that

$$\begin{aligned}&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}F(x)dx\right]^p dy\\&=\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f_\diamond(x)dx\right]^p dy,\end{aligned}\tag{5.19}$$

where $f_\diamond(x)=F(x)/x$. Applying Proposition 2.6 to this function and using the classical Hardy integral inequality, we get

$$\begin{aligned}&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f_\diamond(x)dx\right]^p dy\\&\leq\frac{1}{\beta^p}\int_0^{+\infty} f_\diamond^p(x)dx=\frac{1}{\beta^p}\int_0^{+\infty}\frac{F^p(x)}{x^p}dx\\&\leq\frac{1}{\beta^p}\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx.\end{aligned}\tag{5.20}$$

Putting Equations (5.19) and (5.20) together, we get

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}F(x)dx\right]^p dy\leq\frac{1}{\beta^p}\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx.$$

This completes the proof of Proposition 2.8. □

Proof of Proposition 2.9. It follows from Theorem 2.2 that

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)g(y)dxdy\\&\leq\frac{1}{\beta}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},\end{aligned}$$

with $\beta>0$. Integrating both sides with respect to β with $\beta\in(1,\delta)$ and developing the right-hand side term, we have

$$\begin{aligned}&\int_1^{\delta}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)g(y)dxdy\right]d\beta\\&\leq\left[\int_1^{\delta}\frac{1}{\beta}d\beta\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\&=\log(\delta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.\end{aligned}\tag{5.21}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the order of integration and basic integral calculus, we obtain

$$\begin{aligned}
&\int_1^\delta \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)g(y)dxdy\right]d\beta\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\left[\int_1^\delta \left(\frac{|x-y|}{x+y}\right)^{\beta-1}d\beta\right]f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\left[\frac{1}{\log[|x-y|/(x+y)]}\left(\frac{|x-y|}{x+y}\right)^{\beta-1}\right]_{\beta=1}^{\beta=\delta}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\frac{1}{\log[|x-y|/(x+y)]}\left[\left(\frac{|x-y|}{x+y}\right)^{\delta-1}-1\right]f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\frac{1}{\log[(x+y)/|x-y|]}\left[1-\left(\frac{|x-y|}{x+y}\right)^{\delta-1}\right]f(x)g(y)dxdy. \qquad (5.22)
\end{aligned}$$

Putting Equations (5.21) and (5.22) together, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\frac{1}{\log[(x+y)/|x-y|]}\left[1-\left(\frac{|x-y|}{x+y}\right)^{\delta-1}\right]f(x)g(y)dxdy\\
&\le \log(\delta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

The proof of Proposition 2.9 is concluded. □

Proof of Proposition 2.10. It follows from Theorem 2.2 that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)g(y)dxdy\\
&\le \frac{1}{\beta}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},
\end{aligned}$$

with $\beta>0$. Taking $\beta=\varepsilon i$ with $i\in\mathbb{N}\backslash\{0\}$, multiplying both sides by $i\eta^i$, summing both sides with respect to i and developing the right-hand side term with a basic geometric series formula based on $\eta\in(0,1)$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{+\infty} i\eta^i\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\varepsilon i-1}}{(x+y)^{\varepsilon i+1}}f(x)g(y)dxdy\right]\\
&\le \left[\sum_{i=1}^{+\infty} i\eta^i\frac{1}{\varepsilon i}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&=\frac{1}{\varepsilon}\left[\sum_{i=1}^{+\infty}\eta^i\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&=\frac{\eta}{\varepsilon(1-\eta)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \qquad (5.23)
\end{aligned}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the sum and integrals, and a basic geometric series formula based on $\eta|x-y|^{\varepsilon}/(x+y)^{\varepsilon} \in (0,1)$ almost everywhere, we get

$$\sum_{i=1}^{+\infty} i\eta^i \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\varepsilon i-1}}{(x+y)^{\varepsilon i+1}} f(x)g(y)dxdy\right]$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left\{\sum_{i=1}^{+\infty} i\left[\eta\left(\frac{|x-y|}{x+y}\right)^{\varepsilon}\right]^i\right\} f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\eta\left(\frac{|x-y|}{x+y}\right)^{\varepsilon} \times$$
$$\left\{\sum_{i=1}^{+\infty} i\left[\eta\left(\frac{|x-y|}{x+y}\right)^{\varepsilon}\right]^{i-1}\right\} f(x)g(y)dxdy$$
$$= \eta\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left(\frac{|x-y|}{x+y}\right)^{\varepsilon}\frac{1}{\left[1-\eta|x-y|^{\varepsilon}/(x+y)^{\varepsilon}\right]^2} f(x)g(y)dxdy$$
$$= \eta\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\varepsilon-1}(x+y)^{\varepsilon-1}}{\left[(x+y)^{\varepsilon}-\eta|x-y|^{\varepsilon}\right]^2} f(x)g(y)dxdy. \tag{5.24}$$

Putting Equations (5.23) and (5.24) together, we have

$$\eta\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\varepsilon-1}(x+y)^{\varepsilon-1}}{\left[(x+y)^{\varepsilon}-\eta|x-y|^{\varepsilon}\right]^2} f(x)g(y)dxdy$$
$$\leq \frac{\eta}{\varepsilon(1-\eta)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

This is equivalent to

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\varepsilon-1}(x+y)^{\varepsilon-1}}{\left[(x+y)^{\varepsilon}-\eta|x-y|^{\varepsilon}\right]^2} f(x)g(y)dxdy$$
$$\leq \frac{1}{\varepsilon(1-\eta)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

This ends the proof of Proposition 2.10. □

Proof of Proposition 2.11. It follows from Theorem 2.2 that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x)g(y)dxdy$$
$$\leq \frac{1}{\beta}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\beta > 0$. Taking $\beta = \xi i$ with $i \in \mathbb{N}\backslash\{0\}$, multiplying both sides by $1/i$, summing both sides with respect to i and developing the right-hand side term using the formula $\sum_{i=1}^{+\infty} 1/i^2 = \pi^2/6$, we get

$$\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\xi i-1}}{(x+y)^{\xi i+1}} f(x)g(y)dxdy\right]$$
$$\leq \left[\sum_{i=1}^{+\infty}\frac{1}{i}\frac{1}{\xi i}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\xi}\left[\sum_{i=1}^{+\infty}\frac{1}{i^2}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$

$$= \frac{\pi^2}{6\xi}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{5.25}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the sum and integrals, and a basic logarithmic series formula based on $|x-y|^\xi/(x+y)^\xi \in (0,1)$ almost everywhere, we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\xi i-1}}{(x+y)^{\xi i+1}}f(x)g(y)dxdy\right]\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left\{\sum_{i=1}^{+\infty}\frac{1}{i}\left[\left(\frac{|x-y|}{x+y}\right)^{\xi}\right]^{i}\right\}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left\{-\log\left[1-\left(\frac{|x-y|}{x+y}\right)^{\xi}\right]\right\}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\log\left[\frac{1}{1-|x-y|^{\xi}/(x+y)^{\xi}}\right]f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\log\left[\frac{(x+y)^{\xi}}{(x+y)^{\xi}-|x-y|^{\xi}}\right]f(x)g(y)dxdy. \qquad (5.26)
\end{aligned}$$

Putting Equations (5.25) and (5.26) together, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\log\left[\frac{(x+y)^{\xi}}{(x+y)^{\xi}-|x-y|^{\xi}}\right]f(x)g(y)dxdy\\
&\leq \frac{\pi^2}{6\xi}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Proposition 2.11. □

Proof of Proposition 2.12. It follows from Theorem 2.2 that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}}f(x)g(y)dxdy\\
&\leq \frac{1}{\beta}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},
\end{aligned}$$

with $\beta > 0$. Taking $\beta = \zeta i$ with $i \in \mathbb{N}\backslash\{0\}$, multiplying both sides by $\upsilon^i/[(i-1)!\,]$, summing both sides with respect to i and developing the right-hand side term with a basic exponential series formula, we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty}\frac{\upsilon^i}{(i-1)!}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\zeta i-1}}{(x+y)^{\zeta i+1}}f(x)g(y)dxdy\right]\\
&\leq \left[\sum_{i=1}^{+\infty}\frac{\upsilon^i}{(i-1)!}\frac{1}{\zeta i}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&=\frac{1}{\zeta}\left[\sum_{i=1}^{+\infty}\frac{\upsilon^i}{i!}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&=\frac{\exp(\upsilon)-1}{\zeta}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \qquad (5.27)
\end{aligned}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the sum and integrals, and a basic exponential series formula, we have

$$\sum_{i=1}^{+\infty} \frac{\upsilon^i}{(i-1)!} \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\zeta i-1}}{(x+y)^{\zeta i+1}} f(x)g(y)dxdy\right]$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left\{\sum_{i=1}^{+\infty}\frac{1}{(i-1)!}\left[\upsilon\left(\frac{|x-y|}{x+y}\right)^{\zeta}\right]^{i}\right\} f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\upsilon\left(\frac{|x-y|}{x+y}\right)^{\zeta}\times$$
$$\left\{\sum_{i=1}^{+\infty}\frac{1}{(i-1)!}\left[\upsilon\left(\frac{|x-y|}{x+y}\right)^{\zeta}\right]^{i-1}\right\} f(x)g(y)dxdy$$
$$= \upsilon\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left(\frac{|x-y|}{x+y}\right)^{\zeta}\exp\left[\upsilon\left(\frac{|x-y|}{x+y}\right)^{\zeta}\right] f(x)g(y)dxdy. \tag{5.28}$$

Putting Equations (5.27) and (5.28) together, we get

$$\upsilon\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left(\frac{|x-y|}{x+y}\right)^{\zeta}\exp\left[\upsilon\left(\frac{|x-y|}{x+y}\right)^{\zeta}\right] f(x)g(y)dxdy$$
$$\leq \frac{\exp(\upsilon)-1}{\zeta}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

This is equivalent to

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left(\frac{|x-y|}{x+y}\right)^{\zeta}\exp\left[\upsilon\left(\frac{|x-y|}{x+y}\right)^{\zeta}\right] f(x)g(y)dxdy$$
$$\leq \frac{\exp(\upsilon)-1}{\zeta\upsilon}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

This ends the proof of Proposition 2.12. □

5.3. **Proofs related to the second variation.**

Proof of Theorem 3.1. By a suitable decomposition of the integrand using the relation $1/p + 1/q = 1$ and applying the Hölder integral inequality, we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}\frac{|1-xy|^{(\gamma-1)/p}}{(1+xy)^{(\gamma+1)/p}} f(x)\times y^{1/q}\frac{|1-xy|^{(\gamma-1)/q}}{(1+xy)^{(\gamma+1)/q}} g(y)dxdy$$
$$\leq \left[\int_0^{+\infty}\int_0^{+\infty} x\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f^p(x)dxdy\right]^{1/p}\left[\int_0^{+\infty}\int_0^{+\infty} y\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} g^q(y)dxdy\right]^{1/q}. \tag{5.29}$$

Let us now determine the double integral terms of this bound. For the first double integral term, using the Fubini-Tonelli integral theorem to exchange the order of integration, the change of variables $u = xy$ and

Lemma 5.1 with $\epsilon = \gamma$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f^p(x)dxdy = \int_0^{+\infty} f^p(x)\left[\int_0^{+\infty}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}xdy\right]dx$$

$$= \int_0^{+\infty} f^p(x)\left[\int_0^{+\infty}\frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}du\right]dx = \frac{1}{\gamma}\int_0^{+\infty} f^p(x)dx. \tag{5.30}$$

For the second double integral term, with the same arguments, we find that

$$\int_0^{+\infty}\int_0^{+\infty} y\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} g^q(y)dxdy = \int_0^{+\infty} g^q(y)\left[\int_0^{+\infty}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}ydx\right]dy$$

$$= \int_0^{+\infty} g^q(y)\left[\int_0^{+\infty}\frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}du\right]dy = \frac{1}{\gamma}\int_0^{+\infty} g^p(y)dy. \tag{5.31}$$

Putting Equations (5.29), (5.30) and (5.31) together, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)g(y)dxdy\\ &\leq \left[\frac{1}{\gamma}\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\frac{1}{\gamma}\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\ &= \frac{1}{\gamma}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.\end{aligned}$$

This concludes the proof of Theorem 3.1. □

Proof of Proposition 3.4. We proceed by contradiction. Based on the statement of Theorem 3.1, let us suppose the existence of a constant $\sigma \in (0, 1/\gamma)$ such that, for any $f, g : [0,+\infty) \mapsto [0,+\infty)$,

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)g(y)dxdy\\ &\leq \sigma\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},\end{aligned} \tag{5.32}$$

provided that the two integrals on the right-hand side converge. For any $n \in \mathbb{N}\backslash\{0\}$, we consider the function $f_n : [0,+\infty) \mapsto [0,+\infty)$ defined by $f_n(0) = 0$, $f_n(x) = x^{(1/n-1)/p}$ for any $x \in (0,1)$, and $f_n(x) = 0$ for any $x \in [1,+\infty)$, and the function $g_n : [0,+\infty) \mapsto [0,+\infty)$ defined by $g_n(y) = 0$ for any $y \in [0,1)$, and $g_n(y) = y^{-(1+1/n)/q}$ for any $y \in [1,+\infty)$. We then have

$$\int_0^{+\infty} f_n^p(x)dx = \int_0^1 (x^{(1/n-1)/p})^p dx = \left[nx^{1/n}\right]_{x=0}^{x=1} = n$$

and

$$\int_0^{+\infty} g_n^q(y)dy = \int_1^{+\infty} (y^{-(1+1/n)/q})^q dy = \left[-ny^{-1/n}\right]_{y=1}^{y\to+\infty} = n.$$

By these integral values combined with the relation $1/p + 1/q = 1$ and Equation (5.32), we obtain

$$\begin{aligned}\sigma &= \sigma\frac{1}{n}n^{1/p}n^{1/q} = \frac{1}{n}\left\{\sigma\left[\int_0^{+\infty} f_n^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_n^q(y)dy\right]^{1/q}\right\}\\ &\geq \frac{1}{n}\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f_n(x)g_n(y)dxdy.\end{aligned} \tag{5.33}$$

We now aim to determine the double integral term. Using the definitions of f_n and g_n, the change of variables $x = u/y$, the Fubini-Tonelli integral theorem and the relation $1/p + 1/q = 1$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f_n(x)g_n(y)dxdy \\
&= \int_1^{+\infty}\int_0^{1} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}x^{(1/n-1)/p}y^{-(1+1/n)/q}dxdy \\
&= \int_1^{+\infty}\left[\int_0^{1} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}x^{1/(np)}dx\right]y^{-1/(nq)}dy \\
&= \int_1^{+\infty}\left[\int_0^{y} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}y^{-1/(np)}\frac{1}{y}du\right]y^{-1/(nq)}dy \\
&= \int_1^{+\infty}\left[\int_0^{y} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du\right]y^{-(1+1/n)}dy. \qquad (5.34)
\end{aligned}$$

It follows from the Chasles integral relation at the cutoff value $u = 1$, the Fubini-Tonelli integral theorem, simple integral calculus and the relation $1/p + 1/q = 1$ that

$$\begin{aligned}
&\int_1^{+\infty}\left[\int_0^{y} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du\right]y^{-(1+1/n)}dy \\
&= \int_1^{+\infty}\left[\int_0^{1} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du\right]y^{-(1+1/n)}dy \\
&+ \int_1^{+\infty}\left[\int_1^{y} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du\right]y^{-(1+1/n)}dy \\
&= \left[\int_0^{1} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du\right]\left[\int_1^{+\infty} y^{-(1+1/n)}dy\right] \\
&+ \int_1^{+\infty}\left[\int_u^{+\infty} y^{-(1+1/n)}dy\right]\frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du \\
&= n\left[\int_0^{1} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du\right] + \int_1^{+\infty}(nu^{-1/n})\frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du \\
&= n\left[\int_0^{1} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du + \int_1^{+\infty} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{-1/(nq)}du\right]. \qquad (5.35)
\end{aligned}$$

Putting Equations (5.33), (5.34) and (5.35) together, we get

$$\sigma \geq \int_0^{1} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du + \int_1^{+\infty} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{-1/(nq)}du.$$

This is valid for any $n \in \mathbb{N}\backslash\{0\}$. Applying the inferior limit with respect to n, the Fatou integral lemma, which is possible because the integrand is non-negative, $\liminf_{n\to+\infty} u^{1/(np)} = 1$ for $u \in (0,1)$, $\liminf_{n\to+\infty} u^{-1/(nq)} = 1$ for $u \in [1,+\infty)$, the Chasles integral relation and Lemma 5.1 with $\epsilon = \gamma$, we obtain

$$\begin{aligned}
\sigma &\geq \liminf_{n\to+\infty}\int_0^{1} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{1/(np)}du + \liminf_{n\to+\infty}\int_1^{+\infty} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}u^{-1/(nq)}du \\
&\geq \int_0^{1} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}\left[\liminf_{n\to+\infty} u^{1/(np)}\right]du + \int_1^{+\infty} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}\left[\liminf_{n\to+\infty} u^{-1/(nq)}\right]du \\
&= \int_0^{1} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}du + \int_1^{+\infty} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}du = \int_0^{+\infty} \frac{|1-u|^{\gamma-1}}{(1+u)^{\gamma+1}}du = \frac{1}{\gamma}.
\end{aligned}$$

This contradicts the assumption $\sigma \in (0, 1/\gamma)$. As a result, in the framework of Theorem 3.1, the constant $1/\gamma$ can not be improved; it is optimal. This completes the proof of Proposition 3.4. □

Proof of Proposition 3.5. The Fubini-Tonelli theorem combined with a suitable decomposition of the integrand gives

$$\begin{aligned}
&\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^p dy \\
&= \int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right] \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^{p-1} dy \\
&= \int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x) \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^{p-1} dxdy \\
&= \int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x) g_{\ddagger}(y) dxdy, \qquad (5.36)
\end{aligned}$$

where

$$g_{\ddagger}(y) = \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^{p-1}.$$

Applying Theorem 3.1 to the functions f and $g_{\ddagger}$, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x) g_{\ddagger}(y) dxdy \\
&\leq \frac{1}{\gamma} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g_{\ddagger}^q(y)dy\right]^{1/q}. \qquad (5.37)
\end{aligned}$$

Let us now determine the second integral term. Since $q(p-1) = p$, we have

$$\begin{aligned}
&\int_0^{+\infty} g_{\ddagger}^q(y)dy = \int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^{q(p-1)} dy \\
&= \int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^p dy. \qquad (5.38)
\end{aligned}$$

Putting Equations (5.36), (5.37) and (5.38) together, we get

$$\begin{aligned}
&\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^p dy \\
&\leq \frac{1}{\gamma} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left\{\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^p dy\right\}^{1/q}.
\end{aligned}$$

We thus have

$$\left\{\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^p dy\right\}^{1-1/q} \leq \frac{1}{\gamma} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}.$$

Using the relation $1/p + 1/q = 1$, this is equivalent to

$$\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}} f(x)dx\right]^p dy \leq \frac{1}{\gamma^p} \int_0^{+\infty} f^p(x)dx.$$

The proof of Proposition 3.5 is concluded. □

Proof of Proposition 3.6. Let us notice that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}F(x)G(y)dxdy$$

$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f_\diamond(x)g_\diamond(y)dxdy, \tag{5.39}$$

where $f_\diamond(x) = F(x)/x$ and $g_\diamond(y) = G(y)/y$. Applying Theorem 3.1 to these two functions and the classical Hardy integral inequality to f and g, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f_\diamond(x)g_\diamond(y)dxdy\\
&\le \frac{1}{\gamma}\left[\int_0^{+\infty} f_\diamond^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_\diamond^q(y)dy\right]^{1/q}\\
&= \frac{1}{\gamma}\left[\int_0^{+\infty} \frac{F^p(x)}{x^p}dx\right]^{1/p}\left[\int_0^{+\infty} \frac{G^q(y)}{y^q}dy\right]^{1/q}\\
&\le \frac{1}{\gamma}\left[\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\left(\frac{q}{q-1}\right)^q\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{1}{\gamma}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}\tag{5.40}$$

Putting Equations (5.39) and (5.40) together, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}F(x)G(y)dxdy\\
&\le \frac{1}{\gamma}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Proposition 3.6. □

Proof of Proposition 3.7. Let us notice that

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}F(x)dx\right]^p dy$$

$$= \int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f_\diamond(x)dx\right]^p dy, \tag{5.41}$$

where $f_\diamond(x) = F(x)/x$. Applying Proposition 3.5 to this function and using the classical Hardy integral inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f_\diamond(x)dx\right]^p dy\\
&\le \frac{1}{\gamma^p}\int_0^{+\infty} f_\diamond^p(x)dx = \frac{1}{\gamma^p}\int_0^{+\infty}\frac{F^p(x)}{x^p}dx\\
&\le \frac{1}{\gamma^p}\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx.
\end{aligned}\tag{5.42}$$

Putting Equations (5.41) and (5.42) together, we obtain

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}F(x)dx\right]^p dy \le \frac{1}{\gamma^p}\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx.$$

This completes the proof of Proposition 3.7. □

Proof of Proposition 3.8. It follows from Theorem 3.1 that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)g(y)dxdy \leq \frac{1}{\gamma}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Integrating both sides with respect to γ with $\gamma \in (1,\nu)$ and developing the right-hand side term, we have

$$\begin{aligned}
&\int_1^{\nu}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)g(y)dxdy\right]d\gamma\\
&\leq \left[\int_1^{\nu}\frac{1}{\gamma}d\gamma\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \log(\nu)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \qquad (5.43)
\end{aligned}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the order of integration and basic integral calculus, we get

$$\begin{aligned}
&\int_1^{\nu}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)g(y)dxdy\right]d\gamma\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\left[\int_1^{\nu}\left(\frac{|1-xy|}{1+xy}\right)^{\gamma-1}d\gamma\right]f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\left[\frac{1}{\log[|1-xy|/(1+xy)]}\left(\frac{|1-xy|}{1+xy}\right)^{\gamma-1}\right]_{\gamma=1}^{\gamma=\nu}f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\frac{1}{\log[|1-xy|/(1+xy)]}\left[\left(\frac{|1-xy|}{1+xy}\right)^{\nu-1}-1\right]f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\frac{1}{\log[(1+xy)/|1-xy|]}\left[1-\left(\frac{|1-xy|}{1+xy}\right)^{\nu-1}\right]f(x)g(y)dxdy. \qquad (5.44)
\end{aligned}$$

Putting Equations (5.43) and (5.44) together, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\frac{1}{\log[(1+xy)/|1-xy|]}\left[1-\left(\frac{|1-xy|}{1+xy}\right)^{\nu-1}\right]f(x)g(y)dxdy\\
&\leq \log(\nu)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Proposition 3.8. □

Proof of Proposition 3.9. It follows from Theorem 3.1 that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)g(y)dxdy \leq \frac{1}{\gamma}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\gamma > 0$. Taking $\gamma = \psi i$ with $i \in \mathbb{N}\backslash\{0\}$, multiplying both sides by $i\phi^i$, summing both sides with respect to i and developing the right-hand side term with a basic geometric series formula based on $\phi \in (0,1)$, we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty} i\phi^i \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\psi i-1}}{(1+xy)^{\psi i+1}}f(x)g(y)dxdy\right]\\
&\leq \left[\sum_{i=1}^{+\infty} i\phi^i\frac{1}{\psi i}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{1}{\psi}\left[\sum_{i=1}^{+\infty}\phi^i\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{\phi}{\psi(1-\phi)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \qquad (5.45)
\end{aligned}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the sum and integrals, and a basic geometric series formula based on $\phi|1-xy|^{\psi}/(1+xy)^{\psi}\in(0,1)$ almost everywhere, we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty} i\phi^i \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\psi i-1}}{(1+xy)^{\psi i+1}}f(x)g(y)dxdy\right]\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left\{\sum_{i=1}^{+\infty} i\left[\phi\left(\frac{|1-xy|}{1+xy}\right)^{\psi}\right]^i\right\}f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\phi\left(\frac{|1-xy|}{1+xy}\right)^{\psi}\times\\
&\left\{\sum_{i=1}^{+\infty} i\left[\phi\left(\frac{|1-xy|}{1+xy}\right)^{\psi}\right]^{i-1}\right\}f(x)g(y)dxdy\\
&= \phi\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left(\frac{|1-xy|}{1+xy}\right)^{\psi}\times\\
&\frac{1}{\left[1-\phi|1-xy|^{\psi}/(1+xy)^{\psi}\right]^2}f(x)g(y)dxdy\\
&= \phi\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\psi-1}(1+xy)^{\psi-1}}{\left[(1+xy)^{\psi}-\phi|1-xy|^{\psi}\right]^2}f(x)g(y)dxdy. \qquad (5.46)
\end{aligned}$$

Putting Equations (5.45) and (5.46) together, we get

$$\begin{aligned}
&\phi\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\psi-1}(1+xy)^{\psi-1}}{\left[(1+xy)^{\psi}-\phi|1-xy|^{\psi}\right]^2}f(x)g(y)dxdy\\
&\leq \frac{\phi}{\psi(1-\phi)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\psi-1}(1+xy)^{\psi-1}}{\left[(1+xy)^{\psi}-\phi|1-xy|^{\psi}\right]^2}f(x)g(y)dxdy\\
&\leq \frac{1}{\psi(1-\phi)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

The proof of Proposition 3.9 ends. □

Proof of Proposition 3.10. It follows from Theorem 3.1 that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)g(y)dxdy \leq \frac{1}{\gamma}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\gamma>0$. Taking $\gamma=\varphi i$ with $i\in\mathbb{N}\backslash\{0\}$, multiplying both sides by $1/i$, summing both sides with respect to i and developing the right-hand side term using the formula $\sum_{i=1}^{+\infty}1/i^2=\pi^2/6$, we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\varphi i-1}}{(1+xy)^{\varphi i+1}}f(x)g(y)dxdy\right]\\
&\leq\left[\sum_{i=1}^{+\infty}\frac{1}{i}\frac{1}{\varphi i}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&=\frac{1}{\varphi}\left[\sum_{i=1}^{+\infty}\frac{1}{i^2}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&=\frac{\pi^2}{6\varphi}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}\tag{5.47}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the sum and integrals, and a basic logarithmic series formula based on $|1-xy|^{\varphi}/(1+xy)^{\varphi}\in(0,1)$ almost everywhere, we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\varphi i-1}}{(1+xy)^{\varphi i+1}}f(x)g(y)dxdy\right]\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left\{\sum_{i=1}^{+\infty}\frac{1}{i}\left[\left(\frac{|1-xy|}{1+xy}\right)^{\varphi}\right]^i\right\}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left\{-\log\left[1-\left(\frac{|1-xy|}{1+xy}\right)^{\varphi}\right]\right\}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{1}{1-|1-xy|^{\varphi}/(1+xy)^{\varphi}}\right]f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{(1+xy)^{\varphi}}{(1+xy)^{\varphi}-|1-xy|^{\varphi}}\right]f(x)g(y)dxdy.
\end{aligned}\tag{5.48}$$

Putting Equations (5.47) and (5.48) together, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{(1+xy)^{\varphi}}{(1+xy)^{\varphi}-|1-xy|^{\varphi}}\right]f(x)g(y)dxdy\\
&\leq\frac{\pi^2}{6\varphi}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Proposition 3.10. □

Proof of Proposition 3.11. It follows from Theorem 3.1 that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}f(x)g(y)dxdy \leq \frac{1}{\gamma}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\gamma > 0$. Taking $\gamma = \chi i$ with $i \in \mathbb{N}\backslash\{0\}$, multiplying both sides by $\varpi^i/[(i-1)!]$, summing both sides with respect to i and developing the right-hand side term with a basic exponential series formula, we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty} \frac{\varpi^i}{(i-1)!}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\chi i-1}}{(1+xy)^{\chi i+1}}f(x)g(y)dxdy\right] \\
&\leq \left[\sum_{i=1}^{+\infty}\frac{\varpi^i}{(i-1)!}\frac{1}{\chi i}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} \\
&= \frac{1}{\chi}\left[\sum_{i=1}^{+\infty}\frac{\varpi^i}{i!}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} \\
&= \frac{\exp(\varpi)-1}{\chi}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \qquad (5.49)
\end{aligned}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the sum and integrals, and a basic exponential series formula, we have

$$\begin{aligned}
&\sum_{i=1}^{+\infty} \frac{\varpi^i}{(i-1)!}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\chi i-1}}{(1+xy)^{\chi i+1}}f(x)g(y)dxdy\right] \\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left\{\sum_{i=1}^{+\infty}\frac{1}{(i-1)!}\left[\varpi\left(\frac{|1-xy|}{1+xy}\right)^{\chi}\right]^i\right\}f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\varpi\left(\frac{|1-xy|}{1+xy}\right)^{\chi}\times \\
&\left\{\sum_{i=1}^{+\infty}\frac{1}{(i-1)!}\left[\varpi\left(\frac{|1-xy|}{1+xy}\right)^{\chi}\right]^{i-1}\right\}f(x)g(y)dxdy \\
&= \varpi\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left(\frac{|1-xy|}{1+xy}\right)^{\chi}\exp\left[\varpi\left(\frac{|1-xy|}{1+xy}\right)^{\chi}\right]f(x)g(y)dxdy. \qquad (5.50)
\end{aligned}$$

Putting Equations (5.49) and (5.50) together, we obtain

$$\begin{aligned}
&\varpi\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left(\frac{|1-xy|}{1+xy}\right)^{\chi}\exp\left[\varpi\left(\frac{|1-xy|}{1+xy}\right)^{\chi}\right]f(x)g(y)dxdy \\
&\leq \frac{\exp(\varpi)-1}{\chi}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left(\frac{|1-xy|}{1+xy}\right)^{\chi}\exp\left[\varpi\left(\frac{|1-xy|}{1+xy}\right)^{\chi}\right]f(x)g(y)dxdy \\
&\leq \frac{\exp(\varpi)-1}{\chi\varpi}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

The proof of Proposition 3.11 is completed. □

5.4. **Proofs related to the third variation.**

Proof of Theorem 4.1. We can decompose the integrand as follows:

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}f(x)g(y)h(z)dxdydz \\
&= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}\frac{(1+x)^{\theta}}{(1+y)^{\theta}}\frac{|1-xyz|^{(\sigma-1)/p}}{(1+xyz)^{(\sigma+1)/p}}f(x)\times
\end{aligned}$$

$$(yz)^{1/q}\frac{(1+y)^{\theta}}{(1+z)^{\theta}}\frac{|1-xyz|^{(\sigma-1)/q}}{(1+xyz)^{(\sigma+1)/q}}g(y)\times(xz)^{1/r}\frac{(1+z)^{\theta}}{(1+x)^{\theta}}\frac{|1-xyz|^{(\sigma-1)/r}}{(1+xyz)^{(\sigma+1)/r}}h(z)dxdydz. \tag{5.51}$$

Applying the generalized Hölder integral inequality appropriately, we find that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}\frac{(1+x)^{\theta}}{(1+y)^{\theta}}\frac{|1-xyz|^{(\sigma-1)/p}}{(1+xyz)^{(\sigma+1)/p}}f(x)\times\\
&(yz)^{1/q}\frac{(1+y)^{\theta}}{(1+z)^{\theta}}\frac{|1-xyz|^{(\sigma-1)/q}}{(1+xyz)^{(\sigma+1)/q}}g(y)\times(xz)^{1/r}\frac{(1+z)^{\theta}}{(1+x)^{\theta}}\frac{|1-xyz|^{(\sigma-1)/r}}{(1+xyz)^{(\sigma+1)/r}}h(z)dxdydz\\
&\leq\left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}xy\frac{(1+x)^{\theta p}}{(1+y)^{\theta p}}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}f^p(x)dxdydz\right]^{1/p}\times\\
&\left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}yz\frac{(1+y)^{\theta q}}{(1+z)^{\theta q}}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}g^q(y)dxdydz\right]^{1/q}\times\\
&\left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}xz\frac{(1+z)^{\theta r}}{(1+x)^{\theta r}}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}h^r(z)dxdydz\right]^{1/r}.
\end{aligned} \tag{5.52}$$

Let us determine each of the triple integral terms of this bound. For the first triple integral term, using the Fubini-Tonelli integral theorem to exchange the order of integration, we can write

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}xy\frac{(1+x)^{\theta p}}{(1+y)^{\theta p}}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}f^p(x)dxdydz\\
&=\int_0^{+\infty}(1+x)^{\theta p}f^p(x)\left\{\int_0^{+\infty}\frac{1}{(1+y)^{\theta p}}\left[\int_0^{+\infty}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}xydz\right]dy\right\}dx.
\end{aligned} \tag{5.53}$$

It follows from the change of variables $u=xyz$, Lemma 5.1 with $\epsilon=\sigma$ and a basic integral calculus that

$$\begin{aligned}
&\int_0^{+\infty}(1+x)^{\theta p}f^p(x)\left\{\int_0^{+\infty}\frac{1}{(1+y)^{\theta p}}\left[\int_0^{+\infty}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}xydz\right]dy\right\}dx\\
&=\int_0^{+\infty}(1+x)^{\theta p}f^p(x)\left\{\int_0^{+\infty}\frac{1}{(1+y)^{\theta p}}\left[\int_0^{+\infty}\frac{|1-u|^{\sigma-1}}{(1+u)^{\sigma+1}}du\right]dy\right\}dx\\
&=\frac{1}{\sigma}\int_0^{+\infty}(1+x)^{\theta p}f^p(x)\left[\int_0^{+\infty}\frac{1}{(1+y)^{\theta p}}dy\right]dx\\
&=\frac{1}{\sigma}\int_0^{+\infty}(1+x)^{\theta p}f^p(x)\left[-\frac{1}{(\theta p-1)}\frac{1}{(1+y)^{\theta p-1}}\right]_{y=0}^{y\to+\infty}dx\\
&=\frac{1}{\sigma(\theta p-1)}\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx.
\end{aligned} \tag{5.54}$$

For the second triple integral term, we proceed similarly. We find that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}yz\frac{(1+y)^{\theta q}}{(1+z)^{\theta q}}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}g^q(y)dxdydz\times\\
&=\int_0^{+\infty}(1+y)^{\theta q}g^q(y)\left\{\int_0^{+\infty}\frac{1}{(1+z)^{\theta q}}\left[\int_0^{+\infty}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}yzdx\right]dz\right\}dy\\
&=\int_0^{+\infty}(1+y)^{\theta q}g^q(y)\left\{\int_0^{+\infty}\frac{1}{(1+z)^{\theta q}}\left[\int_0^{+\infty}\frac{|1-u|^{\sigma-1}}{(1+u)^{\sigma+1}}du\right]dz\right\}dy\\
&=\frac{1}{\sigma}\int_0^{+\infty}(1+y)^{\theta q}g^q(y)\left[\int_0^{+\infty}\frac{1}{(1+z)^{\theta q}}dz\right]dy\\
&=\frac{1}{\sigma(\theta q-1)}\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy.
\end{aligned} \tag{5.55}$$

For the third triple integral term, we proceed similarly. We obtain

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} xz\frac{(1+z)^{\theta r}}{(1+x)^{\theta r}}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}h^r(z)dxdydz\times$$
$$= \int_0^{+\infty}(1+z)^{\theta r}h^r(z)\left\{\int_0^{+\infty}\frac{1}{(1+x)^{\theta r}}\left[\int_0^{+\infty}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}xzdy\right]dx\right\}dz$$
$$= \int_0^{+\infty}(1+z)^{\theta r}h^r(z)\left\{\int_0^{+\infty}\frac{1}{(1+x)^{\theta r}}\left[\int_0^{+\infty}\frac{|1-u|^{\sigma-1}}{(1+u)^{\sigma+1}}du\right]dx\right\}dz$$
$$= \frac{1}{\sigma}\int_0^{+\infty}(1+z)^{\theta r}h^r(z)\left[\int_0^{+\infty}\frac{1}{(1+x)^{\theta r}}dx\right]dz$$
$$= \frac{1}{\sigma(\theta r-1)}\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz. \tag{5.56}$$

Putting Equations (5.51), (5.52), (5.53), (5.54), (5.55) and (5.56) together and using the relation $1/p+1/q+1/r=1$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}f(x)g(y)h(z)dxdydz$$
$$\leq \left[\frac{1}{\sigma(\theta p-1)}\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\frac{1}{\sigma(\theta q-1)}\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\times$$
$$\left[\frac{1}{\sigma(\theta r-1)}\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}$$
$$= \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}.$$

This concludes the proof of Theorem 4.1. □

Proof of Proposition 4.2. Let us notice that, for any $x, y, z \in (0, +\infty)$,

$$x^{-1/q}y^{-1/r}z^{-1/p} = (xy)^{1/p}(yz)^{1/q}(xz)^{1/r}x^{-1}y^{-1}z^{-1}.$$

We can therefore write

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}x^{-1/q}y^{-1/r}z^{-1/p}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}\frac{F(x)G(y)H(z)}{(1+x)^{\theta}(1+y)^{\theta}(1+z)^{\theta}}dxdydz$$
$$= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}x^{-1}y^{-1}z^{-1}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}\frac{F(x)G(y)H(z)}{(1+x)^{\theta}(1+y)^{\theta}(1+z)^{\theta}}dxdydz$$
$$= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}f_{\triangle}(x)g_{\triangle}(y)h_{\triangle}(z)dxdydz, \tag{5.57}$$

where $f_{\triangle}(x) = F(x)/[x(1+x)^{\theta}]$, $g_{\triangle}(y) = G(y)/[y(1+y)^{\theta}]$ and $h_{\triangle}(z) = H(z)/[z(1+z)^{\theta}]$. Applying Theorem 4.1 to these three functions and the classical Hardy integral inequality, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}f_{\triangle}(x)g_{\triangle}(y)h_{\triangle}(z)dxdydz$$
$$\leq \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f_{\triangle}^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g_{\triangle}^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h_{\triangle}^r(z)dz\right]^{1/r}$$

$$= \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \times$$
$$\left[\int_0^{+\infty} \frac{F^p(x)}{x^p} dx\right]^{1/p} \left[\int_0^{+\infty} \frac{G^q(y)}{y^q} dy\right]^{1/q} \left[\int_0^{+\infty} \frac{H^r(z)}{z^r} dz\right]^{1/r}$$
$$\leq \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \times$$
$$\left[\left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\left(\frac{q}{q-1}\right)^q \int_0^{+\infty} g^q(y)dy\right]^{1/q} \left[\left(\frac{r}{r-1}\right)^r \int_0^{+\infty} h^r(z)dz\right]^{1/r}$$
$$= \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left(\frac{r}{r-1}\right) \times$$
$$\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty} h^r(z)dz\right]^{1/r}. \tag{5.58}$$

Putting Equations (5.57) and (5.58) together, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} x^{-1/q}y^{-1/r}z^{-1/p} \frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}} \frac{F(x)G(y)H(z)}{(1+x)^{\theta}(1+y)^{\theta}(1+z)^{\theta}} dxdydz$$
$$\leq \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left(\frac{r}{r-1}\right) \times$$
$$\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty} h^r(z)dz\right]^{1/r}.$$

This concludes the proof of Proposition 4.2. □

Proof of Proposition 4.3. Applying Theorem 4.1, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r} \frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}} f(x)g(y)h(z)dxdydz$$
$$\leq \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \times$$
$$\left[\int_0^{+\infty} (1+x)^{\theta p} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} (1+y)^{\theta q} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty} (1+z)^{\theta r} h^r(z)dz\right]^{1/r},$$

with $\sigma > 0$. Integrating both sides with respect to σ with $\sigma \in (1, \omega)$ and developing the right-hand side term, we have

$$\int_1^{\omega}\left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r} \frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}} f(x)g(y)h(z)dxdydz\right] d\sigma$$
$$\leq \frac{1}{(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \left[\int_1^{\omega} \frac{1}{\sigma} d\sigma\right] \times$$
$$\left[\int_0^{+\infty} (1+x)^{\theta p} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} (1+y)^{\theta q} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty} (1+z)^{\theta r} h^r(z)dz\right]^{1/r}$$
$$= \frac{\log(\omega)}{(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \times$$
$$\left[\int_0^{+\infty} (1+x)^{\theta p} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} (1+y)^{\theta q} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty} (1+z)^{\theta r} h^r(z)dz\right]^{1/r}. \tag{5.59}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the order of integration and basic integral calculus, we get

$$\begin{aligned}
&\int_1^{\omega}\left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}f(x)g(y)h(z)dxdydz\right]d\sigma\\
&=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)^2}\times\\
&\left[\int_1^{\omega}\left(\frac{|1-xyz|}{1+xyz}\right)^{\sigma-1}d\sigma\right]f(x)g(y)h(z)dxdydz\\
&=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)^2}\times\\
&\left[\frac{1}{\log[|1-xyz|/(1+xyz)]}\left(\frac{|1-xyz|}{1+xyz}\right)^{\sigma-1}\right]_{\sigma=1}^{\sigma=\omega}f(x)g(y)h(z)dxdydz\\
&=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)^2}\times\\
&\frac{1}{\log[|1-xyz|/(1+xyz)]}\left[\left(\frac{|1-xyz|}{1+xyz}\right)^{\omega-1}-1\right]f(x)g(y)h(z)dxdydz\\
&=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)^2}\times\\
&\frac{1}{\log[(1+xyz)/|1-xyz|]}\left[1-\left(\frac{|1-xyz|}{1+xyz}\right)^{\omega-1}\right]f(x)g(y)h(z)dxdydz. \qquad (5.60)
\end{aligned}$$

Putting Equations (5.59) and (5.60) together, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)^2}\times\\
&\frac{1}{\log[(1+xyz)/|1-xyz|]}\left[1-\left(\frac{|1-xyz|}{1+xyz}\right)^{\omega-1}\right]f(x)g(y)h(z)dxdydz\\
&\leq\frac{\log(\omega)}{(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times\\
&\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}.
\end{aligned}$$

This concludes the proof of Proposition 4.3. □

Proof of Proposition 4.4. Applying Theorem 4.1, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}f(x)g(y)h(z)dxdydz\\
&\leq\frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times\\
&\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r},
\end{aligned}$$

with $\sigma>0$. Taking $\sigma=\varsigma i$ with $i\in\mathbb{N}\backslash\{0\}$, multiplying both sides by $i\iota^i$, summing both sides with respect to i and developing the right-hand side term with a basic geometric series formula based on $\iota\in(0,1)$, we get

$$\sum_{i=1}^{+\infty} i\iota^i \left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\varsigma i-1}}{(1+xyz)^{\varsigma i+1}}f(x)g(y)h(z)dxdydz\right]$$
$$\leq \left[\sum_{i=1}^{+\infty} i\iota^i \frac{1}{\varsigma i(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\right]\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}$$
$$= \frac{1}{\varsigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\left[\sum_{i=1}^{+\infty}\iota^i\right]\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}$$
$$= \frac{\iota}{\varsigma(1-\iota)(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}. \tag{5.61}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the sum and integrals, and a basic geometric series formula based on $\iota|1-xyz|^{\varsigma}/(1+xyz)^{\varsigma}\in(0,1)$ almost everywhere, we get

$$\sum_{i=1}^{+\infty} i\iota^i \left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\varsigma i-1}}{(1+xyz)^{\varsigma i+1}}f(x)g(y)h(z)dxdydz\right]$$
$$= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\times$$
$$\left\{\sum_{i=1}^{+\infty} i\left[\iota\left(\frac{|1-xyz|}{1+xyz}\right)^{\varsigma}\right]^i\right\}f(x)g(y)h(z)dxdydz$$
$$= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\iota\left(\frac{|1-xyz|}{1+xyz}\right)^{\varsigma}\times$$
$$\left\{\sum_{i=1}^{+\infty} i\left[\iota\left(\frac{|1-xyz|}{1+xyz}\right)^{\varsigma}\right]^{i-1}\right\}f(x)g(y)h(z)dxdydz$$
$$= \iota\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\left(\frac{|1-xyz|}{1+xyz}\right)^{\varsigma}\times$$
$$\frac{1}{[1-\iota|1-xyz|^{\varsigma}/(1+xyz)^{\varsigma}]^2}f(x)g(y)h(z)dxdydz$$
$$= \iota\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\varsigma-1}(1+xyz)^{\varsigma-1}}{[(1+xyz)^{\varsigma}-\iota|1-xyz|^{\varsigma}]^2}\times$$
$$f(x)g(y)h(z)dxdydz. \tag{5.62}$$

Putting Equations (5.61) and (5.62) together, we obtain

$$\iota\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\varsigma-1}(1+xyz)^{\varsigma-1}}{[(1+xyz)^{\varsigma}-\iota|1-xyz|^{\varsigma}]^2}f(x)g(y)h(z)dxdydz$$
$$\leq \frac{\iota}{\varsigma(1-\iota)(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}.$$

This is equivalent to

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\varsigma-1}(1+xyz)^{\varsigma-1}}{[(1+xyz)^{\varsigma}-\iota|1-xyz|^{\varsigma}]^2}f(x)g(y)h(z)dxdydz$$
$$\leq \frac{1}{\varsigma(1-\iota)(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}.$$

This ends the proof of Proposition 4.4. □

Proof of Proposition 4.5. Applying Theorem 4.1, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}}f(x)g(y)h(z)dxdydz$$
$$\leq \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r},$$

with $\sigma>0$. Taking $\sigma=\tau i$ with $i\in\mathbb{N}\backslash\{0\}$, multiplying both sides by $1/i$, summing both sides with respect to i and developing the right-hand side term using $\sum_{i=1}^{+\infty}1/i^2=\pi^2/6$, we get

$$\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\tau i-1}}{(1+xyz)^{\tau i+1}}f(x)g(y)h(z)dxdydz\right]$$
$$\leq\left[\sum_{i=1}^{+\infty}\frac{1}{i}\frac{1}{\tau i(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\right]\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}$$
$$=\frac{1}{\tau(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\left[\sum_{i=1}^{+\infty}\frac{1}{i^2}\right]\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}$$
$$=\frac{\pi^2}{6\tau(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times$$
$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}. \tag{5.63}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the sum and integrals, and a basic logarithmic series formula based on $|1-xyz|^{\tau}/(1+xyz)^{\tau}\in(0,1)$ almost everywhere, we get

$$\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\tau i-1}}{(1+xyz)^{\tau i+1}}f(x)g(y)h(z)dxdydz\right]$$
$$=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\times$$
$$\left\{\sum_{i=1}^{+\infty}\frac{1}{i}\left[\left(\frac{|1-xyz|}{1+xyz}\right)^{\tau}\right]^i\right\}f(x)g(y)h(z)dxdydz$$

$$
\begin{aligned}
&= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\times \\
&\left\{-\log\left[1-\left(\frac{|1-xyz|}{1+xyz}\right)^{\tau}\right]\right\} f(x)g(y)h(z)dxdydz \\
&= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\times \\
&\log\left[\frac{1}{1-|1-xyz|^{\tau}/(1+xyz)^{\tau}}\right] f(x)g(y)h(z)dxdydz \\
&= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\times \\
&\log\left[\frac{1}{1-|1-xyz|^{\tau}/(1+xyz)^{\tau}}\right] f(x)g(y)h(z)dxdydz \\
&= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\times \\
&\log\left[\frac{(1+xyz)^{\tau}}{(1+xyz)^{\tau}-|1-xyz|^{\tau}}\right] f(x)g(y)h(z)dxdydz. \qquad (5.64)
\end{aligned}
$$

Putting Equations (5.63) and (5.64) together, we obtain

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{1}{(1+xyz)|1-xyz|}\times \\
&\log\left[\frac{(1+xyz)^{\tau}}{(1+xyz)^{\tau}-|1-xyz|^{\tau}}\right] f(x)g(y)h(z)dxdydz \\
&\leq \frac{\pi^2}{6\tau(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times \\
&\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}.
\end{aligned}
$$

This ends the proof of Proposition 4.5. □

Proof of Proposition 4.6. Applying Theorem 4.1, we get

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\sigma-1}}{(1+xyz)^{\sigma+1}} f(x)g(y)h(z)dxdydz \\
&\leq \frac{1}{\sigma(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\times \\
&\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r},
\end{aligned}
$$

with $\sigma > 0$. Taking $\sigma = \zeta i$ with $i \in \mathbb{N}\backslash\{0\}$, multiplying both sides by $\kappa^i/[(i-1)!\,]$, summing both sides with respect to i and developing the right-hand side term with a basic exponential series formula, we get

$$
\begin{aligned}
&\sum_{i=1}^{+\infty}\frac{\kappa^i}{(i-1)!}\left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|1-xyz|^{\zeta i-1}}{(1+xyz)^{\zeta i+1}} f(x)g(y)h(z)dxdydz\right] \\
&\leq \left[\sum_{i=1}^{+\infty}\frac{\kappa^i}{(i-1)!}\frac{1}{\zeta i(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}}\right]\times \\
&\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}
\end{aligned}
$$

$$= \frac{1}{\zeta(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \left[\sum_{i=1}^{+\infty} \frac{\kappa^i}{i!}\right] \times$$

$$\left[\int_0^{+\infty}(1+x)^{\theta p} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty}(1+y)^{\theta q} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty}(1+z)^{\theta r} h^r(z)dz\right]^{1/r}$$

$$= \frac{\exp(\kappa)-1}{\zeta(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \times$$

$$\left[\int_0^{+\infty}(1+x)^{\theta p} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty}(1+y)^{\theta q} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty}(1+z)^{\theta r} h^r(z)dz\right]^{1/r}. \tag{5.65}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem to exchange the sum and integrals, and a basic exponential series formula, we have

$$\sum_{i=1}^{+\infty} \frac{\kappa^i}{(i-1)!} \left[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r} \frac{|1-xyz|^{\zeta i-1}}{(1+xyz)^{\zeta i+1}} f(x)g(y)h(z)dxdydz\right]$$

$$= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r} \frac{1}{(1+xyz)|1-xyz|} \times$$

$$\left\{\sum_{i=1}^{+\infty} \frac{1}{(i-1)!}\left[\kappa\left(\frac{|1-xyz|}{1+xyz}\right)^{\zeta}\right]^i\right\} f(x)g(y)h(z)dxdydz$$

$$= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r} \frac{1}{(1+xyz)|1-xyz|} \kappa\left(\frac{|1-xyz|}{1+xyz}\right)^{\zeta} \times$$

$$\left\{\sum_{i=1}^{+\infty} \frac{1}{(i-1)!}\left[\kappa\left(\frac{|1-xyz|}{1+xyz}\right)^{\zeta}\right]^{i-1}\right\} f(x)g(y)h(z)dxdydz$$

$$= \kappa\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r} \frac{1}{(1+xyz)|1-xyz|} \left(\frac{|1-xyz|}{1+xyz}\right)^{\zeta} \times$$

$$\exp\left[\kappa\left(\frac{|1-xyz|}{1+xyz}\right)^{\zeta}\right] f(x)g(y)h(z)dxdydz. \tag{5.66}$$

Putting Equations (5.65) and (5.66) together, we get

$$\kappa\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r} \frac{1}{(1+xyz)|1-xyz|} \left(\frac{|1-xyz|}{1+xyz}\right)^{\zeta} \times$$

$$\exp\left[\kappa\left(\frac{|1-xyz|}{1+xyz}\right)^{\zeta}\right] f(x)g(y)h(z)dxdydz$$

$$\leq \frac{\exp(\kappa)-1}{\zeta(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \times$$

$$\left[\int_0^{+\infty}(1+x)^{\theta p} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty}(1+y)^{\theta q} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty}(1+z)^{\theta r} h^r(z)dz\right]^{1/r}.$$

This is equivalent to

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (xy)^{1/p}(yz)^{1/q}(xz)^{1/r} \frac{1}{(1+xyz)|1-xyz|} \left(\frac{|1-xyz|}{1+xyz}\right)^{\zeta} \times$$

$$\exp\left[\kappa\left(\frac{|1-xyz|}{1+xyz}\right)^{\zeta}\right] f(x)g(y)h(z)dxdydz$$

$$\leq \frac{\exp(\kappa)-1}{\kappa\zeta(\theta p-1)^{1/p}(\theta q-1)^{1/q}(\theta r-1)^{1/r}} \times$$

$$\left[\int_0^{+\infty}(1+x)^{\theta p}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}(1+y)^{\theta q}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z)^{\theta r}h^r(z)dz\right]^{1/r}.$$

This ends the proof of Proposition 4.6. □

6. Conclusion and perspectives

In this article, we present new variations of the Hardy-Hilbert integral inequality by introducing weight functions involving sums and products of variables. We extend classical inequalities by incorporating primitives, logarithmic and exponential weight functions, and multiple adjustable parameters. The established results are notable for their originality and versatility. Furthermore, exploring a three-dimensional extension enables us to generalize these inequalities to higher dimensions. The optimality of certain constants has been rigorously demonstrated, thereby strengthening the accuracy of our findings. Thus, these contributions enrich the existing theory of Hardy-Hilbert-type integral inequalities and open up new avenues for further research. One such direction is the study of upper bounds for triple integrals, as defined below, which are closely related to our three-dimensional variations. Let $p,q>1$, $r=pq/(pq-p-q)$, $\sigma>0$ and $f,g,h:[0,+\infty)\mapsto[0,+\infty)$ be three functions. Then we set

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|x-y+z|^{\sigma-1}}{(x+y+z)^{\sigma+1}}f(x)g(y)h(z)dxdydz$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(xy)^{1/p}(yz)^{1/q}(xz)^{1/r}\frac{|x-y-z|^{\sigma-1}}{(x+y+z)^{\sigma+1}}f(x)g(y)h(z)dxdydz.$$

It seems difficult to find suitable upper bounds for these triple integrals using the techniques developed in this article. This mathematical challenge will be considered for the future.

Conflict of interest statement. The author declares that there is no conflict of interests.

References

[1] V. Adiyasuren, T. Batbold, M. Krnić, Multiple Hilbert-type Inequalities Involving Some Differential Operators, Banach J. Math. Anal. 10 (2016), 320–337. https://doi.org/10.1215/17358787-3495561.

[2] V. Adiyasuren, T. Batbold, M. Krnić, Hilbert-type Inequalities Involving Differential Operators, the Best Constants, and Applications, Math. Inequal. Appl. 18 (2015), 111–124. https://doi.org/10.7153/mia-18-07.

[3] L.E. Azar, The Connection Between Hilbert and Hardy Inequalities, J. Inequal. Appl. 2013 (2013), 452. https://doi.org/10.1186/1029-242x-2013-452.

[4] T. Batbold, Y. Sawano, Sharp Bounds for M-linear Hilbert-type Operators on the Weighted Morrey Spaces, Math. Inequal. Appl. 20 (2017), 263–283. https://doi.org/10.7153/mia-20-20.

[5] B. Benaissa, M. Zeki Sarikaya, On the Refinements of Some Important Inequalities with a Finite Set of Positive Numbers, Math. Methods Appl. Sci. 47 (2024), 9589–9599. https://doi.org/10.1002/mma.10084.

[6] Á. Bényi, C.(. Oh, Best Constants for Certain Multilinear Integral Operators, J. Inequal. Appl. 2006 (2006), 28582. https://doi.org/10.1155/jia/2006/28582.

[7] Q. Chen, B. Yang, A Survey on the Study of Hilbert-type Inequalities, Journal of Inequalities and Applications 2015 (2015), 302. https://doi.org/10.1186/s13660-015-0829-7.

[8] C. Chesneau, Some Four-Parameter Trigonometric Generalizations of the Hilbert Integral Inequality, Asia Math. 8 (2024), 45–59. https://doi.org/10.5281/ZENODO.13949386.

[9] C. Chesneau, General Inequalities of the Hilbert Integral Type Using the Method of Switching to Polar Coordinates, Hilbert J. Math. Anal. 3 (2025), 007–026. https://doi.org/10.62918/hjma.v3i1.30.

[10] C. Chesneau, Contributions to the Fractional Hardy Integral Inequality, Univ. J. Math. Appl. 8 (2025), 21–29. https://doi.org/10.32323/ujma.1590154.

[11] C. Chesneau, Two New General Integral Results Related to the Hilbert Integral Inequality, Fundam. J. Math. Appl. 8 (2025), 1–11. https://doi.org/10.33401/fujma.1560482.

[12] H. Du, Y. Miao, Several New Hardy-Hilbert's Inequalities, Filomat 25 (2011), 153–162. https://doi.org/10.2298/FIL1103153D.

[13] G.H. Hardy, Note on a Theorem of Hilbert Concerning Series of Positive Terms, Proc. Lond. Math. Soc. 23 (1925), 45–46.

[14] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge University Press, Cambridge, (1934).

[15] H. Yong, On Multiple Hardy-hilbert Integral Inequalities with Some Parameters, J. Inequal. Appl. 2006 (2006), 94960. https://doi.org/10.1155/jia/2006/94960.

[16] Z. Huang, B. Yang, A Multidimensional Hilbert-type Integral Inequality, Journal of Inequalities and Applications 2015 (2015), 151. https://doi.org/10.1186/s13660-015-0673-9.

[17] S.W. Jian, F.Z. Yang, All-Sided Generalization About Hardy-Hilbert Integral Inequalities, Acta Math. Sin. 44 (2001), 619–626.

[18] Y. Li, Y. Qian, B. He, On Further Analogs of Hilbert's Inequality, Int. J. Math. Math. Sci. 2007 (2007), 76329. https://doi.org/10.1155/2007/76329.

[19] W.T. Sulaiman, On Hardy-Hilbert's Integral Inequality, J. Inequal. Pure Appl. Math. 5 (2004), 1–9.

[20] W.T. Sulaiman, On Three Inequalities Similar to Hardy-Hilbert's Integral Inequality, Acta Math. Univ. Comen. 2 (2007), 273–278.

[21] W.T. Sulaiman, A different type of Hardy-Hilbert's integral inequality, AIP Conf. Proc. 1046 (2008), 135–141.

[22] W.T. Sulaiman, General New Forms of Hardy-Hilbert's Integral Inequality via New Ideas, Int. J. Contemp. Math. Sci. 3 (2008), 1059-1067.

[23] W.T. Sulaiman, New Types of Hardy-Hilbert's Integral Inequality, Gen. Math. Notes, 2 (2011), 111-118.

[24] B. Sun, A Multiple Hilbert-type Integral Inequality with the Best Constant Factor, J. Inequal. Appl. 2007 (2007), 71049. https://doi.org/10.1155/2007/71049.

[25] J.F. Tian, Properties of Generalized Hölder's Inequalities, J. Math. Inequal. 9 (2015), 473–480. https://doi.org/10.7153/jmi-09-40.

[26] D.C. Ullrich, A Simple Elementary Proof of Hilbert's Inequality, Amer. Math. Mon. 120 (2013), 161–164.

[27] J.S. Xu, Hardy-Hilbert's Inequalities With Two Parameters, Adv. Math. 36 (2007), 63–76.

[28] B.C. Yang, On Hilbert's Integral Inequality, J. Math. Anal. Appl. 220 (1998), 778–785. https://doi.org/10.1006/jmaa.1997.5877.

[29] B.C. Yang, On Hardy–Hilbert's Integral Inequality, J. Math. Anal. Appl. 261 (2001), 295–306. https://doi.org/10.1006/jmaa.2001.7525.

[30] B.C. Yang, A Multiple Hardy-Hilbert Integral Inequality, Chin. Ann. Math. Ser. A 24 (2003), 743–750.

[31] B.C. Yang, On the Norm of an Integral Operator and Applications, J. Math. Anal. Appl. 321 (2006), 182–192. https://doi.org/10.1016/j.jmaa.2005.07.071.

[32] B.C. Yang, On the Norm of a Hilbert's Type Linear Operator and Applications, J. Math. Anal. Appl. 325 (2007), 529-541. https://doi.org/10.1016/j.jmaa.2006.02.006.

[33] B.C. Yang, The Norm of Operator and Hilbert-Type Inequalities, Science Press, Beijing, (2009).

[34] B.C. Yang, Hilbert-Type Integral Inequalities, Bentham Science Publishers, (2009).

[35] B.C. Yang, M. Krnić, On the Norm of a Multi-Dimensional Hilbert-Type Operator, Sarajevo J. Math., 7 (2011), 223–243.

[36] W.Y. Zhong, B.C. Yang, On a Multiple Hilbert-type Integral Inequality with the Symmetric Kernel, J. Inequal. Appl. 2007 (2007), 027962. https://doi.org/10.1155/2007/27962.