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RELATIVE PRIME COPRIME GRAPH OF INTEGERS MODULO GROUP AND ITS REVERSE TOPOLOGICAL INDICES

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ABSTRACT. The relationship between edges and vertices is fundamental to graph theory, significantly influencing different graph properties and applications. The topological index is a numerical value calculable by specific techniques and graph properties. Meanwhile, the relative prime coprime graph of a group G for its subgroup H is a graph in which two distinct vertices are adjacent whenever the greatest common divisor of the order of both vertices is equal to one or relative prime and either or both are in H. In this paper, we introduce this definition and describe some properties. Moreover, we also define some reverse topological indices (reverse Harmonic, reverse Randic, reverse SK, reverse SK_1, and reverse SK_2 indices) for the relatively prime coprime graph. In particular, we describe these reverse degree-based topological indices for newly defined relatively prime coprime graphs based on the order of the elements of the integers modulo group.

1. INTRODUCTION

Integrating graph theory with chemistry, chemical graph theory emerged as a novel subfield of mathematical chemistry. This emerging topic aims to learn about molecular graphs' structural properties. Graph theory is a data structure that is significant in multiple domains, including computer science [6], air transportation [14], face recognition [7], and spectral graph properties [18,19]. Graph representation is essential in graph analysis as it influences data storage efficiency and the simplicity of executing operations, such as path finding, vertex degree assessment, or connection evaluation.

Let $\Gamma = (V, E)$ be a graph where V is a non-empty and finite set of vertices and E represents the edge set of Γ. The degree $\deg(v)$ of the vertex $v \in V$ is the number of vertices adjacent to v. The distance between two vertices $u, v \in V$ is the shortest length of the connecting paths. The reverse degree of v, denoted by c_v, is $c_v = \Delta - \deg(v) + 1$, where Δ is the maximum degree of V [9]. This term triggered some researchers to discuss some reverse topological indices, for instance, the reverse Laplacian index [11] and reverse Nirmala index [12]. Two years later, Antalan et al. [8] expanded this study to the comet and double comet graph. The research further investigates reversed degree-based topological indices for diverse materials, including graphene [15], Vanadium Carbide [22], metal-organic frameworks [20], and fullerene cage networks [5].

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These investigations enhance comprehension of the topological qualities and attributes of various materials and substances.

Furthermore, the link among vertices can be established according to specific definitions. A coprime graph is defined so that two vertices are adjacent if and only if the greatest common divisor of their orders is equal to 1 [17]. Additionally, this graph is integrated with a subgroup to formulate a new definition, specifically a relatively coprime graph [1]. In a relative coprime graph of a group G concerning a subgroup H, two vertices are considered adjacent if they are adjacent in the coprime graph and belong to the subgroup H. Moreover, in 2021, the prime coprime graph was introduced by [4] and defined that two distinct vertices are adjacent if and only if the greatest common factor of the two vertex orders is 1 or a prime number.

However, there is a surge of contemporary research exploring the relative prime coprime graph whose vertex sets are group elements. This paper presents a new definition of the relative prime coprime graph of a group. Moreover, there is a marked gap in the available data and publications on reverse topological indices for this graph. Therefore, this current work focuses on the relative prime coprime graph of integers modulo group and examines its reverse topological indices, including reverse Harmonic, reverse Randic, reverse SK, reverse SK_1, and reverse SK_2 indices.

Throughout this work, $\mathbb{Z}_n$ denotes the integers modulo group. The subgroup of $\mathbb{Z}_n$ is the set of elements resulting multiples of an integer d that divides n as given below:

$$\langle d\rangle = \left\{0, d, 2d, \ldots, \left(\frac{p^k}{\gcd(d,n)} - 1\right) d\right\}$$

Furthermore, $\langle d\rangle$ is a generator of group $\mathbb{Z}_n$.

This paper consists of four sections. In Section 2, the new definition of the relative prime coprime graph of a group is defined. We also establish some properties of this graph for the integers modulo group with illustrations. The next result on the reverse topological indices of this graph is shown in Section 3. It contains five indices, which are reverse Harmonic, reverse Randic, reverse SK, reverse SK_1, and reverse SK_2 indices. The last section presents a summary and potential future research suggestions.

2. Relative Prime Coprime Graph

In this section, we establish a new definition of a relative prime coprime graph. We also present the main results regarding some properties of this graph that will benefit the result of the upcoming section.

Definition 2.1. Suppose $\mathbb{Z}_n$ is an integer modulo group and H is a subgroup of $\mathbb{Z}_n$. The relatively prime coprime graph of the group $\mathbb{Z}_n$ for the subgroup H, denoted as $\Gamma_{\mathbb{Z}_n,H}$, is a graph whose vertices are the elements of the group $\mathbb{Z}_n$. Two distict vertices, $x \neq y \in V(\mathbb{Z}_n)$, are said to be adjacent if and only if $\gcd(|x|, |y|) = 1$ or $\gcd(|x|, |y|) = p$ and either x or y is a member of the subgroup H, where $|x|$ is the order of x.

From now, $\Gamma_{\mathbb{Z}_n,H}$ is under discussion in this paper. We denote $V(\Gamma_{\mathbb{Z}_n,H})$ as the vertex set of $\Gamma_{\mathbb{Z}_n,H}$ and $E(\Gamma_{\mathbb{Z}_n,H})$ be the edges set of $\Gamma_{\mathbb{Z}_n,H}$. We can state that $\Gamma_{\mathbb{Z}_n,H}$ is a simple graph.

As an illustration, given an integer group $\mathbb{Z}_8$ with a subgroup $H_1 = \{0, 2, 4, 6\}$ and $H_2 = \{0, 4\}$. Based on the definition of the relatively prime coprime graph with the subgroup H_1, the graph $\Gamma_{\mathbb{Z}_8,H_1}$ is obtained as shown in Figure 1.

On the other hand, the relatively prime coprime graph with the subgroup H_2 is equal to $\Gamma_{\mathbb{Z}_8,H_2}$. It can happen because the vertices adjacent to all other vertices form a subset of the subgroup $\mathbb{Z}_8$. Therefore, for any two adjacent group elements in the prime coprime graph, one of them must be a member of the subgroup.

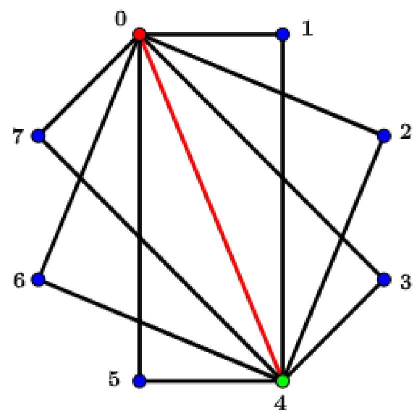

FIGURE 1. Relative Prime Coprime Graph for $\mathbb{Z}_8$

Let $V\left(\Gamma_{\mathbb{Z}_{p^k},H}\right)$ partitioned into two, $V_1 = \left\{0, p^{k-1}, 2p^{k-1}, 3p^{k-1}, \ldots, (p-1)p^{k-1}\right\}$ and $V_2 = \left\{0, 1, 2, 3, \ldots, p^k - 1\right\} \backslash V_1$.

Theorem 2.2. *Given the relatively prime coprime graph $\Gamma_{\mathbb{Z}_{p^k},H}$. Supposed $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ where $V_1 = \left\{0, p^{k-1}, 2p^{k-1}, 3p^{k-1}, \ldots, (p-1)p^{k-1}\right\}$ and $V_2 = \left\{0, 1, 2, 3, \ldots, p^k - 1\right\} \backslash V_1$ as well as $\langle d \rangle$ are a generator of $\mathbb{Z}_{p^k}$ then $V_1 \subseteq \langle d \rangle$.*

Proof. Given $V_1 = \left\{0, p^{k-1}, 2p^{k-1}, 3p^{k-1}, \ldots, (p-1)p^{k-1}\right\}$ and $\langle d \rangle = \left\{0, d, 2d, \ldots, \left(\frac{p^k}{\gcd(d, p^k)} - 1\right) d\right\}$. Take any $ap^{k-1} \in V_1$. In order to $\langle d \rangle$ is a generating group $\mathbb{Z}_{p^k}$, then $ap^{k-1} = (bd) \mod p^k$ for some $b \in \mathbb{Z}$. In other words, $ap^{k-1} \in \langle d \rangle$. Thus, $V_1 \subseteq \langle d \rangle$ $\square$

The following are some theorems related to subgraphs, vertices, distances, and edges.

Theorem 2.3. *Given $\mathbb{Z}_n$, the group of integers modulo, and H a subgroup of $\mathbb{Z}_n$. If $\Gamma_{\mathbb{Z}_n,H}$ is the relatively prime coprime graph with $n = p^k$, p is a prime and $k \geq 2$ is an integer, then the degree of a vertex in the graph $\Gamma_{\mathbb{Z}_n}$ is*

$$\deg(v) = \begin{cases} p^k - 1 & , v \in V_1 \\ p & , v \in V_2 \end{cases}$$

where

$$V_1 = \left\{0, p^{k-1}, 2p^{k-1}, 3p^{k-1}, \ldots, (p-1)p^{k-1}\right\}$$
$$V_2 = \left\{0, 1, 2, 3, \ldots, p^k - 1\right\} \backslash V_1$$

Proof. Take any $v_1, v_2 \in V_1$ and $u_1, u_2 \in V_2$. Consider the following,

a. For $v_1 = 0$
The value $(|v_1|, |v_2|) = 1$ for any $v_2 \in V$ because $|0| = 1$. In other words, the vertex 0 is adjacent to every vertex.

b. for $v_1 = ap^{k-1}$ with some $a \in \mathbb{Z}_n$
Since $\left(ap^{k-1}\right)^p \mod p^k = p \cdot ap^{k-1} \mod p^k = 0$ it follows that $|ap^{k-1}| = p$. Therefore, we obtain $(|v_1|, |v|) = p$ for any $v \in V$. In other words, the vertex v_1 is adjacent to every vertex.

c. For $u_1, u_2 \notin V_1$
Clearly, $|u_1| = |u_2| \neq p$. Therefore, any two distinct vertices in V_2, where $u_1 \neq u_2 \notin V_1$, are not adjacent to each other.

By the three cases, it can be concluded that: (1) The vertices in V_1 are adjacent to every vertex. (2) Two distinct vertices in V_2 are not adjacent to each other. Furthermore, based on Theorem 2.2, we obtain $(|0|, |v_1|) = (|0|, |u_1|) = 1$, $(|v_1|, |v_2|) = (|v_1|, |u_1|) = p$, and $(|u_1|, |u_2|) \neq p$.

Furthermore, since $v_1 \in V_1$ we have $\deg(v_1) = p^k - 1$. Next, since the vertex $v_2 \in V_2$ is only adjacent to vertices in V_1 and the number of vertices in V_1 is p, it follows that $\deg(v_2) = p$. Thus, $\deg(v) = p^k - 1$ for $v \in V_1$ and $\deg(v_2) = p$ for $v \in V_2$. □

Theorem 2.4. *Given $\mathbb{Z}_n$, the group of integers modulo and H is a subgroup of $\mathbb{Z}_n$. If $\Gamma_{\mathbb{Z}_n,H}$ is the relatively prime coprime graph with $n = p^k$, where p is a prime and $k \geq 2$ is an integer, then a subgraph of the graph $\Gamma_{\mathbb{Z}_n}$ is the complete graph K_p and the bipartite graph K_{p,p^k-p}.*

Proof. Based on the explanation in the proof of the Theorem 2.3, it is clear that any two vertices in V_1 are adjacent, and $|V_1| = p$. Therefore, a complete subgraph with p vertices, K_p is obtained.

Next, if the edges connecting two vertices in V_1 are removed, a new graph is formed. Based on Theorem 2.3 and removal of edges only connecting any vertex in V_1, the resulting graph is one where edges only connect any vertex in V_1 with any vertex in V_2. Furthermore, $|V_1| = p^k - p$. Therefore, the resulting subgraph is a complete bipartite graph, K_{p,p^k-p}. □

Theorem 2.5. *Given $\mathbb{Z}_n$ an integer group modulo and H a subgroup of $\mathbb{Z}_n$. If $\Gamma_{\mathbb{Z}_n,H}$ is a relatively prime coprime graph with $n = p^k$, p is a prime, and $k \geq 2$ is an integer, then the distance between of two vertex of graph $\Gamma_{\mathbb{Z}_n,H}$ consists of two, namely*

$$d(u,v) = \begin{cases} 1 & , u \in V_1, v \in V \\ 2 & , u, v \in V_2 \end{cases}$$

where

$$\begin{aligned} V &= \{0, 1, 2, \ldots, p^k - 1\} \\ V_1 &= \{0, p^{k-1}, 2p^{k-1}, 3p^{k-1}, \ldots, (p-1)p^{k-1}\} \\ V_2 &= V \backslash V_1 \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.3 in [3]. □

Theorem 2.6. *Supposed is given $\Gamma_{\mathbb{Z}_n}$, a prime coprime graph and $\Gamma_{\mathbb{Z}_n,H}$, a relatively prime coprime graph of a subgroup H from group $\mathbb{Z}_n$. If $n = p^k$ with p is a prime and $k \geq 2$ is an integer, then, graph $\Gamma_{\mathbb{Z}_n}$ is isomorphic to $\Gamma_{\mathbb{Z}_n,H}$*

Proof. It follows from Theorem 2.3 that the degree of v in $\Gamma_{\mathbb{Z}_n,H}$ is either $p^k - 1$ for $v \in V_1$ or p for $v \in V_2$. This also holds for $\Gamma_{\mathbb{Z}_n}$ which implies that $\Gamma_{\mathbb{Z}_n,H}$ is isomorphic to $\Gamma_{\mathbb{Z}_n}$ □

In order to $\Gamma_{\mathbb{Z}_n}$ isomorphic to $\Gamma_{\mathbb{Z}_n,H}$, then Based on Theorem 3.1 [2] is obtained $|E(\Gamma_{\mathbb{Z}_n,H})| = \frac{1}{2}\left(2p^{k+1} - p^2 - p\right)$. Next, since one of the subgraphs of $\Gamma_{\mathbb{Z}_n,H}$ is the complete graph K_p, the number of edges in this subgraph is $C(p,2) = \frac{1}{2} \cdot p(p-1)$. Therefore, the number of edges connecting $v_1 \in V_1$ and $v_2 \in V_2$ are $p^{k+1} - p^2$.

Next, several reverse indices based on degree will be discussed.

Theorem 2.7. *Given $\mathbb{Z}_n$, is an integer group modulo and H subgroup of $\mathbb{Z}_n$. If $\Gamma_{\mathbb{Z}_n,H}$ is a relatively prime coprime with $n = p^k$, p is a prime and $k \geq 2$ is an integer, then the degree of vertex in the graph $\Gamma_{\mathbb{Z}_n,H}$ is*

$$c_v = \begin{cases} 1 & , v \in V_1 \\ p^k - p & , v \in V_2 \end{cases}$$

where

$$V_1 = \{0, p^{k-1}, 2p^{k-1}, 3p^{k-1}, \dots, (p-1)p^{k-1}\}$$
$$V_2 = \{0, 1, 2, 3, \dots, p^k - 1\} \setminus V_1$$

Proof. Based on Theorem 2.3, is obtained $\Delta = p^k - 1$. Next, based on the definition of reverse degree, clearly $c_u = 1$ and $c_v = p^k - p$ for $u \in V_1$ and $v \in V_2$. □

3. Reverse Topological Indices of Relative Prime Coprime Graph

In this section, we are concerned with the reverse topological indices of $\Gamma_{\mathbb{Z}_n,H}$. By adopting the idea of reverse indices studied in [9, 10, 13], several reverse degree-based topological indices are defined. The topological indices whose reverse versions will be defined include the Harmonic index [23], Randic [16], as well as the SK, SK1, and SK2 indices [21].

Definition 3.1. The reverse harmonic index of $\Gamma_{\mathbb{Z}_n,H}$ is defined as

$$RH(\Gamma_{\mathbb{Z}_n,H}) = \sum_{uv \in E(\Gamma_{\mathbb{Z}_n,H})} \frac{2}{c_u + c_v}.$$

Definition 3.2. The reverse Randic index of $\Gamma_{\mathbb{Z}_n,H}$ is as follows

$$RR(\Gamma_{\mathbb{Z}_n,H}) = \sum_{uv \in E} \frac{1}{\sqrt{c_u \cdot c_v}}.$$

Definition 3.3. The reverse SK index of $\Gamma_{\mathbb{Z}_n,H}$ is

$$RSK(\Gamma_{\mathbb{Z}_n,H}) = \frac{1}{2} \sum_{uv \in E(\Gamma_{\mathbb{Z}_n,H})} (c_u + c_v).$$

Definition 3.4. The reverse SK_1 index of $\Gamma_{\mathbb{Z}_n,H}$ can be written as

$$RSK_1(\Gamma_{\mathbb{Z}_n,H}) = \frac{1}{2} \sum_{uv \in E(\Gamma_{\mathbb{Z}_n,H})} (c_u \cdot c_v).$$

Definition 3.5. The reverse SK_2 index of $\Gamma_{\mathbb{Z}_n,H}$ is defined as

$$RSK_2(\Gamma_{\mathbb{Z}_n,H}) = \frac{1}{4} \sum_{uv \in E(\Gamma_{\mathbb{Z}_n,H})} (c_u + c_v)^2.$$

In the sequel, we use the above definitions to find the next results.

Theorem 3.6. *Let $\mathbb{Z}_n$ be the integer modulo group and H be a subgroup of $\mathbb{Z}_n$. If $\Gamma_{\mathbb{Z}_n,H}$ is the relative prime coprime graph of $\mathbb{Z}_n$ with respect to H, then the reverse Harmonic index of $\Gamma_{\mathbb{Z}_n,H}$ is*

$$RH(\Gamma_{\mathbb{Z}_n,H}) = \frac{1}{2} \cdot p \left((p-1) + (p^k - p) \cdot \frac{4}{p^k - p + 1} \right).$$

Proof. In this proof, the edges will be considered in two parts: first, the edges connecting any two vertices in V_1, and second, the edges connecting a vertex $v_1 \in V_1$ and $v_2 \in V_2$. Take any $v_1, u_1 \in V_1$ and $v_2 \in V_2$. Based on Theorem 2.7, we obtain $c_{u_1} = c_{v_1} = 1$ and $c_{v_2} = p^k - p$. According to Definition 3.1, we have

$$\begin{aligned}
RH(\Gamma_{\mathbb{Z}_n,H}) &= \sum_{u_1v_1 \in E(\Gamma_{\mathbb{Z}_n,H})} \frac{2}{c_{u_1} + c_{v_1}} + \sum_{v_1v_2 \in E(\Gamma_{\mathbb{Z}_n,H})} \frac{2}{c_{v_1} + c_{v_2}} \\
&= \left(\frac{1}{2} \cdot p(p-1) \right) \frac{2}{1+1} + (p^{k+1} - p^2) \frac{2}{1 + (p^k - p)} \\
&= \frac{1}{2} \cdot p \left((p-1) + (p^k - p) \cdot \frac{4}{p^k - p + 1} \right)
\end{aligned}$$

Therefore, the reverse Harmonic index of the relatively prime coprime graph $\Gamma_{\mathbb{Z}_n,H}$ is $RH\left(\Gamma_{\mathbb{Z}_n,H}\right) = \frac{1}{2} \cdot p\left((p-1) + \left(p^k - p\right) \cdot \frac{4}{p^k - p + 1}\right)$. □

Consider Figure 1. The relatively prime coprime graph $\Gamma_{\mathbb{Z}_8,H}$ for any subgroup H of $\mathbb{Z}_8$ has $\Delta = 7$. Using Definition 3.1 the reverse Harmonic index of the relatively prime coprime graph $\Gamma_{\mathbb{Z}_8,H}$ is obtained as

$$\begin{aligned}
RH\left(\Gamma_{\mathbb{Z}_8,H}\right) &= \frac{2}{c_0+c_1} + \frac{2}{c_0+c_2} + \frac{2}{c_0+c_3} + \frac{2}{c_0+c_4} + \frac{2}{c_0+c_5} + \frac{2}{c_0+c_6} + \frac{2}{c_0+c_7} \\
&+ \frac{2}{c_1+c_4} + \frac{2}{c_2+c_4} + \frac{2}{c_3+c_4} + \frac{2}{c_4+c_5} + \frac{2}{c_4+c_6} + \frac{2}{c_4+c_7} \\
&= \frac{2}{1+6} + \frac{2}{1+6} + \frac{2}{1+6} + \frac{2}{1+1} + \frac{2}{1+6} + \frac{2}{1+6} + \frac{2}{1+6} + \frac{2}{6+1} \\
&+ \frac{2}{6+1} + \frac{2}{6+1} + \frac{2}{1+6} + \frac{2}{1+6} + \frac{2}{1+6} \\
&= 1 + 12 \cdot \left(\frac{2}{1+6}\right)
\end{aligned}$$

Using the formula for the reverse Harmonic index in Theorem 3.6, we obtain

$$\begin{aligned}
RH\left(\Gamma_{\mathbb{Z}_8,H}\right) &= \frac{1}{2} \cdot 2\left(\cdot(2-1) + \left(2^3 - 2\right) \cdot \frac{4}{2^3 - 2 + 1}\right) \\
&= 1 \cdot \left(1 + 6 \cdot \left(\frac{4}{1+6}\right)\right) \\
&= 1 + 12 \cdot \left(\frac{2}{1+6}\right)
\end{aligned}$$

Thus, the calculation of the reverse Harmonic index based on Definition 3.1 is the same as the calculating using Theorem 3.6.

Theorem 3.7. *Let $\mathbb{Z}_n$ be the integer modulo group and H be a subgroup of $\mathbb{Z}_n$. If $\Gamma_{\mathbb{Z}_n,H}$ is the relative prime coprime graph of $\mathbb{Z}_n$ with respect to H, then the reverse Randic index of $\Gamma_{\mathbb{Z}_n,H}$ is*

$$RR\left(\Gamma_{\mathbb{Z}_n,H}\right) = \frac{1}{2} \cdot p\left((p-1) + \sqrt{p^k - p}\right)$$

Proof. By using the same reasons on proofing of Theorem 3.6, we obtain

$$\begin{aligned}
RR\left(\Gamma_{\mathbb{Z}_n,H}\right) &= \sum_{u_1v_1 \in E\left(\Gamma_{\mathbb{Z}_n,H}\right)} \frac{1}{\sqrt{c_{u_1} \cdot c_{v_1}}} + \sum_{v_1v_2 \in E\left(\Gamma_{\mathbb{Z}_n,H}\right)} \frac{1}{\sqrt{c_{v_1} \cdot c_{v_2}}} \\
&= \left(\frac{1}{2} \cdot p(p-1)\right) \frac{1}{\sqrt{1 \cdot 1}} + \left(p^{k+1} - p^2\right) \frac{1}{\sqrt{1 \cdot \left(p^k - p\right)}} \\
&= \frac{1}{2} \cdot p\left((p-1) + 2\sqrt{p^k - p}\right)
\end{aligned}$$

Thus, the reverse Randic index of the relatively prime coprime graph $\Gamma_{\mathbb{Z}_n,H}$ is $RR\left(\Gamma_{\mathbb{Z}_n,H}\right) = \frac{1}{2} \cdot p\left((p-1) + \sqrt{p^k - p}\right)$. □

Theorem 3.8. *Given $\mathbb{Z}_n$ is an integer group modulo, and H is a subgroup of $\mathbb{Z}_n$. If $\Gamma_{\mathbb{Z}_n,H}$ is a relatively prime coprime graph, then the reverse SK index of the graph $\Gamma_{\mathbb{Z}_n,H}$ is*

$$RSK\left(\Gamma_{\mathbb{Z}_n,H}\right) = \frac{1}{2} \cdot p\left((p-1) + \left(p^k - p\right)\left(p^k - p + 1\right)\right)$$

Proof. By using the same statements on Proofing of Theorem 3.6, we have

$$\begin{aligned} RSK\left(\Gamma_{\mathbb{Z}_n,H}\right) &= \frac{1}{2}\left(\sum_{u_1v_1\in E(\Gamma_{\mathbb{Z}_n,H})}\left(c_{u_1}+c_{v_1}\right)+\sum_{v_1v_2\in E(\Gamma_{\mathbb{Z}_n,H})}\left(c_{v_1}+c_{v_2}\right)\right)\\ &= \frac{1}{2}\left(\left(\frac{1}{2}\cdot p(p-1)\right)(1+1)+\left(p^{k+1}-p^2\right)\left(1+\left(p^k-p\right)\right)\right)\\ &= \frac{1}{2}\cdot p\left((p-1)+\left(p^k-p\right)\left(p^k-p+1\right)\right) \end{aligned}$$

Thus, reverse SK index of relatively prime $\Gamma_{\mathbb{Z}_n,H}$ is

$$RSK\left(\Gamma_{\mathbb{Z}_n,H}\right)=\frac{1}{2}\cdot p\left((p-1)+\left(p^k-p\right)\left(p^k-p+1\right)\right)$$

□

Theorem 3.9. *Given $\mathbb{Z}_n$ is an integer group modulo, and H is a subgroup of $\mathbb{Z}_n$. If $\Gamma_{\mathbb{Z}_n,H}$ is a relatively prime coprime graph, then reverse SK1 index of graph $\Gamma_{\mathbb{Z}_n,H}$ is*

$$RSK_1\left(\Gamma_{\mathbb{Z}_n,H}\right)=\frac{1}{4}\cdot p\left((p-1)+2\left(p^k-p\right)^2\right)$$

Proof. By using the same reasons on Proofing of Theorem 3.6, it is obtained

$$\begin{aligned} RSK_1\left(\Gamma_{\mathbb{Z}_n,H}\right) &= \frac{1}{2}\left(\sum_{u_1v_1\in E(\Gamma_{\mathbb{Z}_n,H})}\left(c_{u_1}\cdot c_{v_1}\right)+\sum_{v_1v_2\in E(\Gamma_{\mathbb{Z}_n,H})}\left(c_{v_1}\cdot c_{v_2}\right)\right)\\ &= \frac{1}{2}\left(\left(\frac{1}{2}\cdot p(p-1)\right)(1\cdot 1)+\left(p^{k+1}-p^2\right)\left(1\cdot\left(p^k-p\right)\right)\right)\\ &= \frac{1}{2}\left(\frac{1}{2}\cdot p(p-1)+\left(p^{k+1}-p^2\right)\left(p^k-p\right)\right)\\ &= \frac{1}{4}\cdot p\left((p-1)+2\left(p^k-p\right)^2\right) \end{aligned}$$

Thus, reverse SK1 index of a relatively prime coprime graph $\Gamma_{\mathbb{Z}_n,H}$ is $RSK_1\left(\Gamma_{\mathbb{Z}_n,H}\right)=\frac{1}{4}\cdot p\left((p-1)+2\left(p^k-p\right)^2\right)$. □

Theorem 3.10. *Given $\mathbb{Z}_n$ is an integer group modulo, and H is a subgroup of $\mathbb{Z}_n$. If $\Gamma_{\mathbb{Z}_n,H}$ is a relatively prime coprime graph, then reverse SK2 index of graph $\Gamma_{\mathbb{Z}_n,H}$ is*

$$RSK_2\left(\Gamma_{\mathbb{Z}_n,H}\right)=\frac{1}{4}\left(2p(p-1)+\left(p^{k+1}-p^2\right)\left(p^k-p+1\right)^2\right)$$

Proof. By using the same statements on Proofing of Theorem 3.6, it is obtained

$$\begin{aligned} RSK_2\left(\Gamma_{\mathbb{Z}_n,H}\right) &= \frac{1}{4}\left(\sum_{u_1v_1\in E(\Gamma_{\mathbb{Z}_n,H})}\left(c_{u_1}+c_{v_1}\right)^2+\sum_{v_1v_2\in E(\Gamma_{\mathbb{Z}_n,H})}\left(c_{v_1}+c_{v_2}\right)^2\right)\\ &= \frac{1}{4}\left(\left(\frac{1}{2}\cdot p(p-1)\right)(1+1)^2+\left(p^{k+1}-p^2\right)\left(1+\left(p^k-p\right)\right)^2\right)\\ &= \frac{1}{4}\left(2p(p-1)+\left(p^{k+1}-p^2\right)\left(p^k-p+1\right)^2\right) \end{aligned}$$

Thus, reverse SK index of relatively prime graph $\Gamma_{\mathbb{Z}_n,H}$ is

$$RSK\left(\Gamma_{\mathbb{Z}_n,H}\right)=\frac{1}{4}\left(2p(p-1)+\left(p^{k+1}-p^2\right)\left(p^k-p+1\right)^2\right).$$

□

4. CONCLUSIONS

Numerous academic articles have examined the calculation of topological indices across many graph families. This paper has yielded a new graph construction known as a relative prime coprime graph of a group. This finding is further strengthened by the formulation of reverse topological indices, which further enriches the scientific knowledge. One can enhance the exploration of the spectral graph theory of this graph by associating matrices derived from the reverse degree-based approach.

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