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NEUTROSOPHIC COPSON TYPE ROUGH $\mathcal{I}$-STATISTICAL CONVERGENT SEQUENCES

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ABSTRACT. The main objective of work is to study the concept of rough $\mathcal{I}$-statistical convergence for Copson-transformation in the setup of neutrosophic normed spaces by considering algebraic and topological properties of the set of rough ideal statistical limit points for Copson transformation sequences. We also defined the set of rough $\mathcal{I}$-statistical cluster points and investigated some properties for the sequences using Copson-transformation.

1. INTRODUCTION

The Copson operator is a relatively lesser-known but intriguing operator in the study of sequences and series, named after the mathematician Copson [11]. It is utilized in the theory of summability for both series and sequences. Recently, Roopaei [25] has explored the spaces $c_0(\mathcal{C}^n)$, $c(\mathcal{C}^n)$ and $\ell_p(\mathcal{C}^n)$ using the Copson square matrix of order n. This investigation focuses on all sequences whose Copson transformation results in sequences $y = (y_k)$ belonging to as sequence spaces c_0, c, ℓ_p respectively $i.e$,

$$\tilde{\eta}(\mathcal{C}^n) = \left\{ y = (y_j) \in \omega : \lim_{j \to \infty} \left(\sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} \right) y_k \in \tilde{\eta} \right\}$$

for $\eta \in \{c_0, c, \ell_p\}$.

Steinhaus [29] and Fast [14] originally developed the concept of statistical convergence. This idea was later revisited and expanded by Schoenberg [26] with the inclusion of natural density. The natural density of a subset T_n of natural numbers, represented as $D(T_n)$, is defined as $D(T_n) = \lim_{n \to \infty} \frac{|T_n|}{n}$ such that $|T_n|$ represents the number of elements in $T_n = \{r \in \mathbb{N} : r \leq n\}$. A sequence $w = (w_k)$ is considered to be statistically convergent to w_0 if for every $\varepsilon > 0$, the natural density of set $\{k \in \mathbb{N} : |w_k - w_0| \geq \varepsilon\}$ is zero.

Phu [24] initially proposed the concept of rough convergence for sequences in finite-dimensional normed linear spaces. Subsequently, Aytar [4] introduced the concept of rough statistical convergence, further generalizing the idea using natural density. Further research includes studies on double sequences [20], triple sequences [13], lacunary sequences [16], ideals [23], and various spaces [1–3, 8].

Kostyrko $et\,al.$ [18] introduced the concept of ideal convergence ($\mathcal{I}$-convergence) by extending statistical convergence through the use of ideals. For further information, see [5–7, 17, 19]. Utilizing the concept of

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ideals, Das $et\,al.$ [12] defined a new generalization in 2011 called rough ideal statistical convergence in normed spaces.

Smarandache [28] suggested neutrosophic sets, a new generalisation of intuitionistic fuzzy sets. This concept is also used to define neutrosophic metric spaces [15] and neutrosophic soft linear spaces [9]. Bera and Mahapatra [10] started the notion of a neutrosophic norm and established several sequential concepts, such as convergence and Cauchy. Recently, Kirişci and Şimşek [15] expanded the statistical convergence and its properties in neutrosophic normed spaces (NNS).

The present research work on the idea of combinded concept of rough $\mathcal{I}$- statistical convergence using Copson operator on NNS is inspired by notable developments in the field of NNS in recent years. Some important ideas, including topological and algebraic ideas, have been presented and precisely defined. The next section presents the fundamental definitions required for the progress of the study. Sections 3 and 4 focus on all important results of the study. Specifically, Section 3 introduces a rough $\mathcal{I}$-statistical convergence using the Copson operator on the NNS. This section also establishes algebraic and topological properties such as convexity and closedness for the set of rough $\mathcal{I}$-statistical limit points associated with the Copson operator. In Section 4, the concept of rough $\mathcal{I}$-statistical cluster points using the Copson operator on the NNS is established, and the relationship between the limit points and cluster points is examined.

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2. Preliminaries

This section provides a comprehensive review of the essential definitions required for our study. Initially, we examine the neutrosophic normed space (NNS), which was defined by Kirişci and Şimşek [15] utilizing t-norm [27] and t-conorm [22]. Furthermore, we address the established statistical convergence of sequences on the NNS.

Definition 2.1. [15] Consider $\mathcal{N} = \{\tilde{p}, \varsigma(\tilde{p}), \upsilon(\tilde{p}), \tau(\tilde{p}) : \tilde{p} \in \mathbb{W}\}$ as a normed space such that $\varsigma, \upsilon, \tau : \mathbb{W} \times \mathbb{R}^+ \to [0,1]$, $\mathbb{W}$ be a vector space and $\oplus$, $\otimes$ are continuous t-norm and continuous t-conorm respectively. Then, tuple $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ is said to be neutrosophic normed space (NNS) if the following conditions are satisfied for every $\tilde{u}, \tilde{v} \in \mathbb{W}$ and $s, t > 0$ and $\alpha \neq 0$

(1) $0 \leq \varsigma(\tilde{u}, t), \upsilon(\tilde{u}, t), \tau(\tilde{u}, t) \leq 1$ for every $t \in \mathbb{R}^+$;
(2) $\varsigma(\tilde{u}, t) + \upsilon(\tilde{u}, t) + \tau(\tilde{u}, t) \leq 3$ for $t \in \mathbb{R}^+$;
(3) $\varsigma(\tilde{u}, t) = 1, \upsilon(\tilde{u}, t) = 0$ and $\tau(\tilde{u}, t) = 0$ for $t > 0$ iff $\tilde{u} = 0$;
(4) $\varsigma(\tilde{u}, t) = 0, \upsilon(\tilde{u}, t) = 1$ and $\tau(\tilde{u}, t) = 1$ for $t \leq 0$;
(5) $\varsigma(\alpha\tilde{u}, t) = \varsigma\left(\tilde{u}, \frac{t}{|\alpha|}\right), \upsilon(\alpha\tilde{u}, t) = \upsilon\left(\tilde{u}, \frac{t}{|\alpha|}\right)$ and $\tau(\alpha\tilde{u}, t) = \tau\left(\tilde{u}, \frac{t}{|\alpha|}\right)$;
(6) $\varsigma(\tilde{u}, s) \oplus \varsigma(\tilde{u}, t) \leq \varsigma(\tilde{u} + \tilde{v}, s + t)$;
(7) $\varsigma(\tilde{u}, \oplus)$ is continuous non-decreasing function;
(8) $\upsilon(\tilde{u}, s) \otimes \upsilon(\tilde{v}, t) \geq \upsilon(\tilde{u} + \tilde{v}, s + t)$;
(9) $\upsilon(\tilde{u}, \otimes)$ is continuous non-decreasing function;
(10) $\tau(\tilde{u}, s) \otimes \tau(\tilde{v}, t) \geq \tau(\tilde{u} + \tilde{v}, s + t)$;
(11) $\tau(\tilde{u}, \otimes)$ is continuous non-decreasing function;
(12) $\lim_{t\to\infty} \varsigma(\tilde{u}, t) = 1, \lim_{t\to\infty} \upsilon(\tilde{u}, t) = 0$ and $\lim_{t\to\infty} \tau(\tilde{u}, t) = 0$.

Here, $(\varsigma, \upsilon, \tau)$ is the neutrosophic norm on $\mathcal{W}$.

Example 2.2. [15] Take $(\mathbb{W}, \|\,.\,\|)$ as a normed space. For all $t > 0$ and $\tilde{u} \in \mathbb{W}$, if we take
(i) $\varsigma(\tilde{u}; t) = \frac{t}{t+\|\tilde{u}\|}$, $\upsilon(\tilde{u}; t) = \frac{\|\tilde{u}\|}{t+\|\tilde{u}\|}$ and $\tau(\tilde{u}; t) = \frac{\|\tilde{u}\|}{t}$ when $t > \|\tilde{u}\|$,
(ii) $\varsigma(\tilde{u}; t) = 0, \upsilon(\tilde{u}; t) = 1$ and $\tau(\tilde{u}; t) = 1$ when $t \leq \|\tilde{u}\|$.

Along with $\alpha_1 \oplus \alpha_2 = \alpha_1\alpha_2$ and $\alpha_1 \otimes \alpha_2 = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$ for $\alpha_1, \alpha_2 \in [0,1]$.
Then, tuple $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ is a NNS.

Definition 2.3. [15] A sequence $w = (w_k)$ from NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ is said to be statistically convergent with respect to norm $(\varsigma, \upsilon, \tau)$ if for $\varepsilon > 0$ and $\tilde{\eta} \in (0,1)$ there exists $\tilde{\xi} \in \mathbb{W}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : \varsigma(w_k - \tilde{\xi}, \varepsilon) \le 1 - \tilde{\eta} \text{ or } \upsilon(w_k - \tilde{\xi}, \varepsilon) \ge \tilde{\eta} \text{ or } \tau(w_k - \tilde{\xi}, \varepsilon) \ge \tilde{\eta}\}\right| = 0,$$

$$i.e\ D(\{k \in \mathbb{N} : \varsigma(w_k - \tilde{\xi}, \varepsilon) \le 1 - \tilde{\eta} \text{ or } \upsilon(w_k - \tilde{\xi}, \varepsilon) \ge \tilde{\eta} \text{ or } \tau(w_k - \tilde{\xi}, \varepsilon) \ge \tilde{\eta}\}) = 0.$$

3. Rough ideal statistical convergence using Copson-transformation on NNS

In this section, we define the concept of rough statistical convergence and rough ideal statistical convergence with the help of a Copson transformation of a sequence $y = (y_m)$ on NNS as follows:

$$\widetilde{w} = C^n y_m = \lim_{j\to\infty} \left(\sum_{m=j}^{\infty} \frac{\binom{n+m-j-1}{m-j}}{\binom{n+m}{m}} \right) y_m,$$

and then proved some significant results. For the sake of mathematical writing convenience, we will write $\widetilde{w} = C^n(y_m) = (\widetilde{w}_k)$ throughout the paper.

Definition 3.1. Let $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ be NNS, $\widetilde{w} = (\widetilde{w}_k)$, Copson transformation sequence is said to be rough statistical convergent to $\tilde{\xi} \in \mathbb{W}$ with respect to norm $(\varsigma, \upsilon, \tau)$ for $r \ge 0$ if for every $\varepsilon > 0$ and $\tilde{\eta} \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \varsigma(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \le 1 - \tilde{\eta} \text{or } \upsilon(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \ge \tilde{\eta} \text{or } \tau(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \ge \tilde{\eta}\}| = 0,$$

or

$$D(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \le 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \ge \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \ge \tilde{\eta}\}) = 0.$$

It is denoted by $\widetilde{w}_k \xrightarrow{r - st_{(\varsigma,\upsilon,\tau)}} \tilde{\xi}$ or $r - st_{(\varsigma,\upsilon,\tau)} - \lim_{p\to\infty} \widetilde{w}_k = \tilde{\xi}$. Here, r denotes the degree of roughness.
Let $st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$ denotes the the set of all rough statistical limit points of the sequence $\widetilde{w} = (\widetilde{w}_k)$.

Remark 3.2. For $r = 0$, the notion rough statistical convergence is equivalent to the statistical convergence sequence $\widetilde{w} = (\widetilde{w}_k)$ on NNS.

The $r - st_{(\varsigma,\upsilon,\tau)}$-limit of Copson-transformation sequence may not be unique on NNS. So, consider the set of rough statistical limit points of $\widetilde{w} = (\widetilde{w}_k)$ as $st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} = \left[\tilde{\xi} : \widetilde{w}_k \xrightarrow{r - st_{(\varsigma,\upsilon,\tau)}} \tilde{\xi}\right]$. Any sequence $\widetilde{w} = (\widetilde{w}_k)$ is $r_{(\varsigma,\upsilon,\tau)}$-convergent if $LIM^{r(\varsigma,\upsilon,\tau)}_{\widetilde{w}_k} \ne \emptyset$, where

$$LIM^{r(\varsigma,\upsilon,\tau)}_{\widetilde{w}_k} = \left[\tilde{\xi}^* \in \mathbb{W} : \widetilde{w}_k \xrightarrow{r_{(\varsigma,\upsilon,\tau)}} \tilde{\xi}^*\right].$$

Example 3.3. Consider $(\mathbb{W}, \|.\|)$ as real normed space. For every $t > 0$ and for all $\widetilde{w} \in \mathbb{W}$, take $\varsigma(\widetilde{w}, t) = \frac{t}{t + ||\widetilde{w}||}$, $\upsilon(\widetilde{w}, t) = \frac{\widetilde{w}}{t + ||\widetilde{w}||}$ and $\tau(\widetilde{w}, t) = \frac{\widetilde{w}}{t + ||\widetilde{w}||}$. Then, $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ is NNS. Consider a sequence $\widetilde{w} = (\widetilde{w}_k)$ as

$$\widetilde{w}_k = \begin{cases} (-1)^k & \text{if } k \ne n^2 \\ k & \text{if } k = n^2 . \end{cases}$$

Then $\widetilde{w}_k = (-1, 2, 3, 1, 5, 6, 7, 8, -1.........)$ and clearly for every $\varepsilon > 0$ and $\tilde{\eta} \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \varsigma(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \le 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \ge \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \ge \tilde{\eta}\}| = 0.$$

Also,

$$st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} = \begin{cases} \emptyset & r < 1 \\ [1-r, r-1] & \text{otherwise.} \end{cases}$$

For unbounded sequences, $LIM^{r(\varsigma,\upsilon,\tau)}_{\widetilde{w}_k} = \emptyset$. But $st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} \neq \emptyset$, so sequence is rough statistical convergent. Above example justifies that Copson-transformated sequence is rough statistically convergent but may not be rough convergent.

Definition 3.4. Let $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ be NNS, $\widetilde{w} = (\widetilde{w}_k)$ in $\mathbb{W}$ is said to be $\mathcal{I}$-statistically convergent to $\tilde{\xi} \in \mathbb{W}$ with respect to the norm $(\varsigma, \upsilon, \tau)$ if for every $\varepsilon > 0$, $\tilde{\eta} \in (0, 1)$ and $\delta > 0$

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - \tilde{\xi}; \varepsilon) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \tilde{\xi}; \varepsilon) \geq \tilde{\eta} \text{ or}$$
$$\tau(\widetilde{w}_k - \tilde{\xi}; \varepsilon) \geq \tilde{\eta}\}| \geq \delta\} \in \mathcal{I}.$$

It is denoted by $\widetilde{w}_k \xrightarrow{\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}} \tilde{\xi}$.
Let $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM_{\widetilde{w}_k}$ represents the collection of all ideal statistical limit points of sequence $\widetilde{w} = (\widetilde{w}_k)$.

Next, we consider rough ideal statistical convergence using Copson-transformation of sequences in neutrosophic normed spaces.

Definition 3.5. Let $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ be NNS, $\widetilde{w} = (\widetilde{w}_k)$ in $\mathbb{W}$ is said to be rough $\mathcal{I}$-statistically convergent to $\tilde{\xi} \in \mathbb{W}$ with respect to neutrosophic norm $(\varsigma, \upsilon, \tau)$ for some non-negative number r if for every $\varepsilon > 0$, $\tilde{\eta} \in (0, 1)$ and $\delta > 0$ satisfies

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \geq \tilde{\eta}$$
$$\text{or } \tau(\widetilde{w}_k - \tilde{\xi}; r + \varepsilon) \geq \tilde{\eta}\}| \geq \delta\} \in \mathcal{I}.$$

It is denoted by $\widetilde{w}_k \xrightarrow{r-\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}} \tilde{\xi}$.
Let $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$ represents the collection of all rough ideal statistical limit points sequence $\widetilde{w} = (\widetilde{w}_k)$.

Remark 3.6. For $r = 0$, notion of rough $\mathcal{I}$-statistical convergence agrees with $\mathcal{I}$-statistical convergence on NNS.

Example 3.7. Let $(\mathbb{W}, \|\,.\,\|)$ be a real normed space. For every $t > 0$ and for all $\widetilde{w} \in \mathbb{W}$, define $\varsigma(\widetilde{w}, t) = \frac{t}{t+||\widetilde{w}||}$, $\upsilon(\widetilde{w}, t) = \frac{\widetilde{w}}{t+||\widetilde{w}||}$ and $\tau(\widetilde{w}, t) = \frac{\widetilde{w}}{t+||\widetilde{w}||}$. Then $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ is NNS. Consider admissible ideal $\mathcal{I}_D = \{A \in \mathbb{N} \text{ with } D(A) = 0\}$. Define $\widetilde{w} = (\widetilde{w}_k)$ such that

$$\widetilde{w}_k = \begin{cases} k, & \text{if } k \in A \\ (-1)^k, & \text{otherwise.} \end{cases}$$

Then,

$$\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} = \begin{cases} \emptyset & r < 1 \\ [1-r, r-1] & \text{otherwise.} \end{cases}$$

Hence, $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} \neq \emptyset$.

Definition 3.8. A sequence $\widetilde{w} = (\widetilde{w}_k)$ on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ is said to be $\mathcal{I}$ statistically bounded ($\mathcal{I}$-st bounded) if for $\varepsilon > 0$, $\tilde{\eta} \in (0, 1)$ and $\delta > 0$, there exists $\mathcal{K} > 0$ such that

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k; \mathcal{K}) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k; \mathcal{K}) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k; \mathcal{K}) \geq \tilde{\eta}\}| \geq \delta\} \in \mathcal{I}.$$

We get the following results for $\widetilde{w} = (\widetilde{w}_k)$ sequence using above mentioned definitions on NNS.

Theorem 3.9. *Let $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ be NNS with neutrosophic norm $(\varsigma, \upsilon, \tau)$. A sequence $\widetilde{w} = (\widetilde{w}_k)$ from $\mathbb{W}$ is $\mathcal{I}$-st-bounded iff $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} \neq \emptyset$ for some $r > 0$.*

Proof. Necessary Part:-
Consider any sequence $\widetilde{w} = (\widetilde{w}_k)$ which is $\mathcal{I}$-st-bounded on NNS $\mathbb{W} = (W, \mathcal{N}, \oplus, \otimes)$. Then, for every $\varepsilon > 0$, $\tilde{\eta} \in (0, 1)$, $\delta > 0$ and some $r > 0$, $\exists \mathcal{K} > 0$ such that

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \varsigma(\widetilde{w}_k; \mathcal{K}) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k; \mathcal{K}) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k; \mathcal{K}) \geq \tilde{\eta}\}| \geq \delta\} \in \mathcal{I}.$$

As $\mathcal{I}$ an admissible ideal, therefore $M = \mathbb{N} \setminus A \neq \emptyset$, where

$$A = \left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \varsigma(\widetilde{w}_k; \mathcal{K}) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k; \mathcal{K}) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k; \mathcal{K}) \geq \tilde{\eta}\}| \geq \delta\right\}.$$

Take $n \in M$, then

$$\frac{1}{n} |\{k \leq n : \varsigma(\widetilde{w}_k; \mathcal{K}) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k; \mathcal{K}) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k; \mathcal{K}) \geq \tilde{\eta}\}| < \delta.$$

$$\implies \frac{1}{n} |\{k \leq n : \varsigma(\widetilde{w}_k; \mathcal{K}) > 1 - \tilde{\eta} \text{ and } \upsilon(\widetilde{w}_k; \mathcal{K}) < \tilde{\eta} \text{ or } \tau(\widetilde{w}_k; \mathcal{K}) \geq \tilde{\eta}\}| \geq 1 - \delta. \tag{3.1}$$

Let $B_n = \{k \leq n : \varsigma(\widetilde{w}_k; \mathcal{K}) > 1 - \tilde{\eta} \text{ and } \upsilon(\widetilde{w}_k; \mathcal{K}) < \tilde{\eta} \text{ and } \tau(\widetilde{w}_k; \mathcal{K}) < \tilde{\eta}\}$.
Also for $k \in B_n$,

$$\begin{aligned} \varsigma(\widetilde{w}_k; r + \mathcal{K}) &\geq \min\{\varsigma(0, r), \varsigma(\widetilde{w}_k, \mathcal{K})\} \\ &= \min\{1, \varsigma(\widetilde{w}_k; \mathcal{K})\} \\ &> 1 - \tilde{\eta}, \end{aligned}$$

$$\begin{aligned} \upsilon(\widetilde{w}_k; r + \mathcal{K}) &\leq \max\{\upsilon(0, r), \upsilon(\widetilde{w}_k, \mathcal{K})\} \\ &= \max\{0, \upsilon(\widetilde{w}_k; \mathcal{K})\} \\ &< \tilde{\eta}, \end{aligned}$$

$$\begin{aligned} \tau(\widetilde{w}_k; r + \mathcal{K}) &\leq \max\{\tau(0, r), \tau(\widetilde{w}_k, \mathcal{K})\} \\ &= \max\{0, \tau(\widetilde{w}_k; \mathcal{K})\} \\ &< \tilde{\eta}. \end{aligned}$$

Thus, $B_n \subset \{k \leq n : \varsigma(\widetilde{w}_k; r + \mathcal{K}) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k; r + \mathcal{K}) < \tilde{\eta}, \tau(\widetilde{w}_k; r + \mathcal{K}) < \tilde{\eta}\}$.
Using (3.1), we have

$$1 - \delta \leq \frac{|B_n|}{n} \leq \frac{1}{n} |\{k \leq n : \varsigma(\widetilde{w}_k; r + \mathcal{K}) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k; r + \mathcal{K}) < \tilde{\eta}, \tau(\widetilde{w}_k; r + \mathcal{K}) < \tilde{\eta}\}|.$$

Consequently,

$$\frac{1}{n} |\{k \leq n : \varsigma(\widetilde{w}_k; r + \mathcal{K}) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k; r + \mathcal{K}) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k; r + \mathcal{K}) < \tilde{\eta}\}| < \delta.$$

For $n \in \mathbb{N}$,

$$\frac{1}{n} |\{k \leq n : \varsigma(w_p; r + \mathcal{K}) \leq 1 - \tilde{\eta} \text{ or } \upsilon(w_p; r + \mathcal{K}) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k; r + G) \geq \tilde{\eta}\}| \geq \delta.$$

Hence, $0 \in \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}}$. Therefore, $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}} \neq \emptyset$.

Sufficient Part:-

Let $\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^{r}_{\widetilde{w}_k} \neq \emptyset$ for some $r > 0$. Then, there exists some $\tilde{\xi} \in \mathbb{W}$ such that $\tilde{\xi} \in \mathcal{I}-st_{(\varsigma,\upsilon)}-LIM^{r}_{\widetilde{w}_k}$. For every $\varepsilon > 0$ and $\tilde{\eta} \in (0,1)$, we have

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) \leq 1-\tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) \geq \tilde{\eta} \text{ or}$$

$$\tau(\widetilde{w}_k; r+\varepsilon) < \tilde{\eta}\}| \geq \delta\} \in \mathcal{I}.$$

Therefore, almost all $\widetilde{w}_k$'s are enclosed in a ball with centre $\tilde{\xi}$ on NNS, which imply that $\widetilde{w} = (\widetilde{w}_k)$ is $\mathcal{I}$-statistically bounded on NNS. □

Further, we present the algebraic characterization for rough ideal statistical convergence on NNS.

Theorem 3.10. *Let $\widetilde{v} = (\widetilde{v}_k)$ and $\widetilde{w} = (\widetilde{w}_k)$ be two sequences on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$, then*
(i) if $\widetilde{w}_k \xrightarrow{r-\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}} \tilde{\xi}$ and $\beta \in \mathbb{R}$ then $\beta\widetilde{w}_k \xrightarrow{r-\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}} \beta\tilde{\xi}$.
(ii) if $\widetilde{v}_k \xrightarrow{r-\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}} \tilde{\xi}$ and $\widetilde{w}_k \xrightarrow{r-\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}} \tilde{\rho}$ then $(\widetilde{v}_k + \widetilde{w}_k) \xrightarrow{r-\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}} (\tilde{\xi} + \tilde{\rho})$

Proof. (i) If $\beta = 0$ then the result is obvious. So, assume $\beta \neq 0$. As $\widetilde{w}_k \xrightarrow{r-\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}} \tilde{\xi}$ then for given $\tilde{\eta} > 0$ and $r \geq 0$. Consider $A \in \mathcal{I}$ where

$$A = \{n \in \mathbb{N}: \frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) \leq 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) < \tilde{\eta}\}| \geq \delta\}.$$

As $\mathcal{I}$ an admissible ideal, therefore $M = \mathbb{N} \setminus A \neq \emptyset$. Choose $n \in M$, then

$$\frac{1}{n}\left|\{k \leq n : \varsigma(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) \leq 1-\tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) < \tilde{\eta}\}\right| < \delta,$$

$$\frac{1}{n}\left|\{k \leq n : \varsigma(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) > 1-\tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) < \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) < \tilde{\eta}\}\right| \geq 1-\delta.$$

$$\Longrightarrow \frac{1}{n}|B_n| \geq 1-\delta. \tag{3.2}$$

Where

$$B_n = \left\{k \leq n : \varsigma(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) < \tilde{\eta}\right\}.$$

It sufficent to demonstrate that

$$B_n \subset \left\{k \in \mathbb{N} : \varsigma(\beta\widetilde{w}_k - \beta\tilde{\xi}; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\beta\widetilde{w}_k - \beta\tilde{\xi}; r+\varepsilon) < \tilde{\eta}, \tau(\beta\widetilde{w}_k - \beta\tilde{\xi}; r+\varepsilon) < \tilde{\eta}\right\}.$$

Let $k \in B_n$, then $\varsigma(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) > 1-\tilde{\eta}$, $\upsilon(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) < \tilde{\eta}$ and $\tau(\widetilde{w}_k - \tilde{\xi}; r+\varepsilon) < \tilde{\eta}$.
Then,

$$\begin{aligned}
\varsigma(\beta\widetilde{w}_k - \beta\tilde{\xi}; r+\varepsilon) &= \varsigma\left(\widetilde{w}_k - \tilde{\xi}, \frac{r+\varepsilon}{|\beta|}\right) \\
&\geq \min\left\{\varsigma(\widetilde{w}_k - \tilde{\xi}, r+\varepsilon), \varsigma\left(0, \frac{r+\varepsilon}{|\beta|} - (r+\varepsilon)\right)\right\} \\
&\geq \min\left\{\varsigma(\widetilde{w}_k - \tilde{\xi}, r+\varepsilon), 1\right\} \\
&= \varsigma(\widetilde{w}_k - \tilde{\xi}, r+\varepsilon) > 1-\tilde{\eta},
\end{aligned}$$

$$\begin{aligned}
\upsilon(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon) &= \upsilon\left(\widetilde{w}_k-\tilde{\xi},\frac{r+\varepsilon}{|\beta|}\right)\\
&\leq \max\left\{\upsilon(\widetilde{w}_k-\tilde{\xi},r+\varepsilon),\upsilon\left(0,\frac{r+\varepsilon}{|\beta|}-(r+\varepsilon)\right)\right\}\\
&\leq \max\left\{\upsilon(\widetilde{w}_k-L,r+\varepsilon),0\right\}\\
&= \upsilon(\widetilde{w}_k-\tilde{\xi},r+\varepsilon)<\tilde{\eta}.\\
\text{and } \tau(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon) &= \tau\left(\widetilde{w}_k-\tilde{\xi},\frac{r+\varepsilon}{|\beta|}\right)\\
&\leq \max\left\{\tau(\widetilde{w}_k-\tilde{\xi},r+\varepsilon),\tau\left(0,\frac{r+\varepsilon}{|\beta|}-(r+\varepsilon)\right)\right\}\\
&\leq \max\left\{\tau(\widetilde{w}_k-\tilde{\xi},r+\varepsilon),0\right\}\\
&= \tau(\widetilde{w}_k-\tilde{\xi},r+\varepsilon)<\tilde{\eta}.
\end{aligned}$$

which gives;

$$B_n\subset\{k\in\mathbb{N}:\varsigma(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)>1-\tilde{\eta},\upsilon(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)<\tilde{\eta},\tau(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)<\tilde{\eta}\}.$$

Using (3.2), we have

$$1-\delta\leq\frac{|B_n|}{n}\leq\frac{1}{n}|\{k\leq n:\varsigma(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)>1-\tilde{\eta},\upsilon(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)<\tilde{\eta},$$
$$\tau(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)<\tilde{\eta}\}|.$$

Consequently,

$$\frac{1}{n}|\{k\leq n:\varsigma(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)\leq 1-\tilde{\eta}\text{ or }\upsilon(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)\geq\tilde{\eta},$$
$$\text{or }\tau(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)\geq\tilde{\eta}\}|<\delta.$$

Then

$$\{n\in\mathbb{N}:\frac{1}{n}|\{k\leq n:\varsigma(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)\leq 1-\tilde{\eta}\text{ or }\upsilon(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)\geq\tilde{\eta},$$
$$\text{or }\tau(\beta\widetilde{w}_k-\beta\tilde{\xi};r+\varepsilon)\geq\tilde{\eta}\}|\geq\delta\}\subset A\in\mathcal{I}.$$

Therefore, $\beta\widetilde{w}_k\xrightarrow{r-\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}}\beta\tilde{\xi}$.

(ii) By similar way, we can prove (ii) part. □

Next, we will show the set $\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}_k}$ is closed.

Theorem 3.11. *The set $\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}_k}$ of Copson transformation sequence $\widetilde{w}=(\widetilde{w}_k)$ is a closed set on NNS $\mathcal{W}=(\mathbb{W},\mathcal{N},\oplus,\otimes)$.*

Proof. If $\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}_k}=\emptyset$ then $\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}}$ is either $\emptyset$ or singleton set.
Assume $\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}_k}\neq\emptyset$ for some $r>0$.
Consider $\widetilde{x}=(\widetilde{x}_k)$ be a convergent sequence in $\mathcal{W}=(\mathbb{W},\mathcal{N},\oplus,\otimes)$ with respect to $(\varsigma,\upsilon,\tau)$, which converges to $x_0\in\mathbb{W}$. For $\varepsilon>0$ and $\tilde{\eta}\in(0,1)$ then there exists $p_0\in\mathbb{N}$ such that

$$\varsigma\left(\widetilde{x}_k-x_0;\frac{\varepsilon}{2}\right)>1-\tilde{\eta},\upsilon\left(\widetilde{x}_k-x_0;\frac{\varepsilon}{2}\right)<\tilde{\eta},\tau(\widetilde{x}_k-x_0;\frac{\varepsilon}{2})<\tilde{\eta}\text{ for all }k\geq p_0.$$

Take $\widetilde{x}_{p_1}\in\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}_k}$ with $p_1>p_0$ and $\delta>0$ such that

$$A=\{n\in\mathbb{N}:\frac{1}{n}|\{k\leq n:\varsigma(\widetilde{w}_k-\widetilde{x}_{p_1};r+\frac{\varepsilon}{2})\leq 1-\tilde{\eta}\text{ or }\upsilon(\widetilde{w}_k-\widetilde{x}_{p_1};r+\frac{\varepsilon}{2})\geq\tilde{\eta},$$
$$\text{or }\tau(\widetilde{w}_k-\widetilde{x}_{p_1};r+\frac{\varepsilon}{2})\geq\tilde{\eta}\}|\geq\delta\}\in\mathcal{I}.$$

Since $\mathcal{I}$ an admissible so $M = \mathbb{N} \setminus A \neq \emptyset$. Choose $n \in M$, then

$$\frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}) \geq \tilde{\eta}$$

$$\text{or } \tau(\widetilde{w}_k - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}) \geq \tilde{\eta}\}| < \delta.$$

$$\Rightarrow \frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}) < \tilde{\eta},$$

$$\tau(\widetilde{w}_k - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}) < \tilde{\eta}\}| \geq 1 - \delta.$$

$$\text{Let } B_n = \{k \leq n : \varsigma(\widetilde{w}_k - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}) < \tilde{\eta},$$

$$\tau(\widetilde{w}_k - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}) < \tilde{\eta}\}.$$

For $j \in B_n$ and $j \geq p_0$, we have

$$\begin{aligned}
\varsigma(\widetilde{w}_j - x_0; r + \varepsilon) \geq & \min \left\{\varsigma\left(\widetilde{w}_j - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}\right), \varsigma\left(\widetilde{x}_{p_1} - x_0; \frac{\varepsilon}{2}\right)\right\} \\
& > 1 - \tilde{\eta}, \\
\upsilon(\widetilde{w}_j - x_0; r + \varepsilon) \leq & \max \left\{\upsilon\left(\widetilde{w}_j - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}\right), \upsilon\left(\widetilde{x}_{p_1} - x_0; \frac{\varepsilon}{2}\right)\right\} \\
& < \tilde{\eta}, \\
\tau(\widetilde{w}_j - x_0; r + \varepsilon) \leq & \max \left\{\tau\left(\widetilde{w}_j - \widetilde{x}_{p_1}; r + \frac{\varepsilon}{2}\right), \tau\left(\widetilde{x}_{p_1} - x_0; \frac{\varepsilon}{2}\right)\right\} \\
& < \tilde{\eta}.
\end{aligned}$$

Therefore,

$$j \in \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - x_0; r + \varepsilon) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k - x_0; r + \varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - x_0; r + \varepsilon) < \tilde{\eta}\}.$$

Consequently,

$$B_n \subset \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - x_0; r + \varepsilon) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k - x_0; r + \varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - x_0; r + \varepsilon) < \tilde{\eta}\}$$

which implies

$$1 - \delta \leq \frac{|B_n|}{n} \leq \frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - x_0; r + \varepsilon) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k - x_0; r + \varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - x_0; r + \varepsilon) < \tilde{\eta}\}|.$$

Therefore,

$$\frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - x_0; r + \varepsilon) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - x_0; r + \varepsilon) \geq \tilde{\eta}$$

$$\text{or } \tau(\widetilde{w}_k - x_0; r + \varepsilon) \geq \tilde{\eta}\}| < 1 - (1 - \delta) = \delta.$$

Then

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - x_0; r + \varepsilon) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - x_0; r + \varepsilon) \geq \tilde{\eta} \text{ or}$$

$$\tau(\widetilde{w}_k - x_0; r + \varepsilon) \geq \tilde{\eta}\}| \geq \delta\} \subset A \in \mathcal{I},$$

which shows that $x_0 \in \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$ in $(\mathbb{W}, \mathcal{N}, \oplus, \otimes)$. □

The convexity of the set $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}}$ is explained as below.

Theorem 3.12. *The set $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$ of Copson-transformation sequence $\widetilde{w} = (\widetilde{w}_k)$ on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ is a convex set for some non-negative number r.*

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$. For convexity we have to show that

$$(1-\lambda)\varphi_1 + \lambda\varphi_2 \in \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}}$$

for any real number $\lambda \in (0,1)$.
Since $\varphi_1, \varphi_2 \in \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$, then there exists $k \in \mathbb{N}$ for every $\varepsilon > 0$ and $\tilde{\eta} \in (0,1)$ such that

$$M_1 = \{k \in \mathbb{N} : \varsigma\left(\widetilde{w}_k - \varphi_1; \frac{r+\varepsilon}{2(1-\lambda)}\right) \leq 1 - \tilde{\eta} \text{ or } \upsilon\left(\widetilde{w}_k - \varphi_1; \frac{r+\varepsilon}{2(1-\lambda)}\right) \geq \tilde{\eta}$$
$$\text{or } \tau\left(\widetilde{w}_k - \varphi_1; \frac{r+\varepsilon}{2(1-\lambda)}\right) \geq \tilde{\eta}\},$$

and

$$M_2 = \{k \in \mathbb{N} : \varsigma\left(\widetilde{w}_k - \varphi_2; \frac{r+\varepsilon}{2\lambda}\right) \leq 1 - \tilde{\eta} \text{ or } \upsilon\left(\widetilde{w}_k - \varphi_2; \frac{r+\varepsilon}{2\lambda}\right) \geq \tilde{\eta}$$
$$\text{or } \tau\left(\widetilde{w}_k - \varphi_2; \frac{r+\varepsilon}{2\lambda}\right) \geq \tilde{\eta}\}.$$

For $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in M_1 \cup M_2\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Now, choose $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$. Let

$$A = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in M_1 \cup M_2\}\right| \geq \delta_1\right\} \in \mathcal{I}.$$

For $n \notin A$

$$\frac{1}{n}\left|\{k \leq n : k \in M_1 \cup M_2\}\right| < 1 - \delta_1.$$
$$\frac{1}{n}\left|\{k \leq n : k \notin M_1 \cup M_2\}\right| \geq \delta_1.$$

This implies $\{k \leq n : k \notin M_1 \cup M_2\} \neq \emptyset$.
Let $j \in (M_0 \cup M_1)^c = M_1^c \cap M_2^c$.
Then

$$\begin{aligned}
\varsigma(\widetilde{w}_j - [(1-\lambda)\varphi_1 + \lambda\varphi_2]; r+\varepsilon) &= \varsigma[(1-\lambda)(\widetilde{w}_j - \varphi_1) + \lambda(\widetilde{w}_j - \varphi_2); r+\varepsilon] \\
&\geq \min\left\{\varsigma\left((1-\lambda)(\widetilde{w}_j - \varphi_1); \frac{r+\varepsilon}{2}\right), \varsigma\left(\lambda(\widetilde{w}_j - \varphi_2); \frac{r+\varepsilon}{2}\right)\right\} \\
&= \min\left\{\varsigma\left(\widetilde{w}_j - \varphi_1; \frac{r+\varepsilon}{2(1-\lambda)}\right), \varsigma\left(\widetilde{w}_j - \varphi_2; \frac{r+\varepsilon}{2\lambda}\right)\right\} \\
&> 1 - \tilde{\eta}, \\
\upsilon(\widetilde{w}_j - [(1-\lambda)\varphi_1 + \lambda\varphi_2]; r+\varepsilon) &= \upsilon[(1-\lambda)(\widetilde{w}_j - \varphi_1) + \lambda(\widetilde{w}_j - \varphi_2); r+\varepsilon] \\
&\leq \max\left\{\upsilon\left((1-\lambda)(\widetilde{w}_j - \varphi_1); \frac{r+\varepsilon}{2}\right), \upsilon\left(\lambda(\widetilde{w}_j - \varphi_2); \frac{r+\varepsilon}{2}\right)\right\} \\
&= \max\left\{\upsilon\left(\widetilde{w}_j - \varphi_1; \frac{r+\varepsilon}{2(1-\lambda)}\right), \upsilon\left(\widetilde{w}_j - \varphi_2; \frac{r+\varepsilon}{2\lambda}\right)\right\} \\
&< \tilde{\eta}, \\
\tau(\widetilde{w}_j - [(1-\lambda)\varphi_1 + \lambda\varphi_2]; r+\varepsilon) &= \tau[(1-\lambda)(\widetilde{w}_j - \varphi_1) + \lambda(\widetilde{w}_j - \varphi_2); r+\varepsilon] \\
&\leq \max\left\{\tau\left((1-\lambda)(\widetilde{w}_j - \varphi_1); \frac{r+\varepsilon}{2}\right), \tau\left(\lambda(\widetilde{w}_j - \varphi_2); \frac{r+\varepsilon}{2}\right)\right\} \\
&= \max\left\{\tau\left(\widetilde{w}_j - \varphi_1; \frac{r+\varepsilon}{2(1-\lambda)}\right), \tau\left(\widetilde{w}_j - \varphi_2; \frac{r+\varepsilon}{2\lambda}\right)\right\} \\
&< \tilde{\eta}.
\end{aligned}$$

This implies $M_1^c \cap M_2^c \subset B^c$ where

$$B=\{k\in\mathbb{N}:\varsigma(\widetilde{w}_k-[(1-\lambda)\varphi_1+\lambda\varphi_2];r+\varepsilon)\leq 1-\tilde{\eta} \text{ or } \upsilon\left(\widetilde{w}_k-[(1-\lambda)\varphi_1+\lambda\varphi_2];r+\varepsilon\right)\geq\tilde{\eta}$$

$$\text{or } \tau\left(\widetilde{w}_k-[(1-\lambda)\varphi_1+\lambda\varphi_2];r+\varepsilon\right)\geq\tilde{\eta}\}.$$

For $n \notin A$,

$$\delta_1 \leq \frac{1}{n}\left|\{k \leq n : p \notin M_1 \cup M_2\}\right| \leq \frac{1}{n}\left|\{k \leq n : k \notin B\}\right|$$

or

$$\frac{1}{n}\left|\{k \leq n : k \in B\}\right| < 1-\delta_1 < \delta.$$

Therefore, we obtain $A^c \subset \left\{n \in \mathbb{N} : \frac{1}{n}\left|k \leq n : k \in B\right| < \delta\right\}$.
Since $A^c \in \mathcal{F}(\mathcal{I})$, then $\left\{n \in \mathbb{N} : \frac{1}{n}\left|k \leq n : k \in B\right| < \delta\right\} \in \mathcal{F}(\mathcal{I})$,
$i.e$ $\left\{n \in \mathbb{N} : \frac{1}{n}\left|k \leq n : k \in B\right| \geq \delta\right\} \in \mathcal{I}$. This implies that $\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}_k}$ is a convex set. □

Theorem 3.13. *A Copson-transformation sequence $\widetilde{w}=(\widetilde{w}_k)$ on NNS $\mathcal{W}=(\mathbb{W},\mathcal{N},\oplus,\otimes)$ is rough $\mathcal{I}$-statistically convergent to $\rho\in\mathbb{W}$ with respect to norm $(\varsigma,\upsilon,\tau)$ for some $r>0$ if there exists a sequence $\widetilde{z}=(\widetilde{z}_k)$ in $\mathbb{W}$such that $\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM_{\widetilde{z}_k}=\rho$ in $\mathbb{W}$ and for every $\tilde{\eta}\in(0,1)$ we have $\varsigma(\widetilde{w}_k-\widetilde{z}_k;r+\varepsilon)>1-\tilde{\eta}, \upsilon(\widetilde{w}_k-\widetilde{z}_k;r+\varepsilon)<\tilde{\eta}, \tau(\widetilde{w}_k-\widetilde{z}_k;r+\varepsilon)<\tilde{\eta}$ for all $k\in\mathbb{N}$.*

Proof. Since $\widetilde{z}=(\widetilde{z}_k)$ is a Copson-transformation sequence in $\mathbb{W}$, which is $\mathcal{I}$-statistically convergent to $\rho\in\mathbb{W}$ and $\varsigma(\widetilde{w}_k-\widetilde{z}_k;r+\varepsilon)>1-\tilde{\eta}, \upsilon(\widetilde{w}_k-\widetilde{z}_k;r+\varepsilon)<\tilde{\eta}, \tau(\widetilde{w}_k-\widetilde{z}_k;r+\varepsilon)<\tilde{\eta}$ for all $k\in\mathbb{N}$ and $\tilde{\eta}\in(0,1)$.
By definition, for any $\varepsilon,\delta>0$ and $\tilde{\eta}\in(0,1)$, we have

$$\{n\in\mathbb{N}:\frac{1}{n}|\{k\leq n:\varsigma(\widetilde{z}_k-\rho;\varepsilon)\leq 1-\tilde{\eta} \text{ or } \upsilon(\widetilde{z}_k-\rho;\varepsilon)\geq\tilde{\eta}$$

$$\text{or } \tau(\widetilde{z}_k-\rho;\varepsilon)\geq\tilde{\eta}\}|\geq\delta\}\in\mathcal{I}.$$

Define

$$M_1=\{k\in\mathbb{N}:\varsigma(\widetilde{z}_k-\rho;\varepsilon)\leq 1-\tilde{\eta} \text{ or } \upsilon(\widetilde{z}_k-\rho;\varepsilon)\geq\tilde{\eta} \text{ or } \tau(\widetilde{z}_k-\rho;\varepsilon)\geq\tilde{\eta}\},$$

$$M_2=\{k\in\mathbb{N}:\varsigma(\widetilde{w}_k-\widetilde{z}_p;r)\leq 1-\tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k-\widetilde{z}_p;r)\geq\tilde{\eta} \text{ or } \tau(\widetilde{w}_k-\widetilde{z}_k;r)\geq\tilde{\eta}\}.$$

For $\delta>0$, we have

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:k\in M_1\cup M_2\}\right|\geq\delta\right\}\in\mathcal{I}.$$

take $0<\delta_1<1$ such that $0<1-\delta_1<\delta$. Let

$$A=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:k\in M_1\cup M_2\}\right|\geq\delta_1\right\}\in\mathcal{I}.$$

For $n\notin A$

$$\frac{1}{n}\left|\{k\leq n:k\in M_1\cup M_2|\right\}<1-\delta_1$$

$$\frac{1}{n}\left|\{k\leq n:k\notin M_1\cup M_2|\right\}\geq\delta_1.$$

Thus, $\{k\leq n:k\notin M_1\cup M_2\}\neq\emptyset$.
Let $j\in(M_1\cup M_2)^c=M_1^c\cap M_2^c$

Then

$$\begin{aligned}\varsigma\left(\widetilde{w}_j-\rho;r+\varepsilon\right) &\geq \min\left\{\varsigma(\widetilde{w}_j-\widetilde{z}_j;r),\varsigma(\widetilde{z}_j-\rho;\varepsilon)\right\}\\ &> 1-\tilde{\eta},\\ \upsilon\left(\widetilde{w}_j-\rho;r+\varepsilon\right) &\leq \max\left\{\upsilon(\widetilde{w}_j-\widetilde{z}_j;r),\upsilon(\widetilde{z}_j-\rho;\varepsilon)\right\}\\ &< \tilde{\eta},\\ \tau\left(\widetilde{w}_j-\rho;r+\varepsilon\right) &\leq \max\left\{\tau(\widetilde{w}_j-\widetilde{z}_j;r),\tau(\widetilde{z}_j-\rho;\varepsilon)\right\}\\ &< \tilde{\eta}.\end{aligned}$$

Therefore, $M_1^c\cap M_2^c\subset B^c$, where

$$B=\{k\in\mathbb{N}:\varsigma(\widetilde{w}_k-\rho;r+\varepsilon)\leq 1-\tilde{\eta}\text{ or }\upsilon(\widetilde{w}_k-\rho;r+\varepsilon)\geq\tilde{\eta}\text{ or }\tau(\widetilde{w}_k-\rho;r+\varepsilon)\geq\tilde{\eta}\}.$$

Hence, for $n\notin A$,

$$\delta_1\leq\frac{1}{n}|\{k\leq n:k\notin M_1\cup M_2\}|\leq\frac{1}{n}|\{k\leq n:k\notin B\}|$$

or

$$\frac{1}{n}|\{k\leq n:k\in B\}|<1-\delta_1<\delta.$$

Therefore, we obtain $A^c\subset\{n\in\mathbb{N}:\frac{1}{n}|k\leq n:k\in B|<\delta\}$. Since $A^c\in\mathcal{F}(\mathcal{I})$, then $\left\{n\in\mathbb{N}:\frac{1}{n}\left|k\leq n:k\in B\right|<\delta\right\}\in\mathcal{F}(\mathcal{I})$, which implies $\left\{n\in\mathbb{N}:\frac{1}{n}\left|k\leq n:k\in B\right|\geq\delta\right\}\in\mathcal{I}$.
Finally, $\widetilde{w}_k\xrightarrow{r-\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}}\rho$ on NNS $\mathcal{W}=(\mathbb{W},\mathcal{N},\oplus,\otimes)$. □

Theorem 3.14. *Let $\widetilde{w}=(\widetilde{w}_k)$ be a sequence on NNS $\mathcal{W}=(\mathbb{W},\mathcal{N},\oplus,\otimes)$. There does not exist two elements $\tilde{\xi}_1,\tilde{\xi}_2\in\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}}$ for $r>0$ and $\tilde{\eta}\in(0,1)$ such that $\varsigma(\tilde{\xi}_1-\tilde{\xi}_2;\tilde{c}r)\leq 1-\tilde{\eta}$ or $\upsilon(\tilde{\xi}_1-\tilde{\xi}_2;cr)\geq\tilde{\eta}$ or $\tau(\tilde{\xi}_1-\tilde{\xi}_2;\tilde{c}r)\geq\tilde{\eta}$ for $\tilde{c}>2$.*

Proof. If possible, suppose there exists two elements $\tilde{\xi}_1,\tilde{\xi}_2\in\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}}$ such that

$$\varsigma(\tilde{\xi}_1-\tilde{\xi}_2;\tilde{c}r)\leq 1-\tilde{\eta}\text{ or }\upsilon(\tilde{\xi}_1-\tilde{\xi}_2;\tilde{c}r)\geq\tilde{\eta}\text{ or }\tau(\tilde{\xi}_1-\tilde{\xi}_2;cr)\geq\tilde{\eta}\text{ for }\tilde{c}>2. \tag{3.3}$$

As $\tilde{\xi}_1,\tilde{\xi}_2\in\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}-LIM^r_{\widetilde{w}}$ then for every $\varepsilon>0$ and $\tilde{\eta}\in(0,1)$. Define,

$$\begin{aligned}M_1=\{k\in\mathbb{N}:\varsigma\left(\widetilde{w}_k-\tilde{\xi}_1;r+\frac{\varepsilon}{2}\right)\leq 1-\tilde{\eta}\text{ or }\upsilon\left(\widetilde{w}_k-\tilde{\xi}_1;r+\frac{\varepsilon}{2}\right)\geq\tilde{\eta}\\ \text{or }\tau\left(\widetilde{w}_k-\tilde{\xi}_1;r+\frac{\varepsilon}{2}\right)\geq\tilde{\eta}\}\\ M_2=\{k\in\mathbb{N}:\varsigma\left(\widetilde{w}_k-\tilde{\xi}_2;r+\frac{\varepsilon}{2}\right)\leq 1-\tilde{\eta}\text{ or }\upsilon\left(\widetilde{w}_k-\tilde{\xi}_2;r+\frac{\varepsilon}{2}\right)\geq\tilde{\eta}\\ \text{or }\tau\left(\widetilde{w}_k-\tilde{\xi}_2;r+\frac{\varepsilon}{2}\right)\geq\tilde{\eta}\}.\end{aligned}$$

Then,

$$\frac{1}{n}|\{k\leq n:p\in M_1\cup M_2\}|\leq\frac{1}{n}|\{k\leq n:p\in M_1\}|+\frac{1}{n}|\{k\leq n:p\in M_2\}|.$$

By the property of $\mathcal{I}$*-convergence*, we get

$$\begin{aligned}\mathcal{I}-\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n:p\in M_1\cup M_2\}\right|\leq\mathcal{I}-\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n:p\in M_1\}\right|\\ +\mathcal{I}-\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n:p\in M_2\}\right|=0.\end{aligned}$$

For $\delta>0$

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n\in\mathbb{N}:k\in M_1\cup M_2\}\right|\geq\delta\right\}\in\mathcal{I}.$$

Now choose $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$.
Let

$$A = \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : k \in M_1 \cup M_2 \} \right| \geq \delta_1 \right\} \in \mathcal{I}.$$

For $n \notin A$

$$\frac{1}{n} \left| \{ k \leq n : k \in M_1 \cup M_2 \} \right| < 1 - \delta_1$$
$$\frac{1}{n} \left| \{ k \leq n : k \notin M_1 \cup M_2 | \} \right. \geq \delta_1.$$

This implies $\{k \leq n : k \notin M_1 \cup M_2\} \neq \emptyset$. Then for $k \in M_1^c \cap M_2^c$ we have

$$\begin{aligned} \varsigma\left(\tilde{\xi}_1 - \tilde{\xi}_2; 2r + \varepsilon\right) &\geq \min\left\{\varsigma\left(\widetilde{w}_k - \tilde{\xi}_2; r + \frac{\varepsilon}{2}\right), \varsigma\left(\widetilde{w}_k - \tilde{\xi}_1; r + \frac{\varepsilon}{2}\right)\right\} \\ &> 1 - \tilde{\eta}, \\ \upsilon\left(\tilde{\xi}_1 - \tilde{\xi}_2; 2r + \varepsilon\right) &\leq \max\left\{\upsilon\left(\widetilde{w}_k - \tilde{\xi}_2; r + \frac{\varepsilon}{2}\right), \upsilon\left(\widetilde{w}_k - \tilde{\xi}_1; r + \frac{\varepsilon}{2}\right)\right\} \\ &< \tilde{\eta}, \\ \tau\left(\tilde{\xi}_1 - \tilde{\xi}_2; 2r + \varepsilon\right) &\leq \max\left\{\tau\left(\widetilde{w}_k - \tilde{\xi}_2; r + \frac{\varepsilon}{2}\right), \tau\left(\widetilde{w}_k - \tilde{\xi}_1; r + \frac{\varepsilon}{2}\right)\right\} \\ &< \tilde{\eta}. \end{aligned}$$

Hence,

$$\varsigma\left(\tilde{\xi}_1 - \tilde{\xi}_2; 2r + \varepsilon\right) > 1 - \tilde{\eta}, \upsilon\left(\tilde{\xi}_1 - \tilde{\xi}_2; 2r + \varepsilon\right) < \tilde{\eta}, \tau\left(\tilde{\xi}_1 - \tilde{\xi}_2; 2r + \varepsilon\right) < \tilde{\eta}. \tag{3.4}$$

Using (3.4), we have

$$\varsigma\left(\tilde{\xi}_1 - \tilde{\xi}_2; \tilde{c}r\right) > 1 - \tilde{\eta}, \upsilon\left(\tilde{\xi}_1 - \tilde{\xi}_2; \tilde{c}r\right) < \tilde{\eta}, \tau\left(\tilde{\xi}_1 - \tilde{\xi}_2; \tilde{c}r\right) < \tilde{\eta} \text{ for } \tilde{c} > 2$$

which leads contradiction to (3.3). Hence, there does not exist two elements such that $\varsigma(\tilde{\xi}_1 - \tilde{\xi}_2; \tilde{c}r) \leq 1 - \tilde{\eta}$ or $\upsilon(\tilde{\xi}_1 - \tilde{\xi}_2; \tilde{c}r) \geq \tilde{\eta}$ or $\tau(\tilde{\xi}_1 - \tilde{\xi}_2; \tilde{c}r) \geq \tilde{\eta}$ for $\tilde{c} > 2$. □

4. Rough ideal statistical cluster points for Copson-transformation sequence on NNS

Definition 4.1. Let $\mathcal{W} = (\mathbb{W}, N, \oplus, \otimes)$ be NNS. Then $\tilde{\gamma} \in \mathbb{W}$ is called rough $\mathcal{I}$-statistical cluster point of the sequence $\widetilde{w} = (\widetilde{w}_k)$ in $\mathbb{W}$ with respect to norm $(\varsigma, \upsilon, \tau)$ for some $r > 0$ if for every $\varepsilon > 0$ and $\tilde{\eta} \in (0, 1)$

$$D_{\mathcal{I}}(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \tilde{\gamma}; r + \varepsilon) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k - \tilde{\gamma}; r + \varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \tilde{\gamma}; r + \varepsilon) < \tilde{\eta}\}) \neq 0,$$

where $D_{\mathcal{I}}(A) = \mathcal{I} - \lim\limits_{n \to \infty} \frac{1}{n} |\{k \leq n : p \in A\}|$ if exists. In this case, $\tilde{\gamma}$ is known as r-$\mathcal{I}$-statistical cluster point of a sequence $\widetilde{w} = (\widetilde{w}_k)$.

Let $\mathit{\Gamma}^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$ indicates the set of all r-$\mathcal{I}$-statistical cluster points with respect to the norm $(\varsigma, \upsilon, \tau)$ of a sequence $\widetilde{w}_k$ on NNS $\mathbb{W} = (W, \mathcal{N}, \oplus, \otimes)$. If $r = 0$, then the notion stands for only $\mathcal{I}$-statistical cluster point with respect to the norm $(\varsigma, \upsilon, \tau)$ on NNS $\mathbb{W} = (W, \mathcal{N}, \oplus, \otimes)$, symbolically; $\mathit{\Gamma}^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) = \mathit{\Gamma}^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$.

In the next result, we have derived the closedness of the set $\mathit{\Gamma}^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$ for sequence $\widetilde{w}_k$ in $\mathbb{W}$.

Theorem 4.2. *The set $\mathit{\Gamma}^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$ of Copson- transformation sequence $\widetilde{w} = (\widetilde{w}_k)$ on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ is closed for some $r > 0$.*

Proof. If $\mathit{\Gamma}^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) = \emptyset$, then the result is obvious.
Let $\mathit{\Gamma}^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \neq \emptyset$. Consider $\widetilde{x} = (\widetilde{x}_k)$ be Copson-transformation sequence such that

$$(\widetilde{x}) \subseteq \mathit{\Gamma}^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \text{ and } \widetilde{x}_k \xrightarrow{(\varsigma,\upsilon)} x_0.$$

To prove closedness, it is sufficient to show that $x_0 \in \Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$.
As $\widetilde{x}_k \xrightarrow{(\varsigma,\upsilon,\tau)} x_0$,then for every $\varepsilon > 0$ and $\tilde{\eta} \in (0,1)$, there exists $p_\varepsilon \in \mathbb{N}$ such that
$\varsigma(\widetilde{x}_k - x_0; \frac{\varepsilon}{2}) > 1-\tilde{\eta}, \upsilon(\widetilde{x}_k - x_0; \frac{\varepsilon}{2}) < \tilde{\eta}, \tau(\widetilde{x}_k - x_0; \frac{\varepsilon}{2}) < \tilde{\eta}$ for $k \geq p_\varepsilon$.
Choose some $p_0 \in \mathbb{N}$ such that $p_0 \geq p_\varepsilon$. Then, we have

$$\varsigma(\widetilde{x}_{p_0} - x_0; \frac{\varepsilon}{2}) > 1-\tilde{\eta}, \upsilon(\widetilde{x}_{p_0} - x_0; \frac{\varepsilon}{2}) < \tilde{\eta}, \tau(\widetilde{x}_{p_0} - x_0; \frac{\varepsilon}{2}) < \tilde{\eta}.$$

Again as $\widetilde{x} = (\widetilde{x}_k) \subseteq \Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$, we have $\widetilde{x}_{p_0} \in \Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$.

$$\implies D_{\mathcal{I}}(\{k \in \mathbb{N} : \varsigma\left(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}\right) > 1-\tilde{\eta}, \upsilon\left(\widetilde{w}_k - \widetilde{x}_{p_0}); r + \frac{\varepsilon}{2}\right) < \tilde{\eta},$$

$$\tau\left(\widetilde{w}_k - \widetilde{x}_{p_0}); r + \frac{\varepsilon}{2}\right) < \tilde{\eta}\}) \neq 0. \tag{4.1}$$

Consider

$$A = \{k \in \mathbb{N} : \varsigma\left(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}\right) > 1-\tilde{\eta}, \upsilon\left(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}\right) < \tilde{\eta}, \tau\left(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}\right) < \tilde{\eta}\}.$$

Choose $j \in A$, then we have $\varsigma(\widetilde{w}_j - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_j - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}) < \tilde{\eta}, \tau(\widetilde{w}_j - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}) < \tilde{\eta}$.
Now,

$$\begin{aligned}
\varsigma(\widetilde{w}_j - x_0; r+\varepsilon) &\geq \min\left\{\varsigma\left(\widetilde{w}_j - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}\right), \varsigma\left(\widetilde{x}_{p_0} - x_0; r + \frac{\varepsilon}{2}\right)\right\} \\
&> 1-\tilde{\eta}, \\
\upsilon(\widetilde{w}_j - x_0; r+\varepsilon) &\leq \max\left\{\upsilon\left(\widetilde{w}_j - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}\right), \upsilon\left(\widetilde{x}_{p_0} - y_0; r + \frac{\varepsilon}{2}\right)\right\} \\
&< \tilde{\eta}, \\
\tau(\widetilde{w}_j - x_0; r+\varepsilon) &\leq \max\left\{\tau\left(\widetilde{w}_j - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}\right), \tau\left(\widetilde{x}_{p_0} - y_0; r + \frac{\varepsilon}{2}\right)\right\} \\
&< \tilde{\eta}.
\end{aligned}$$

Thus,

$$j \in \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - x_0; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - x_0; r+\varepsilon) < \tilde{\eta}, (\widetilde{w}_k - x_0; r+\varepsilon) < \tilde{\eta}\}.$$

Hence,

$$\begin{aligned}
&\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}) < \tilde{\eta}, \tau(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}) < \tilde{\eta}\} \\
&\subseteq \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - x_0; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - x_0; r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - x_0; r+\varepsilon) < \tilde{\eta}\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&D_{\mathcal{I}}(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}) < \tilde{\eta}, \tau(\widetilde{w}_k - \widetilde{x}_{p_0}; r + \frac{\varepsilon}{2}) < \tilde{\eta}\}) \\
&\leq D_{\mathcal{I}}(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - x_0; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - x_0; r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - x_0; r+\varepsilon) < \tilde{\eta}\}).
\end{aligned} \tag{4.2}$$

Using (4.1), we get

$$D_{\mathcal{I}}\left(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - x_0; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - x_0; r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - x_0; r+\varepsilon) < \tilde{\eta}\}\right) \neq 0,$$

as the set on left side hand of (4.2) possesses natural density more than zero.
Therefore, $x_0 \in \Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$. □

Theorem 4.3. *Let $\Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$ be the collection of all $\mathcal{I}$-statistical cluster points of the sequence $\widetilde{w} = (\widetilde{w}_k)$ on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$. Then, for any arbitrary $\nu \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k), r \geq 0$ and $\tilde{\eta} \in (0,1)$, we have $\varsigma(\zeta - \nu; r) > 1-\tilde{\eta}, \upsilon(\zeta - \nu; r) < \tilde{\eta}, \tau(\zeta - \nu; r) < \tilde{\eta}$ for all $\zeta \in \Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$.*

Proof. Since $\nu \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$ then for $\varepsilon > 0$ and $\tilde{\eta} \in (0,1)$, we have

$$D_{\mathcal{I}}\left(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \nu;\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}\}\right) \neq 0. \tag{4.3}$$

Now it is sufficient to show that if any $\zeta \in \mathbb{W}$ satisfying $\varsigma(\zeta-\nu;\varepsilon) > 1-\tilde{\eta}, \upsilon(\zeta-\nu;\varepsilon) < \tilde{\eta}, \tau(\zeta-\nu;\varepsilon) < \tilde{\eta}$ then $\zeta \in \Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$.
Suppose $j \in \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \nu;\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}\}$ then

$$\varsigma(\widetilde{w}_k - \nu;\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}.$$

Now,

$$\begin{aligned}
\varsigma\left(\widetilde{w}_k - \zeta; r+\varepsilon\right) &\geq \min\left\{\varsigma\left(\widetilde{w}_k - \nu;\varepsilon\right), \varsigma\left(\zeta-\nu;r\right)\right\}\\
&> 1-\tilde{\eta},\\
\upsilon\left(\widetilde{w}_k - \zeta; r+\varepsilon\right) &\leq \max\left\{\upsilon\left(\widetilde{w}_k - \nu;\varepsilon\right), \upsilon\left(\zeta-\nu;r\right)\right\}\\
&< \tilde{\eta},\\
\tau\left(\widetilde{w}_k - \zeta; r+\varepsilon\right) &\leq \max\left\{\tau\left(\widetilde{w}_k - \nu;\varepsilon\right), \tau\left(\zeta-\nu;r\right)\right\}\\
&< \tilde{\eta}.
\end{aligned}$$

Thus, $j \in \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \zeta; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \zeta; r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \zeta; r+\varepsilon) < \tilde{\eta}\}$ which provides inclusion

$$\begin{aligned}
&\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \nu;\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}\}\\
&\subseteq \left\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \zeta; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \zeta; r+\varepsilon) < \tilde{\eta}, \upsilon(\widetilde{w}_k - \zeta; r+\varepsilon) < \tilde{\eta}\right\}.
\end{aligned}$$

Then,

$$\begin{aligned}
&D_{\mathcal{I}}\left(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \nu;\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \nu;\varepsilon) < \tilde{\eta}\}\right)\\
&\leq D_{\mathcal{I}}\left(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \zeta; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \zeta; r+\varepsilon) < \tilde{\eta}, \upsilon(\widetilde{w}_k - \zeta; r+\varepsilon) < \tilde{\eta}\}\right).
\end{aligned}$$

Therefore, from (4.3)

$$D_{\mathcal{I}}\left(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \zeta; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \zeta; r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \zeta; r+\varepsilon) < \tilde{\eta}\}\right) \neq 0.$$

Hence, $\zeta \in \Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$. □

Theorem 4.4. *Let $\widetilde{w} = (\widetilde{w}_k)$ be Copson-transformation sequence on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$ and $\overline{B(\rho,\tilde{\eta},r)} = \{\widetilde{w} \in \mathbb{W} : \varsigma(\widetilde{w}-\rho;r) \geq 1-\tilde{\eta}, \upsilon(\widetilde{w}-\rho;r) \leq \tilde{\eta}, \tau(\widetilde{w}-\rho;r) \leq \tilde{\eta}\}$, represents the closure of the open ball $B(\rho,\tilde{\eta},r) = \{\widetilde{w} \in \mathbb{W} : \varsigma(\widetilde{w}-\rho;r) > 1-\tilde{\eta}, \upsilon(\widetilde{w}-\rho;r) < \tilde{\eta}, \tau(\widetilde{w}-\rho;r) < \tilde{\eta}\}$ for some $r > 0$ and $\tilde{\eta} \in (0,1)$ and fixed $\rho \in \mathbb{W}$ then $\Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) = \bigcup_{\rho \in \Gamma^{I}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho,\tilde{\eta},r)}$.*

Proof. Let $\zeta \in \bigcup_{\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho,\tilde{\eta},r)}$ then there exists $\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$ for $r > 0$, and $\tilde{\eta} \in (0,1)$ such that $\varsigma(\rho-\zeta;r) > 1-\tilde{\eta}, \upsilon(\rho-\zeta;r) < \tilde{\eta}, \tau(\rho-\zeta;r) < \tilde{\eta}$.
As $\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$ then there exists a set

$$M = \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - \rho;\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - \rho;\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \rho;\varepsilon) < \tilde{\eta}\}$$

with $D_{\mathcal{I}}(M) \neq 0$. For $p \in M$,

$$\begin{aligned} \varsigma\left(\widetilde{w}_k-\zeta;r+\varepsilon\right) &\geq \min\left\{\varsigma\left(\widetilde{w}_k-\rho;\varepsilon\right),\varsigma\left(\rho-\zeta;r\right)\right\} \\ &> 1-\tilde{\eta}, \\ \upsilon\left(\widetilde{w}_k-\zeta;r+\varepsilon\right) &\leq \max\left\{\upsilon\left(\widetilde{w}_k-\rho;\varepsilon\right),\upsilon\left(\rho-\zeta;r\right)\right\} \\ &< \tilde{\eta}, \\ \tau\left(\widetilde{w}_k-\zeta;r+\varepsilon\right) &\leq \max\left\{\tau\left(\widetilde{w}_k-\rho;\varepsilon\right),\tau\left(\rho-\zeta;r\right)\right\} \\ &< \tilde{\eta}. \end{aligned}$$

Consequently,
$D_{\mathcal{I}}(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k-\zeta;r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k-\zeta;r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k-\zeta;r+\varepsilon) < \tilde{\eta}\}) \neq 0$.
Hence, $\zeta \in \Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$. So, $\bigcup_{\rho \in \Gamma^{I}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho,\tilde{\eta},r)} \subseteq \Gamma^{r(I)}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$.

Conversely,
Take $\zeta \in \Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$. If possible let $\zeta \notin \bigcup_{\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho,\tilde{\eta},r)}$ $i.e$ $\zeta \notin \overline{B\left(\rho,\tilde{\eta},r\right)}$ for all $\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$.
Then, for $\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$, we have $\varsigma(\zeta-\rho;r) \leq 1-\tilde{\eta}$ or $\upsilon(\zeta-\rho;r) \geq \tilde{\eta}$ or $\tau(\zeta-\rho;r) \geq \tilde{\eta}$. According to theorem (4.3) for any arbitrary $\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$, we have $\varsigma(\zeta-\rho;r) > 1-\tilde{\eta}, \upsilon(\zeta-\rho;r) < \tilde{\eta}, \tau(\zeta-\rho;r) < \tilde{\eta}$ which is contradiction to our supposition. Hence, $\zeta \in \bigcup_{\rho \in \Gamma^{I}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho,\tilde{\eta},r)}$. So $\Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \subseteq \bigcup_{\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho,\tilde{\eta},r)}$. This completes the proof. □

Theorem 4.5. *Let $\widetilde{w} = (\widetilde{w}_k)$ be Copson-transformation sequence on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$, Then for $\tilde{\eta} \in (0,1)$ and $r > 0$,*
(i) If $\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$ then $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^{r}_{\widetilde{w}_k} \subseteq \overline{B\left(\rho,\tilde{\eta},r\right)}$.
(ii) $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^{r}_{\widetilde{w}_k} = \bigcap_{\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho,\tilde{\eta},r)} = \left\{\tilde{\xi} \in \mathbb{W} : \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \subseteq \overline{B\left(\tilde{\xi},\tilde{\eta},r\right)}\right\}$.

Proof. Let $\tilde{\xi} \in \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^{r}_{\widetilde{w}_k}$ and $\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$
For $\varepsilon > 0$ and $\tilde{\eta} \in (0,1)$,
Consider

$$A = \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k-\tilde{\xi};r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k-\tilde{\xi};r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k-\tilde{\xi};r+\varepsilon) < \tilde{\eta}\}$$

and

$$B = \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k-\rho;\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k-\rho;\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k-\rho;\varepsilon) < \tilde{\eta}\}$$

with $D_{\mathcal{I}}(A^c) = 0$ and $D_{\mathcal{I}}(B) \neq 0$ respectively. For $k \in A \cap B$,

$$\begin{aligned} \varsigma\left(\tilde{\xi}-\rho;r\right) &\geq \min\left\{\varsigma\left(\widetilde{w}_k-\rho;\varepsilon\right),\varsigma\left(\widetilde{w}_k-\tilde{\xi};r+\varepsilon\right)\right\} \\ &> 1-\tilde{\eta}, \\ \upsilon\left(\tilde{\xi}-\rho;r\right) &\leq \max\left\{\upsilon\left(\widetilde{w}_k-\rho;\varepsilon\right),\upsilon\left(\widetilde{w}_k-\tilde{\xi};r+\varepsilon\right)\right\} \\ &< \tilde{\eta}, \\ \tau(\tilde{\xi}-\rho;r) &\leq \max\left\{\tau(\widetilde{w}_k-\rho;\varepsilon),\tau(\widetilde{w}_k-\tilde{\xi};r+\varepsilon)\right\} \\ &< \tilde{\eta}. \end{aligned}$$

Thus, $\tilde{\xi} \in \overline{B(\rho, \tilde{\eta}, r)}$. Hence, $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} \subseteq \overline{B(\rho, \tilde{\eta}, r)}$.
(ii) From (i) part we have $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} \subseteq \bigcap_{\rho \in \Gamma^I_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho, \tilde{\eta}, r)}$.
Take $y \in \bigcap_{\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho, \tilde{\eta}, r)}$ then $\varsigma(y-\rho; r) \geq 1-\tilde{\eta}$, $\upsilon(y-\rho; r) \leq \tilde{\eta}$, $\tau(y-\rho; r) \leq \tilde{\eta}$ for all $\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$.
This implies that
$\Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \subseteq \overline{B(y, \tilde{\eta}, r)}$ $i.e$ $\bigcap_{\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho, \tilde{\eta}, r)} \subseteq \{\tilde{\xi} \in \mathbb{W} : \Gamma^I_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \subseteq \overline{B\left(\tilde{\xi}, \tilde{\eta}, r\right)}\}$.
Now assume $y \notin \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$, then for $\tilde{\eta} \in (0,1)$ and $\varepsilon > 0$, we have

$$D_{\mathcal{I}}(\{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - y; r+\varepsilon) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - y; r+\varepsilon) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - y; r+\varepsilon) \geq \tilde{\eta}\}) \neq 0.$$

Then, there exists some $\mathcal{I}$-statistical cluster point ρ for sequence $\widetilde{w} = (\widetilde{w}_k)$ such that $\varsigma(y - \rho; r+\varepsilon) \leq 1 - \tilde{\eta}$ or $\upsilon(y - \rho; r+\varepsilon) \geq \tilde{\eta}$ or $\tau(y - \rho; r+\varepsilon) \geq \tilde{\eta}$.
Thus, $\Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \subseteq \overline{B(y, \tilde{\eta}, r)}$ does not hold and $y \notin \{\tilde{\xi} \in \mathbb{W} : \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \subseteq \overline{B(\tilde{\xi}, \tilde{\eta}, r)}\}$
Hence, $\{\tilde{\xi} \in \mathbb{W} : \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \subseteq \overline{B\left(\tilde{\xi}, \tilde{\eta}, r\right)}\} \subseteq \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$.
And $\bigcap_{\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon)}(\widetilde{w}_k)} \overline{B(\rho, \tilde{\eta}, r)} \subseteq \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$.
So, $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} = \bigcap_{\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho, \tilde{\eta}, r)} = \{\tilde{\xi} \in \mathbb{W} : \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) \subseteq \overline{B(\tilde{\xi}, \tilde{\eta}, r)}\}$. □

Theorem 4.6. *Let $\widetilde{w} = (\widetilde{w}_k)$ be Copson-transformation sequence on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$, which is ideal statistically convergent to ρ then $\overline{B(\rho, \tilde{\eta}, r)} = \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$.*

Proof. Since $\widetilde{w}_k$ is ideal statistically convergent to ρ with respect to the norm $(\varsigma, \upsilon, \tau)$ $i.e$ $\widetilde{w}_k \xrightarrow{I-st(\varsigma,\upsilon,\tau)} \rho$, then $A \in \mathcal{I}$ where

$$A = \left\{ n : \frac{1}{n} \left|\{k \leq n : \varsigma(\widetilde{w}_k - \rho; \varepsilon) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \rho; \varepsilon) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - \rho; \varepsilon) \geq \tilde{\eta}\}\right| > \delta \right\}.$$

Since $\mathcal{I}$ an admissible ideal so $M = \mathbb{N} \setminus A \neq \emptyset$. For $n \in M^c$,

$$\frac{1}{n} \left|\{k \leq n : \varsigma(\widetilde{w}_k - \rho; \varepsilon) \leq 1 - \tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - \rho; \varepsilon) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - \rho; \varepsilon) \geq \tilde{\eta}\}\right| < \delta$$

$$\Rightarrow \frac{1}{n} \left|\{k \leq n : \varsigma(\widetilde{w}_k - \rho; \varepsilon) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k - \rho; \varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \rho; \varepsilon) < \tilde{\eta}\}\right| \geq 1 - \delta.$$

Put $B_n = \{k \leq n : \varsigma(\widetilde{w}_k - \rho; \varepsilon) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_k - \rho; \varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - \rho; \varepsilon) < \tilde{\eta}\}$.
Now for $j \in B_n$, we have $\varsigma(\widetilde{w}_j - \rho; \varepsilon) > 1 - \tilde{\eta}, \upsilon(\widetilde{w}_j - \rho; \varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_j - \rho; \varepsilon) < \tilde{\eta}$.
Let $y \in \overline{B(\rho, \tilde{\eta}, r)}$. We will prove $y \in \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$.

$$\begin{aligned}
\varsigma(\widetilde{w}_j - y; r+\varepsilon) &\geq \min\{\varsigma(\widetilde{w}_j - \rho, \varepsilon), \varsigma(y - \rho, r)\} \\
&> 1 - \tilde{\eta}, \\
\upsilon(\widetilde{w}_j - y; r+\varepsilon) &\leq \max\{\upsilon(\widetilde{w}_j - \rho, \varepsilon), \upsilon(y - \rho, r)\} \\
&< \tilde{\eta}, \\
\tau(\widetilde{w}_j - y; r+\varepsilon) &\leq \max\{\tau(\widetilde{w}_j - \rho, \varepsilon), \tau(y - \rho, r)\} \\
&< \tilde{\eta}.
\end{aligned}$$

Hence, $B_n \subset \{k \in \mathbb{N} : \varsigma(\widetilde{w}_k - y; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - y; r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - y; r+\varepsilon) < \tilde{\eta}\}$. Consequently,

$$\begin{aligned} 1-\delta &\leq \frac{|B_n|}{n} \\ &\leq \frac{1}{n}\left|\{k \leq n : \varsigma(\widetilde{w}_k - y; r+\varepsilon) > 1-\tilde{\eta}, \upsilon(\widetilde{w}_k - y; r+\varepsilon) < \tilde{\eta}, \tau(\widetilde{w}_k - y; r+\varepsilon) < \tilde{\eta}\}\right|. \end{aligned}$$

Therefore,

$$\frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - y; r+\varepsilon) \leq 1-\tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - y; r+\varepsilon) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - y; r+\varepsilon) \geq \tilde{\eta}\}| < \delta.$$

Then,

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \varsigma(\widetilde{w}_k - y; r+\varepsilon) \leq 1-\tilde{\eta} \text{ or } \upsilon(\widetilde{w}_k - y; r+\varepsilon) \geq \tilde{\eta} \text{ or } \tau(\widetilde{w}_k - y; r+\varepsilon) \geq \tilde{\eta}\}| \geq \delta\} \subset \mathbb{A} \in \mathcal{I},$$

which shows that $y \in \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$ on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$.
Hence, $\overline{B(\rho, \tilde{\eta}, r)} \subseteq \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$. Also $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} \subseteq \overline{B(\rho, \tilde{\eta}, r)}$
Therefore,

$$\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} = \overline{B(\rho, \tilde{\eta}, r)}.$$

□

The subsequent result will investigate the connection between rough $\mathcal{I}$-statistical limit points and rough $\mathcal{I}$-statistical cluster points of the Copson-transformation sequence within neutrosophic normed spaces.

Theorem 4.7. *Let $\widetilde{w} = (\widetilde{w}_k)$ be Copson-transformation sequence on NNS $\mathcal{W} = (\mathbb{W}, \mathcal{N}, \oplus, \otimes)$, which is ideal statistically convergent to $\tilde{\xi}$ then $\Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) = \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$.*

Proof. Firstly, assume $w_k \xrightarrow{\mathcal{I}-st_{(\varsigma,\upsilon,\tau)}} \tilde{\xi}$, which gives $\Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) = \{\tilde{\xi}\}$. Then for $r > 0$ and $\tilde{\eta} \in (0,1)$ by Theorem (4.4) , we have $\Gamma^{r(\mathcal{I})}_{st_{(\varsigma,\upsilon,\tau)}}(\widetilde{w}_k) = \overline{B\left(\tilde{\xi}, \tilde{\eta}, r\right)}$. Also from Theorem (4.6), $\overline{B\left(\tilde{\xi}, \tilde{\eta}, r\right)} = \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$.
Hence, $\Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) = \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$.
Conversely,
Assume $\Gamma^{r(\mathcal{I})}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) = \mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k}$, then using Theorem (4.4) and (4.5)(ii),

$$\bigcap_{\tilde{\xi} \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho, \tilde{\eta}, r)} = \bigcup_{\tilde{\xi} \in \Gamma^{I}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho, \tilde{\eta}, r)}$$

This is possible only if either $\Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k) = \emptyset$ or $\Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$ singlton set. Then, $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} = \bigcap_{\rho \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)} \overline{B(\rho, \tilde{\eta}, r)} = \overline{B\left(\tilde{\xi}, \tilde{\eta}, r\right)}$ for some $\tilde{\xi} \in \Gamma^{\mathcal{I}}_{st(\varsigma,\upsilon,\tau)}(\widetilde{w}_k)$. Also, by Theorem (4.4), $\mathcal{I} - st_{(\varsigma,\upsilon,\tau)} - LIM^r_{\widetilde{w}_k} = \tilde{\xi}$. □

Conclusions

The idea of rough ideal statistical convergence using Copson transformation is more generalized convergence in summability theory. For this type of convergence various properties like statistical boundness, algebraic properties, closedness, convexity and relations of rough $\mathcal{I}$-statistical limit points, rough $\mathcal{I}$-statistical cluster points have been obtained. Our work gives new insights and techniques for the analysis of sequences in neutrosophic normed spaces by extending these classical principles to rough ideal statistical convergence.

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