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PURE FRACTIONAL OPTIMAL CONTROL OF PARTIAL DIFFERENTIAL EQUATIONS: NONLINEAR, DELAY AND TWO-DIMENSIONAL PDES

IMAN MALMIR

ABSTRACT. Novel methods for solving the optimal control problems of different and new types of the fractional partial differential equations (PDEs) as: nonlinear PDEs, delay PDEs, and two-dimensional PDEs, are introduced in this paper. These problems are formulated with the constant Riemann–Liouville performance indices together with tracking, constant Riemann–Liouville performance indices in which there are additional terms for the tracking optimal control of PDEs. These pure fractional optimal control problems are transformed into quadratic programing ones and there is no need to derive any optimality conditions. Some challenging optimal control problems of PDEs that have applications in real-world systems are investigated. As an unprecedented constraint, we introduce Riemann–Liouville, two-dimensional isoperimetric constraint in the PDE optimal control problem.

1. INTRODUCTION

A partial differential equation (PDE) is an equation involving one or more partial derivatives of an unknown function that depends on two or more variables, often time t and one or several variables in space [1]. PDEs like ordinary differential equations are classified as linear or nonlinear [2]. PDEs serve as models for real-world systems and there many applications of PDEs in mechanics and science, for example, see [1–6]. In this work, by providing new theoretical results, we are going to develop the idea and results of the previous works [7,8] for the pure fractional PDE optimal control problems of three different types as nonlinear, delayed or stretched and two-dimensional PDEs. In [7], we proposed a QP method for the optimal control of PDEs defined with fractional linear PDEs and the integer performance indices in the ordinary fractional sense, not in pure fractional sense. The concepts 'pure fractional' and 'Riemann–Liouville isoperimetric constraints' were introduced in [8]. The QP is an invaluable tool that simplifies the solutions of the optimal control problems. We can see in the work that the QP method has the ability to solve the optimal control problems with complicated constraints, which cannot be solved by many of the existing methods, even problems with fractional constraints, for instance, see Example 5 in [8]. In real-world optimal control problems, we often deal with constraints, for example, see [9]. In most of control system engineering the traditional methods are not applicable and the complexity of optimization problems increases exponentially [10].

We will study nonlinear PDEs, delay PDEs and two-dimensional PDEs with a new fractional performance index. Thus, compared with previous work, there are Riemann–Liouville integral orders in the performance index of the problems called the constant Riemann–Liouville quadratic performance index. We

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AEROSPACE AND MECHANICAL ENGINEERING GROUP, RONIN INSTITUTE, MONTCLAIR, USA
E-mail address: iman.malmir@outlook.com.
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derive the continuous models of the original problems by using parameterizing methods. A system may be either delayed from its descriptions or from an induction within the control loop. The delayed systems are more complex than non-delay systems [11,12]. We use the wavelets for the delayed PDEs. One can see that the wavelets theory has many application in practical problems. For example, the wavelet algorithm can improve the accuracy of multi-scale modeling in power electronic system and reduce the simulation time [13]. Research in tracking optimal control of PDEs is of vital importance. Hence, we formulate the problems with tracking criteria for possible tracking purposes.

The new PDE optimization problems are defined as follows.

Problem 1: three-dimensional optimal control of fractional two-dimensional PDEs. In this work, we are going to investigate the three-dimensional optimal control of PDEs with the constant Riemann–Liouville quadratic performance index or cost function given by

$$\begin{aligned} J &= \tfrac{1}{2}a_0x^2(z_f,y_f,t_f) + \tfrac{1}{2}\,{}^{RL}_{\ 0}I^{\bar\alpha_t}_{t_f}\,{}^{RL}_{\ 0}I^{\bar\alpha_y}_{y_f}\,{}^{RL}_{\ 0}I^{\bar\alpha_z}_{z_f}\left[a_1(z)a_2(y)a_3(t)\left\{x^2(z,y,t)+u^2(z,y,t)\right\}\right] \\ &\quad + \tfrac{1}{2}a_4\,{}^{RL}_{\ 0}I^{\bar\alpha_y}_{y_f}\,{}^{RL}_{\ 0}I^{\bar\alpha_z}_{z_f}\left[x(z,y,t_f)-x_d(z,y)\right]^2 \\ &= \tfrac{1}{2}a_0x^2(z_f,y_f,t_f) + \tfrac{1}{2}\frac{1}{\Gamma(\bar\alpha_z)}\frac{1}{\Gamma(\bar\alpha_y)}\frac{1}{\Gamma(\bar\alpha_t)}\int_0^{t_f}(t_f-t)^{\bar\alpha_t-1}\int_0^{y_f}(y_f-y)^{\bar\alpha_y-1}\int_0^{z_f}(z_f-z)^{\bar\alpha_z-1} \\ &\quad a_1(z)a_2(y)a_3(t)\left\{x^2(z,y,t)+u^2(z,y,t)\right\}dz\,dy\,dt \\ &\quad + \tfrac{1}{2}\frac{1}{\Gamma(\bar\alpha_z)}\frac{1}{\Gamma(\bar\alpha_y)}\int_0^{y_f}(y_f-y)^{\bar\alpha_y-1}\int_0^{z_f}(z_f-z)^{\bar\alpha_z-1}a_4\left(x(z,y,t_f)-x_d(z,y)\right)^2dz\,dy, \end{aligned} \tag{1.1}$$

where $a_0 \geq 0$, $\Gamma(\cdot)$ is the gamma function, $\bar\alpha_{t,y,z}$ are the orders of the Riemann–Liouville performance index, a_1, a_2, and a_3 are known continuous functions, a_4 is the weighting parameter, which is used in the tracking systems to adjust the error and $a_4 \geq 0$, y, z, and t are three independent variables as the spatial and temporal variables, y_f, z_f, and t_f are fixed, $x(z,y,t)$ and $u(z,y,t)$ are the state and the control of the systems and $x_d(z,y)$ is a separable, continuous function of y and z. The performance index J consists of the terminal cost term, the total energy of the system and the squared terminal error of the system. In the third term of J, we are interested in minimizing the error between the desired state $x_d(z,y)$ and the actual state $x(z,y,t_f)$ at the final (terminal) time t_f as the target state, that it, we want the system tracks the desired two-dimensional reference at t_f. The state equation of the fractional partial differential equations based on the Caputo sense is given in a general form as

$$\frac{\partial^{\alpha_t}x(z,y,t)}{\partial t^{\alpha_t}} = b(z)\frac{\partial^{\alpha_z}x(z,y,t)}{\partial z^{\alpha_z}} + c(y)\frac{\partial^{\alpha_y}x(z,y,t)}{\partial y^{\alpha_y}} + d(z)e(y)f_t(t)u(z,y,t) + g_z(z)g_y(y)g_t(t), \tag{1.2}$$

where $\alpha_{t,y,z}$ are the Caputo derivative orders, $0 < \alpha_{t,y,z} \leq 4$, $b, c, d, e, f_t, g_z, g_y$, and g_t are known continuous functions of the given independent variables and the state variables are separable. In (1.1), we define

$$\bar\alpha_. = 1 + \alpha_. - \lceil \alpha_. \rceil.$$

Obviously, for $\bar\alpha_{t,y,z} = 1$, we have the convectional performance index as

$$\begin{aligned} J =& \tfrac{1}{2}a_0x^2(z_f,y_f,t_f) + \tfrac{1}{2}\int_0^{t_f}\int_0^{y_f}\int_0^{z_f} a_1(z)a_2(y)a_3(t)\left\{x^2(z,y,t)+u^2(z,y,t)\right\}dz\,dy\,dt \\ &+ \tfrac{1}{2}\int_0^{y_f}\int_0^{z_f} a_4\left(x(z,y,t_f)-x_d(z,y)\right)^2dz\,dy, \end{aligned} \tag{1.3}$$

thus, the quadratic performance index (1.3) is a special case of the Riemann–Liouville quadratic performance index (1.1). We may have a combination of the initial and terminal conditions in the problem, for example,

$$x(0,y,t) = h_y(y)h_t(t), \quad x(z,0,t) = i_z(z)i_t(t), \quad x(z,y,0) = j_z(z)j_y(y), \tag{1.4}$$

$$x(z_f, y, t) = h_y(y)h_t(t), \quad x(z, y_f, t) = i_z(z)i_t(t), \quad x(z, y, t_f) = j_z(z)j_y(y), \tag{1.5}$$

or

$$\begin{aligned} &\frac{\partial^\gamma x(z,y,t)}{\partial z^\gamma}|_{(0,y,t)} = h_y(y)h_t(t), \frac{\partial^\gamma x(z,y,t)}{\partial y^\gamma}|_{(z,0,t)} = i_z(z)i_t(t), \frac{\partial^\gamma x(z,y,t)}{\partial t^\gamma}|_{(z,y,0)} = j_z(z)j_y(y), \\ &\frac{\partial^\gamma x(z,y,t)}{\partial z^\gamma}|_{(z_f,y,t)} = h_y(y)h_t(t), \frac{\partial^\gamma x(z,y,t)}{\partial y^\gamma}|_{(z,y_f,t)} = i_z(z)i_t(t), \frac{\partial^\gamma x(z,y,t)}{\partial t^\gamma}|_{(z,y,t_f)} = j_z(z)j_y(y), \end{aligned} \tag{1.6}$$

where h_y, h_t, i_z, i_t, j_z and j_y are known real-valued continuous functions and γ is a fractional or an integer order derivative. The problem is finding the optimal control $u^*(z, y, t)$ and optimal state $x^*(z, y, t)$, which when applied to the two-dimensional PDE plant described by (1.2) with the indicated conditions like those in (1.4)–(1.6), give an optimal performance index J^* described by (1.1).

Problem 2: optimal control of fractional nonlinear PDEs. Also, we consider the optimal control of fractional nonlinear partial differential equations with a constant Riemann–Liouville quadratic performance index as

$$\begin{aligned} J(x, u, \bar{\alpha}_t, \bar{\alpha}_y) &= \tfrac{1}{2}a_0 x^2(y_f, t_f) + \tfrac{1}{2}\, {}^{RL}_{\ 0}I^{\bar{\alpha}_t}_{t_f}\, {}^{RL}_{\ 0}I^{\bar{\alpha}_y}_{y_f} \left[a_2(y)a_3(t)\left\{x^2(y,t) + u^2(y,t)\right\}\right] \\ &\quad + \tfrac{1}{2}a_4\, {}^{RL}_{\ 0}I^{\bar{\alpha}_y}_{y_f} \left[x(y, t_f) - x_d(y)\right]^2 \\ &= \tfrac{1}{2}a_0 x^2(y_f, t_f) + \tfrac{1}{2}\frac{1}{\Gamma(\bar{\alpha}_y)}\frac{1}{\Gamma(\bar{\alpha}_t)} \int_0^{t_f} (t_f - t)^{\bar{\alpha}_t - 1} \int_0^{y_f} (y_f - y)^{\bar{\alpha}_y - 1} a_2(y)a_3(t) \\ &\quad \times \left\{x^2(y,t) + u^2(y,t)\right\} dy\, dt \\ &\quad + \tfrac{1}{2}\frac{1}{\Gamma(\bar{\alpha}_y)} \int_0^{y_f} (y_f - y)^{\bar{\alpha}_y - 1} a_4 \left(x(y, t_f) - x_d(y)\right)^2 dy, \end{aligned} \tag{1.7}$$

where the previous statements hold, and with the state equation of the fractional nonlinear partial differential equation given in a general form as (based on the Caputo sense)

$$\frac{\partial^{\alpha_t} x(y,t)}{\partial t^{\alpha_t}} = \Im\big(\frac{\partial^{\alpha_y} x(y,t)}{\partial y^{\alpha_y}}, x(y,t), u(y,t), y, t\big), \tag{1.8}$$

where $\Im$ is continuous and satisfies local Lipschitz conditions and the state variables are separable.

We have a combination of the following initial and terminal conditions as

$$x(0, t) = h(t), \quad x(y, 0) = i(y), \tag{1.9}$$

$$x(y_f, t) = h(t), \quad x(y, t_f) = i(y), \tag{1.10}$$

or a combination of

$$\begin{aligned} &\frac{\partial^\gamma x(y,t)}{\partial y^\gamma}|_{(0,t)} = h(t), \quad \frac{\partial^\gamma x(y,t)}{\partial t^\gamma}|_{(y,0)} = i(y), \\ &\frac{\partial^\gamma x(y,t)}{\partial y^\gamma}|_{(y_f,t)} = h(t), \quad \frac{\partial^\gamma x(y,t)}{\partial t^\gamma}|_{(y,t_f)} = i(y), \end{aligned} \tag{1.11}$$

where γ if a fractional derivative or an integer order derivative, and h and i are known real-valued continuous functions. The problem is finding the optimal control $u^*(y, t)$ and optimal state $x^*(y, t)$, which when applied to the nonlinear plant described by (1.8) with the indicated conditions, give an optimal performance index J^* described by (1.7).

Problem 3: optimal control of fractional delay PDEs. Consider a fractional delay partial differential equation based on the Caputo sense as

$$\begin{aligned}\frac{\partial^{\alpha_t} x(y,t)}{\partial t^{\alpha_t}} =& b_1(y)b_2(t)\frac{\partial^{\alpha_y} x(y,t)}{\partial y^{\alpha_y}} + c(t)x(y,t-\tau) + d(t)x(y,t-\tau(t)) + e(t)x(y,\frac{1}{\lambda}t)) \\ &+ f_1(y)f_2(t)u(y,t) + g_1(y)g_2(t),\end{aligned} \tag{1.12}$$

where $0 < \alpha_{t,y} \leq 4$, $b_1, b_2, c, d, e, f_1, f_2, g_1$, and g_2 are known continuous functions of the given independent variable, τ is a constant delay, $\tau(t)$ is a time-varying or a piecewise delay, and $1/\lambda$ is a stretch. In addition, there exists the initial function or the history function in the delay PDE as

$$x(y,t) = \zeta(y)\theta(t), \quad t < 0, \quad 0 < y < y_f, \tag{1.13}$$

where $\zeta(y)$ and $\theta(t)$ are known, continuous functions. The history function is defined for a delayed system and it has obvious effects on the solutions. We may have a combination of the initial and terminal conditions given in (1.9)–(1.11) in the problem. The problem is finding the optimal control $u^*(y,t)$ and optimal state $x^*(y,t)$, which when applied to the delayed plant described by (1.12) with the indicated conditions, give an optimal performance index J^* described by (1.7).

In some cases of all given problems, we may face with constraints such as those presented in [7]. Here, we introduce a new type of isoperimetric constraints, which has applications in the optimal control theory [14].

Two-dimensional Riemann–Liouville isoperimetric constraint. Like the Riemann–Liouville isoperimetric constraint introduced in [8], as a new constraint, we define a two-dimensional fractional isoperimetric constraint for PDE optimization as

$${}_{0}^{RL}I_{t_f}^{\bar{\alpha}_{i,t}}\,{}_{0}^{RL}I_{y_f}^{\bar{\alpha}_{i,y}}[a_5(y)a_6(t)x(y,t) - a_7(y)a_8(t)u(y,t)] \leq c_i, \tag{1.14}$$

where $\bar{\alpha}_{i,t}, \bar{\alpha}_{i,y}$ are chosen similarly to $\bar{\alpha}_t$ and $\bar{\alpha}_y$, a_5, a_6, a_7, and a_8 are known continuous functions and c_i is a constant; we name the new constraint as two-dimensional Riemann–Liouville isoperimetric constraint.

One knows the fact that such problems do not have exact solutions or there is no method to find the exact solutions. PDE control problems are complex enough for domains of one dimension, but many physical PDE problems exist which evolve in two and three dimensions [15]. The behavior of many dynamical systems depends upon their past histories and they can be induced by the presence of time delays [16] in their state equations. In traditional methods for finding the optimal control of PDE, we must derive the optimality condition for the optimization problem [17]. In [18], the PDE optimal control problem was discretized by the method of lines and transformed into a nonlinear programming problem, where the resulting system of ordinary differential algebraic equations was solved by a standard integrative routine. To get an optimal solution in an iterative way, a sequential quadratic programming method was used. It is shown in [19] that that the finite difference method applied to PDEs enables us to obtain Roesser discrete state-space models. In [20], the turnpike phenomenon for optimal control problems of ODEs and PDEs was discussed. In [21], the optimal control and the parameter identification of systems governed by PDEs with random input data were presented. The solution methods for partial differential equations with time delay have been investigated in some work, for example, [22–24]. In [25], a general formulation and numerical scheme for the fractional optimal control problem of distributed systems in spherical and cylindrical coordinates was presented. [26] presented a numerical scheme for optimal control problem governed by time fractional diffusion equation based on a Legendre pseudo-spectral method for space discretization and finite difference method for time discretization. In [27], optimal control of a stochastic delay partial differential equation by means of the associated backward stochastic differential equations. A new technique for computing the optimal control of delay-differential-algebraic dynamic systems was introduced in [28]. An iterative proper orthogonal decomposition method by using the finite element and the backward Euler methods for a parabolic optimal control problem is investigated in [29]. Some of the engineering models are described with two-dimensional PDEs [30]. Ref. [31] studied the approximation of optimally controlled PDEs for inverse problems in optimal design. In this reference, solutions to various applications in optimal material design were presented. Some interested optimal control problems of PDEs were presented in [32]. Using the traditional method, one must derive the optimality conditions, for example, see [17,33]. In [18], an approach was presented to compute optimal control functions in dynamic models based on one-dimensional partial differential algebraic equations. Ref. [34] discussed the regional tracking problem of the bilinear wave equation with bounded controls acting on the velocity term of the system. Ref. [35] by defining continuous 2D models of discrete systems, presented the optimality conditions and feedback representation.

2. Preliminaries

First, the definitions of the fractional operators are given. Then, the concepts of the wavelets are presented.

Definition 2.1. The Riemann–Liouville integral of order α for a function $f(t)$ “ ${}^{RL}_{0}I^{\alpha}_{t}f(t)$” is defined by

$$ {}^{RL}_{0}I^{\alpha}_{t}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\rho)^{\alpha-1} f(\rho)\, d\rho, \tag{2.1}$$

where, as usual, $\Gamma(\alpha)$ is the gamma function [36].

Definition 2.2. The constant Riemann–Liouville integral of order α for a function $f(t)$ “ ${}^{RL}_{0}I^{\alpha}_{t_f}f(t)$” is defined by

$$ {}^{RL}_{0}I^{\alpha}_{t_f}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^{t_f} (t_f-t)^{\alpha-1} f(t)\, dt, \tag{2.2}$$

where t_f is a finite constant and $t_f > 0$ [8]. We can see that the constant Riemann–Liouville α-integral for a function $f(t)$ is a constant value. We use this definition for defining the new fractional Riemann–Liouville performance indices and isoperimetric constraints, for example, see [8,37].

Definition 2.3. The Caputo fractional derivative of order α for a function $f(t)$ “ ${}^{C}_{0}D^{\alpha}_{t}f(t)$” is defined by

$$ {}^{C}_{0}D^{\alpha}_{t}f(t) = \begin{cases} \frac{1}{\Gamma(k-\alpha)}\int_0^t (t-\rho)^{k-\alpha-1} f^{(k)}(\rho)\, d\rho, & k-1<\alpha<k \\ \frac{d^k}{dt^k}f(t), & \alpha = k, \end{cases} \tag{2.3}$$

where $k \in \mathbb{N}$ [36]. Here, the subscript “t” denotes the independent variable. For a three-dimensional function as $x(z,y,t)$, we have

$$ \frac{\partial^\alpha x(z,y,t)}{\partial t^\alpha} = {}^{C}_{0}D^{\alpha}_{t}x(z,y,t). $$

We usually apply (2.1) to the state equations defined with (2.3) see [37–39], but we need (2.2) to define the new, fractional, Riemann–Liouville performance index and Riemann–Liouville isoperimetric constraint.

Definition 2.4. Legendre and Chebyshev wavelets

Legendre wavelets ϕ_{nm} are defined on $[0,1]$ as [40,42]

$$ \phi_{nm}(t) = \begin{cases} \sqrt{N}c_m P_m(2Nt-2n+1), & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right] \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

where P_m are the well-known Legendre polynomials, a finite value $N \in \mathbb{N}_{\geq 2}$ is an arbitrarily selected scaling parameter and specifies the number of subintervals, $n = 1, 2, \ldots, N$ refers to the number of subinterval and specifies the location of the subinterval, $m = 0, 1, \ldots$ is the degree of P_m, $c_m = \sqrt{2m+1}$, and $t \in [0,1]$

is as an independent variable. In (2.4), $\bar{\phi}_{nm}(t) := \sqrt{N}c_m P_m(2Nt-2n+1)$ is Legendre scaling function. Legendre wavelets form an orthogonal basis with respect to the weight functions $\varsigma_{nlw}(t)=1$.

Chebyshev wavelets ψ_{nm} are defined on $[0,1]$ as [41,42]

$$\psi_{nm}(t) = \begin{cases} \sqrt{2N}\mathrm{c}_m T_m(2Nt-2n+1), & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right] \\ 0, & \text{otherwise,} \end{cases} \tag{2.5}$$

where T_m are the well-known Chebyshev polynomials, N, n, and m are the same as before and $\mathrm{c}_0 = 1/\sqrt{\pi}$, $\mathrm{c}_{m\neq 0} = \sqrt{2}/\sqrt{\pi}$. In (2.5), $\bar{\psi}_{nm}(t) := \sqrt{2N}\mathrm{c}_m T_m(2Nt-2n+1)$ is Chebyshev scaling function. Chebyshev wavelets form an orthogonal basis with respect to $\varsigma_{ncw}(t) = 1/\sqrt{1-(2Nt-2n+1)^2}$.

Definition 2.5. Wavelet expansions

As we all know, we can expand a function $f(t)$ in a series of Legendre or Chebyshev polynomials denoted by $\{\varphi_m(t)\}$ as $f(t) = \sum_{m=0}^{\infty} f_m\varphi_m(t)$. If we use the dilation and scaling properties of the wavelets, we can expand a function $f(t)$ in a series of Legendre or Chebyshev wavelets denoted by $\{w^{\xi}_{nm}(t)\}$ as

$$f(t) = \sum_{n=1}^{N}\sum_{m=0}^{\infty} f_{nm}w_{nm}(t),$$

where N is a (large enough) finite value, $N \ll \infty$. By truncating the power series, say, the Mth term in N subintervals, we have

$$f(t) \cong \sum_{n=1}^{N}\sum_{m=0}^{M-1} f_{nm}w_{nm}(t). \tag{2.6}$$

The constant coefficients of the scaling functions $\{\bar{w}_{nm}(t)\}$ can be obtained from

$$f_{nm} = \int_{\frac{n-1}{N}}^{\frac{n}{N}} f(t)\, w_{nm}(t)\varsigma_{nw}(t)\, dt. \tag{2.7}$$

In (2.6), the application of the first summation differs from that of the second one, the first summation indicates the expansion is piecewise-defined function of N sub-functions. By

$$\begin{aligned} &\mathbf{f}_{\mathrm{w}} := [f_{10}, \dots, f_{1M-1}, f_{20}, \dots, f_{2M-1}, \dots, f_{N0}, \dots, f_{NM-1}], \\ &\mathfrak{w}(t) := [w_{10}(t), w_{11}(t), \dots, w_{1M-1}(t), \dots, w_{N0}(t), \dots, w_{NM-1}(t)]^{\top}, \end{aligned}$$

where $\mathbf{f}_{\mathrm{w}}$ is a $1 \times NM$ vector consists of constants, $\mathfrak{w}(t)$ as a vector consisting of any of the wavelets is a $NM \times 1$ vector, we are able to write (2.6) in a simple form as

$$f(t) = \mathbf{f}_{\mathrm{w}}\mathfrak{w}(t). \tag{2.8}$$

We use the subscript "w" throughout this text to refer to both wavelets.

Remark 2.6. In using (2.4) and (2.5) or any wavelet for a series expansion, we must consider several important points as follows:

- $N \in \mathbb{N}_{\geq 2}$ is a finite value, $N \ll \infty$. The scaling functions are defined on $[(n-1)/N, n/N]$ and if $N \to \infty$, the lenght of subinervald tends to zero. Also, we cannot use infinite value due to presence of N in the coefficients and arguments of the wavelets, that is, in $\sqrt{N}$ and $(2Nt-2n+1)$. We cannot choose $N = 1$ in the wavelets concepts, because we revert to polynomials concepts and some of the wavelets properties are eliminated.
- For different values of N like N_1 and N_2, we have different scaling functions, different subintervals, different definitions, different constant coefficients and different weight functions defined on different subintervals.

- The application of the first summation differs from the second summation. The first summation indicates this the obtained function is a piecewise-defined function. For example, by expanding the first summation, we have

$$f(t)=\sum_{n=1}^{N}\sum_{m=0}^{M-1} f_{nm}w_{nm}(t)=\begin{cases}\sum_{m=0}^{M-1} f_{1m}w_{1m}(t), & \text{when} \quad 0\le t\le \frac{1}{N}\\ \vdots & \vdots\\ \sum_{m=0}^{M-1} f_{nm}w_{nm}(t), & \text{when} \quad \frac{n-1}{N}\le t\le \frac{n}{N}\\ \vdots & \vdots\\ \sum_{m=0}^{M-1} f_{Nm}w_{Nm}(t), & \text{when} \quad \frac{N-1}{N}\le t\le 1\end{cases},$$

while by expanding the second summation we have

$$f(t)=\sum_{n=1}^{N}\sum_{m=0}^{M-1} f_{nm}w_{nm}(t)=\sum_{n=1}^{N}(f_{n0}w_{n0}(t)+f_{n1}w_{n1}(t)+\ldots+f_{n,M-1}w_{n,M-1}(t)).$$

Thus we cannot treat the first summation $\sum_n$ as the second summation $\sum_m$. Again, we find the fact that N is a finite value, $N\ll\infty$.

- One should note that for $N_1\neq N_2$, the expression

$$\sum_{n=1}^{N_1}\sum_{m=0}^{\infty} f_{nm}w_{nm}(t)-\sum_{n=1}^{N_2}\sum_{m=0}^{M-1} f_{nm}w_{nm}(t)$$

cannot be simplified, because for different N, we have different coefficients and different scaling functions defined on different subintervals.

- Although the wavelets are constructed from the polynomials, but there are many differences between the wavelets and the polynomials. For example, they have different properties on the interval $[0,t_f]$. For the same M, the wavelets are more accurate than the polynomials.

2.1. **Properties of Legendre and Chebyshev wavelets.**

Theorem 2.7. *The Caputo fractional derivative operational matrix of these wavelets $\mathfrak{D}_{\mathrm{w}}^{\alpha}$ simplifies the derivative operation as*

$$ {}_0^C D_t^{\alpha}\mathfrak{w}(t)\cong\mathfrak{D}_{\mathrm{w}}^{\alpha}\mathfrak{w}(t). \tag{2.9}$$

The Caputo derivative operational matrix of Legendre wavelets $\mathfrak{D}_{\mathrm{lw}}^{\alpha}$ for $0<\alpha\le 1$ as

$$ {}_0^C D_t^{\alpha}\boldsymbol{\phi}(t)\cong\mathfrak{D}_{\mathrm{lw}}^{\alpha}\boldsymbol{\phi}(t), \tag{2.10}$$

is obtained from

$$\mathfrak{D}_{\mathrm{lw}}^{\alpha}=(2N)^{\alpha}\begin{bmatrix}\mathbf{S}^{\alpha} & \mathbf{V}_1^{\alpha} & \mathbf{V}_2^{\alpha} & \mathbf{V}_3 & \mathbf{V}_4^{\alpha} & \cdots & \mathbf{V}_{N-2}^{\alpha} & \mathbf{V}_{N-1}^{\alpha}\\ \mathbf{0} & \mathbf{S}^{\alpha} & \mathbf{V}_1^{\alpha} & \mathbf{V}_2^{\alpha} & \mathbf{V}_3 & \cdots & \mathbf{V}_{N-3}^{\alpha} & \mathbf{V}_{N-2}^{\alpha}\\ \mathbf{0} & \mathbf{0} & \mathbf{S}^{\alpha} & \mathbf{V}_1^{\alpha} & \mathbf{V}_2^{\alpha} & \cdots & \mathbf{V}_{N-4}^{\alpha} & \mathbf{V}_{N-3}^{\alpha}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{S}^{\alpha} & \mathbf{V}_1^{\alpha}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{S}^{\alpha}\end{bmatrix}, \tag{2.11}$$

where

$$\mathbf{S}^{\alpha}=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0\\ s_0^{\alpha 1} & s_1^{\alpha 1} & s_2^{\alpha 1} & \cdots & s_{M-1}^{\alpha 1}\\ s_0^{\alpha 2} & s_1^{\alpha 2} & s_2^{\alpha 2} & \cdots & s_{M-1}^{\alpha 2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ s_0^{\alpha M-1} & s_1^{\alpha M-1} & s_2^{\alpha M-1} & \cdots & s_{M-1}^{\alpha M-1}\end{bmatrix},\ \mathbf{V}_{\eta}^{\alpha}=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0\\ v_{\eta 0}^{\alpha 1} & v_{\eta 1}^{\alpha 1} & v_{\eta 2}^{\alpha 1} & \cdots & v_{\eta M-1}^{\alpha 1}\\ v_{\eta 0}^{\alpha 2} & v_{\eta 1}^{\alpha 2} & v_{\eta 2}^{\alpha 2} & \cdots & v_{\eta M-1}^{\alpha 2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ v_{\eta 0}^{\alpha M-1} & v_{\eta}^{\alpha M-1} & v_{\eta 2}^{\alpha M-1} & \cdots & v_{\eta M-1}^{\alpha M-1}\end{bmatrix}, \tag{2.12}$$

$$s_i^{\alpha m}=(-1)^m\tfrac{c_m c_i}{2}\int_{-1}^{1}\sum_{j=1}^{m}\frac{(m+j)!\Gamma(j+1)}{(-2)^j(m-j)!(j!)^2\Gamma(j-\alpha+1)}(\varrho+1)^{j-\alpha}P_i(\varrho)\,d\varrho, \tag{2.13}$$

$$\begin{aligned} v_{\eta i}^{\alpha m}=(-1)^m\tfrac{c_m c_i}{2}\int_{-1}^{1}\sum_{j=1}^{m}\frac{(m+j)!\Gamma(j+1)}{(-2)^j(m-j)!(j!)^2\Gamma(j-\alpha+1)}&\{(\varrho+2\eta+1)^{j-\alpha}\\ &-(-1)^{m-j}(\varrho+2\eta-1)^{j-\alpha}\}P_i(\varrho)\,d\varrho.\end{aligned} \tag{2.14}$$

Also, the Caputo derivative operational matrix of Chebyshev wavelets $\mathfrak{D}_{\mathrm{cw}}^{\alpha}$ *for* $0<\alpha\le 1$ *as*

$${}_0^C D_t^{\alpha}\boldsymbol{\psi}(t)\cong\mathfrak{D}_{\mathrm{cw}}^{\alpha}\boldsymbol{\psi}(t), \tag{2.15}$$

is obtained from

$$\mathfrak{D}_{\mathrm{cw}}^{\alpha}=(2N)^{\alpha}\begin{bmatrix} \mathbf{S}^{\alpha} & \mathbf{V}_1^{\alpha} & \mathbf{V}_2^{\alpha} & \mathbf{V}_3 & \mathbf{V}_4^{\alpha} & \cdots & \mathbf{V}_{N-2}^{\alpha} & \mathbf{V}_{N-1}^{\alpha}\\ \mathbf{0} & \mathbf{S}^{\alpha} & \mathbf{V}_1^{\alpha} & \mathbf{V}_2^{\alpha} & \mathbf{V}_3 & \cdots & \mathbf{V}_{N-3}^{\alpha} & \mathbf{V}_{N-2}^{\alpha}\\ \mathbf{0} & \mathbf{0} & \mathbf{S}^{\alpha} & \mathbf{V}_1^{\alpha} & \mathbf{V}_2^{\alpha} & \cdots & \mathbf{V}_{N-4}^{\alpha} & \mathbf{V}_{N-3}^{\alpha}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{S}^{\alpha} & \mathbf{V}_1^{\alpha}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{S}^{\alpha}\end{bmatrix}, \tag{2.16}$$

where

$$\mathbf{S}^{\alpha}=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0\\ s_0^{\alpha 1} & s_1^{\alpha 1} & s_2^{\alpha 1} & \cdots & s_{M-1}^{\alpha 1}\\ s_0^{\alpha 2} & s_1^{\alpha 2} & s_2^{\alpha 2} & \cdots & s_{M-1}^{\alpha 2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ s_0^{\alpha M-1} & s_1^{\alpha M-1} & s_2^{\alpha M-1} & \cdots & s_{M-1}^{\alpha M-1}\end{bmatrix},\ \mathbf{V}_{\eta}^{\alpha}=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0\\ v_{\eta 0}^{\alpha 1} & v_{\eta 1}^{\alpha 1} & v_{\eta 2}^{\alpha 1} & \cdots & v_{\eta M-1}^{\alpha 1}\\ v_{\eta 0}^{\alpha 2} & v_{\eta 1}^{\alpha 2} & v_{\eta 2}^{\alpha 2} & \cdots & v_{\eta M-1}^{\alpha 2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ v_{\eta 0}^{\alpha M-1} & v_{\eta}^{\alpha M-1} & v_{\eta 2}^{\alpha M-1} & \cdots & v_{\eta M-1}^{\alpha M-1}\end{bmatrix}, \tag{2.17}$$

$$s_i^{\alpha m}=(-1)^m m\mathrm{c}_m\mathrm{c}_i\int_0^{\pi}\sum_{j=1}^{m}\frac{(-2)^j(m+j-1)!\Gamma(j+1)}{(m-j)!(2j)!\Gamma(j-\alpha+1)}(\cos\theta+1)^{j-\alpha}\ \cos(i\theta)\,d\theta, \tag{2.18}$$

$$\begin{aligned} v_{\eta i}^{\alpha m}=(-1)^m m\mathrm{c}_m\mathrm{c}_i\int_0^{\pi}\sum_{j=1}^{m}\frac{(-2)^j(m+j-1)!\Gamma(j+1)}{(m-j)!(2j)!\Gamma(j-\alpha+1)}&\{(\cos\theta+2\eta+1)^{j-\alpha}\\ &-(-1)^{m-j}(\cos\theta+2\eta-1)^{j-\alpha}\}\ \cos(i\theta)\,d\theta.\end{aligned} \tag{2.19}$$

Proof. By setting $N=\xi^{k-1}$ in the derived results of [43], we reach the given results in (2.10)–(2.19). □

2.2. Constant Riemann–Liouville integration operational matrix of wavelets.

Theorem 2.8. *The constant Riemann–Liouville fractional integral matrices of product of two Legendre and Chebyshev wavelets vectors are obtained directly from these wavelets as*

$${}_0^{RL}I_1^{\alpha_t}\{\boldsymbol{\mathfrak{w}}(t)\boldsymbol{\mathfrak{w}}^{\top}(t)\}=\boldsymbol{\Gamma}_{\mathrm{w}}^{\alpha}, \tag{2.20}$$

where

$$\boldsymbol{\Gamma}_{\mathrm{lw}}^{\alpha}=\mathrm{blkdiag}\Big(\breve{\boldsymbol{\Gamma}}_1^{\alpha},\breve{\boldsymbol{\Gamma}}_2^{\alpha},\breve{\boldsymbol{\Gamma}}_3^{\alpha},\ldots,\breve{\boldsymbol{\Gamma}}_N^{\alpha}\Big),$$

$$\mathbf{\Gamma}^{\alpha}_{\mathrm{cw}} = \mathrm{blkdiag}\Big(\breve{\mathbf{\Gamma}}^{\alpha}_1, \breve{\mathbf{\Gamma}}^{\alpha}_2, \breve{\mathbf{\Gamma}}^{\alpha}_3, \ldots, \breve{\mathbf{\Gamma}}^{\alpha}_N\Big),$$

in which for Legendre wavelets

$$\breve{\mathbf{\Gamma}}^{\alpha}_n = [\gamma^n_{ij}], i, j = 1, 2, \ldots, M, \gamma^n_{ij} = \tfrac{c_{i-1}c_{j-1}}{2}\frac{1}{\Gamma(\alpha)}\int_{-1}^{1}(\frac{2N - z - 2n + 1}{2N})^{\alpha-1}P_{i-1}(z)P_{j-1}(z)\,dz$$

and for Chebyshev wavelets

$$\breve{\mathbf{\Gamma}}^{\alpha}_n = [\gamma^n_{ij}], i, j = 1, 2, \ldots, M,$$
$$\gamma^n_{ij} = \mathrm{c}_{i-1}\mathrm{c}_{j-1}\frac{1}{\Gamma(\alpha)}\int_0^{\pi}(\frac{2N - \cos(\theta) - 2n + 1}{2N})^{\alpha-1}\cos((i-1)\theta)\cos((j-1)\theta)\sin\theta\,d\theta.$$

Proof. The proof is given in [37]. □

We also have the following operational properties of these wavelets:

$$\mathbf{f}_{\mathrm{w}}\boldsymbol{\mathfrak{w}}(t)\boldsymbol{\mathfrak{w}}^{\top}(t) \cong \boldsymbol{\mathfrak{w}}^{\top}(t)\tilde{\mathbf{f}}_{\mathrm{w}}, \tag{2.21}$$

$$\boldsymbol{\mathfrak{w}}(\frac{1}{\lambda}t) = \mathbf{S}_{\mathrm{w}}\boldsymbol{\mathfrak{w}}(t), \tag{2.22}$$

$$\boldsymbol{\mathfrak{w}}(t - \tau_\iota) = \begin{cases} \mathbf{0}, & 0 \le t < \tau_\iota \\ \mathbf{D}_\iota\boldsymbol{\mathfrak{w}}(t), & \tau_\iota \le t \le 1, \end{cases} \tag{2.23}$$

$$\boldsymbol{\mathfrak{w}}(t - \tau(t)) = \begin{cases} \mathbf{0}, & 0 \le t < \tau(t) \\ \mathbf{D}^t\boldsymbol{\mathfrak{w}}(t), & \tau(t) \le t \le 1, \end{cases} \tag{2.24}$$

where $\tilde{\mathbf{f}}_{\mathrm{w}}$ is the product operational matrix of the desired wavelets for $\mathbf{f}_{\mathrm{w}}$, $\mathbf{S}_{\mathrm{w}}$ is the stretch operational matrix for a stretch $1/\lambda$, $\mathbf{D}_\iota$ is the delay operational matrix for a time-delay τ_ι defined for $n_{d_\iota} = \tau_\iota N$ as

$$\mathbf{D}_\iota = \left[\begin{array}{cc} \mathbf{0}_{(N-n_{d_\iota})M \times n_{d_\iota}M} & \mathbf{I}_{(N-n_{d_\iota})M} \\ \hdashline \multicolumn{2}{c}{\mathbf{0}_{n_{d_\iota}M \times MN}} \end{array}\right],$$

and $\mathbf{D}^t$ is the piecewise delay operational matrix for a piecewise delay $\tau(t)$. Since $\mathbf{D}_\iota$ and $\mathbf{D}^t$ have the same structure for both wavelets, we do not use the subscript "w" for them. For more details, see [38,40,41] and set $N = \xi^{k-1}$ in their formulas. We use the general form for the important properties (2.15) and (2.10) as (2.9). One can find the fact that these operational matrices simplify the operational processes.

In the next sections, we will model the given problems. It should be noted that one must rescale the problem before modeling it by the given procedures. Since we use polynomials concepts for modeling the first and second problems, we will not discuss on the initial and final conditions; the reader is referred to see the procedures given in [7] for more details. In the third problem, we use the wavelets concepts that have many differences with the polynomials concepts and we model the conditions with the wavelets.

3. Modeling process for optimization of two-dimensional PDEs

From [7], for the shifted Legendre polynomials or shifted Chebyshev polynomials denoted by $\{\varphi_m(t)\}$, we have

$$f(t) = \sum_{m=0}^{M-1} f^{\varphi}_m\varphi_m(t)$$
$$= \mathbf{f}_{\varphi}\boldsymbol{\varphi}(t), \tag{3.1}$$

$$ {}_0^C D_t^{\alpha} \boldsymbol{\varphi}(t) \cong \mathfrak{D}_{\varphi}^{\alpha} \boldsymbol{\varphi}(t), \tag{3.2}$$

$$ \mathbf{f}_{\varphi} \boldsymbol{\varphi}(t) \boldsymbol{\varphi}^{\top}(t) \cong \boldsymbol{\varphi}^{\top}(t) \tilde{\mathbf{f}}_{\varphi} \tag{3.3}$$

In addition to the concepts given in [7], we need the following property.

Theorem 3.1. *The constant Riemann–Liouville integration operational matrix of the product of these shifted polynomials simplifies the Riemann–Liouville α-integral operation as*

$$ {}_0^{RL} I_1^{\alpha} \{ \boldsymbol{\varphi}(t) \boldsymbol{\varphi}^{\top}(t) \} = {}_0^1 \boldsymbol{\Gamma}_{\varphi}^{\alpha}, \tag{3.4}$$

where ${}_0^{t_f} \boldsymbol{\Gamma}_{\varphi}^{\alpha}$ is the $M \times M$, exact, left constant Riemann–Liouville integration operational matrix of the product of any of these polynomials and for $m, m' = 0, 1, \dots, M-1$,

$$ {}_0^1 \boldsymbol{\Gamma}_{SC}^{\alpha} = \frac{2}{\pi} \left[\gamma_{m+1,m'+1}^{SC} \right], \gamma_{m+1,m'+1}^{SC} = \begin{cases} \frac{1}{\Gamma(1+\alpha)}, & m = m' = 0 \\ \sqrt{2} \sum_{j=0}^{m'} a_{SC}(m', j) \frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}, & m = 0 \\ \gamma_{1,n}, n = 1, \dots, M, & m' = 0 \\ 2 \sum_{i=0}^{m} a_{SC}(m, i) \sum_{j=0}^{m'} a_{SC}(m', j) \frac{\Gamma(i+j+1)}{\Gamma(i+j+\alpha+1)}, & m, m' > 0, \end{cases}$$

$$ {}_0^1 \boldsymbol{\Gamma}_{SL}^{\alpha} = \left[\gamma_{m+1,m'+1}^{SL} \right], \gamma_{m+1,m'+1}^{SL} = \begin{cases} \frac{1}{\Gamma(1+\alpha)}, & m = m' = 0 \\ c_{m'} \sum_{j=0}^{m'} a_{SL}(m', j) \frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}, & m = 0 \\ \gamma_{1,n}, n = 1, \dots, M, & m' = 0 \\ c_m c_{m'} \sum_{i=0}^{m} a_{SL}(m, i) \sum_{j=0}^{m'} a_{SL}(m', j) \frac{\Gamma(i+j+1)}{\Gamma(i+j+\alpha+1)}, & m, m' > 0, \end{cases}$$

in which $a_{SC}(m, i) = (-1)^m a_m \sum_{i=0}^{m} \frac{(-2)^i (a_m + i - 1)!}{(m-i)!(2i)!} 2^i$ and $a_{SL}(m, i) = (-1)^m \sum_{i=0}^{m} \frac{(m+i)!}{(-2)^i (m-i)!(i!)^2} 2^i$.

Proof. The proof is given in [8]. □

Assume that $\boldsymbol{\varphi}(z)$ is of size $M_1 \times 1$, $\boldsymbol{\varphi}(y)$ is of size $M_2 \times 1$ and $\boldsymbol{\varphi}(t)$ is of size $M_3 \times 1$. Like the procedure given in [7], we express the state and the control of the system as products of separable functions of z, y and t such that

$$ x(z, y, t) = (\boldsymbol{\varphi}^{\top}(z) \otimes (\boldsymbol{\varphi}^{\top}(y) \otimes \boldsymbol{\varphi}^{\top}(t))) \mathbf{x}_{\varphi}, \tag{3.5}$$

$$ u(z, y, t) = (\boldsymbol{\varphi}^{\top}(z) \otimes (\boldsymbol{\varphi}^{\top}(y) \otimes \boldsymbol{\varphi}^{\top}(t))) \mathbf{u}_{\varphi}. \tag{3.6}$$

Using (3.5) and (3.6), applying (3.1) to $a_1(z)$, $a_2(y)$, $a_3(t)$ and $x_d(z, y) = \mu(z)\nu(y)$ (as a separable function), and using (3.3) and (3.4) we can write

$$\begin{aligned} J &= \tfrac{1}{2} a_0 x^2(z_f, y_f, t_f) + \tfrac{1}{2} {}_0^{RL} I_{t_f}^{\bar{\alpha}_t} {}_0^{RL} I_{y_f}^{\bar{\alpha}_y} {}_0^{RL} I_{z_f}^{\bar{\alpha}_z} \left[a_1(z) a_2(y) a_3(t) \left\{ x^2(z, y, t) + u^2(z, y, t) \right\} \right] \\ &\quad + \tfrac{1}{2} a_4 {}_0^{RL} I_{y_f}^{\bar{\alpha}_y} {}_0^{RL} I_{z_f}^{\bar{\alpha}_z} \left[x(z, y, t_f) - x_d(z, y) \right]^2 \\ &= \tfrac{1}{2} a_0 x^2(z_f, y_f, t_f) + \tfrac{1}{2} z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} \frac{1}{\Gamma(\bar{\alpha}_z)} \frac{1}{\Gamma(\bar{\alpha}_y)} \frac{1}{\Gamma(\bar{\alpha}_t)} \int_0^1 (1-t)^{\bar{\alpha}_t - 1} \int_0^1 (1-y)^{\bar{\alpha}_y - 1} \int_0^1 (1-z)^{\bar{\alpha}_z - 1} \\ &\quad \left\{ \mathbf{x}_{\varphi}^{\top} (\boldsymbol{\varphi}(z) \otimes (\boldsymbol{\varphi}(y) \otimes \boldsymbol{\varphi}(t))) (a_1(z) \boldsymbol{\varphi}^{\top}(z) \otimes (a_2(y) \boldsymbol{\varphi}^{\top}(y) \otimes a_3(t) \boldsymbol{\varphi}^{\top}(t))) \mathbf{x}_{\varphi} \right. \\ &\quad \left. + \mathbf{u}_{\varphi}^{\top} (\boldsymbol{\varphi}(z) \otimes (\boldsymbol{\varphi}(y) \otimes \boldsymbol{\varphi}(t))) (a_1(z) \boldsymbol{\varphi}^{\top}(z) \otimes (a_2(y) \boldsymbol{\varphi}^{\top}(y) \otimes a_3(t) \boldsymbol{\varphi}^{\top}(t))) \mathbf{u}_{\varphi} \right\} dz \, dy \, dt \\ &\quad + c + \tfrac{1}{2} a_4 z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} \frac{1}{\Gamma(\bar{\alpha}_z)} \frac{1}{\Gamma(\bar{\alpha}_y)} \int_0^1 (1-y)^{\bar{\alpha}_y - 1} \int_0^1 (1-z)^{\bar{\alpha}_z - 1} \\ &\quad \left\{ \mathbf{x}_{\varphi}^{\top} (\boldsymbol{\varphi}(z) \otimes (\boldsymbol{\varphi}(y) \otimes \boldsymbol{\varphi}(1))) (\boldsymbol{\varphi}^{\top}(z) \otimes (\boldsymbol{\varphi}^{\top}(y) \otimes \boldsymbol{\varphi}^{\top}(1))) \mathbf{x}_{\varphi} \right. \end{aligned}$$

$$
\begin{aligned}
&- 2(\boldsymbol{\mu}_\varphi \boldsymbol{\varphi}(z) \times \boldsymbol{\nu}_\varphi \boldsymbol{\varphi}(y))(\boldsymbol{\varphi}^\top(z) \otimes (\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi\big\}dz\,dy \\
= {}&\tfrac{1}{2}a_0 x^2(z_f, y_f, t_f) + \tfrac{1}{2} z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} \frac{1}{\Gamma(\bar{\alpha}_z)} \frac{1}{\Gamma(\bar{\alpha}_y)} \frac{1}{\Gamma(\bar{\alpha}_t)} \int_0^1 (1-t)^{\bar{\alpha}_t - 1} \int_0^1 (1-y)^{\bar{\alpha}_y - 1} \int_0^1 (1-z)^{\bar{\alpha}_z - 1} \\
&\big\{\mathbf{x}_\varphi^\top(\boldsymbol{\varphi}(z) \otimes (\boldsymbol{\varphi}(y) \otimes \boldsymbol{\varphi}(t)))(\boldsymbol{\varphi}(z)\tilde{\mathbf{a}}_{1\varphi} \otimes (\boldsymbol{\varphi}^\top(y)\tilde{\mathbf{a}}_{2\varphi} \otimes \boldsymbol{\varphi}^\top(t)\tilde{\mathbf{a}}_{3\varphi}))\mathbf{x}_\varphi \\
&+ \mathbf{u}_\varphi^\top(\boldsymbol{\varphi}(z) \otimes (\boldsymbol{\varphi}(y) \otimes \boldsymbol{\varphi}(t)))(\boldsymbol{\varphi}(z)\tilde{\mathbf{a}}_{1\varphi} \otimes (\boldsymbol{\varphi}^\top(y)\tilde{\mathbf{a}}_{2\varphi} \otimes \boldsymbol{\varphi}^\top(t)\tilde{\mathbf{a}}_{3\varphi}))\mathbf{u}_\varphi\big\}dz\,dy\,dt \\
&+ c + \tfrac{1}{2}a_4 z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} \frac{1}{\Gamma(\bar{\alpha}_z)} \frac{1}{\Gamma(\bar{\alpha}_y)} \int_0^1 (1-y)^{\bar{\alpha}_y - 1} \int_0^1 (1-z)^{\bar{\alpha}_z - 1} \\
&\big\{\mathbf{x}_\varphi^\top(\boldsymbol{\varphi}(z)\boldsymbol{\varphi}^\top(z) \otimes (\boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi \\
&- 2(\boldsymbol{\mu}_\varphi \boldsymbol{\varphi}(z)\boldsymbol{\varphi}^\top(z) \otimes (\boldsymbol{\nu}_\varphi \boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi\big\}dz\,dy \\
= {}&\tfrac{1}{2}a_0 x^2(z_f, y_f, t_f) + \tfrac{1}{2} z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} \frac{1}{\Gamma(\bar{\alpha}_z)} \frac{1}{\Gamma(\bar{\alpha}_y)} \frac{1}{\Gamma(\bar{\alpha}_t)} \int_0^1 (1-t)^{\bar{\alpha}_t - 1} \int_0^1 (1-y)^{\bar{\alpha}_y - 1} \int_0^1 (1-z)^{\bar{\alpha}_z - 1} \\
&\big\{\mathbf{x}_\varphi^\top(\boldsymbol{\varphi}(z)\boldsymbol{\varphi}^\top(z) \otimes (\boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^\top(t)))(\tilde{\mathbf{a}}_{1\varphi} \otimes (\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}))\mathbf{x}_\varphi \\
&+ \mathbf{u}_\varphi^\top(\boldsymbol{\varphi}(z)\boldsymbol{\varphi}^\top(z) \otimes (\boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^\top(t)))(\tilde{\mathbf{a}}_{1\varphi} \otimes (\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}))\mathbf{u}_\varphi\big\}dz\,dy\,dt \\
&+ c + \tfrac{1}{2}a_4 z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} \frac{1}{\Gamma(\bar{\alpha}_z)} \frac{1}{\Gamma(\bar{\alpha}_y)} \int_0^1 (1-y)^{\bar{\alpha}_y - 1} \int_0^1 (1-z)^{\bar{\alpha}_z - 1} \\
&\big\{\mathbf{x}_\varphi^\top(\boldsymbol{\varphi}(z)\boldsymbol{\varphi}^\top(z) \otimes (\boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi \\
&- 2(\boldsymbol{\mu}_\varphi \boldsymbol{\varphi}(z)\boldsymbol{\varphi}^\top(z) \otimes (\boldsymbol{\nu}_\varphi \boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi\big\}dz\,dy \\
= {}&\tfrac{1}{2}a_0 x^2(z_f, y_f, t_f) + \tfrac{1}{2} z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} \frac{1}{\Gamma(\bar{\alpha}_y)} \frac{1}{\Gamma(\bar{\alpha}_t)} \int_0^1 (1-t)^{\bar{\alpha}_t - 1} \int_0^1 (1-y)^{\bar{\alpha}_y - 1} \\
&\big\{\mathbf{x}_\varphi^\top({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes (\boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^\top(t))(\tilde{\mathbf{a}}_{1\varphi} \otimes (\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}))\mathbf{x}_\varphi \\
&+ \mathbf{u}_\varphi^\top({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes (\boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^\top(t)))(\tilde{\mathbf{a}}_{1\varphi} \otimes (\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}))\mathbf{u}_\varphi\big\}dy\,dt \\
&+ c + \tfrac{1}{2}a_4 z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} \int_0^1 (1-y)^{\bar{\alpha}_y - 1}\big\{\mathbf{x}_\varphi^\top({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes (\boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi \\
&- 2(\boldsymbol{\mu}_\varphi {}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes (\boldsymbol{\nu}_\varphi \boldsymbol{\varphi}(y)\boldsymbol{\varphi}^\top(y) \otimes \boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi\big\}dy \\
= {}&\tfrac{1}{2}a_0 \mathbf{x}_\varphi^\top(\boldsymbol{\varphi}(1) \otimes (\boldsymbol{\varphi}(1) \otimes \boldsymbol{\varphi}(1)))(\boldsymbol{\varphi}^\top(1) \otimes (\boldsymbol{\varphi}^\top(1) \otimes \boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi \\
&+ \tfrac{1}{2} z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} \frac{1}{\Gamma(\bar{\alpha}_t)} \int_0^1 (1-t)^{\bar{\alpha}_t - 1}\big\{\mathbf{x}_\varphi^\top({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes ({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^\top(t))(\tilde{\mathbf{a}}_{1\varphi} \otimes (\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}))\mathbf{x}_\varphi \\
&+ \mathbf{u}_\varphi^\top({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes ({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^\top(t)))(\tilde{\mathbf{a}}_{1\varphi} \otimes (\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}))\mathbf{u}_\varphi\big\}dt \\
&+ c + \tfrac{1}{2}a_4 z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y}\big\{\mathbf{x}_\varphi^\top({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes ({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes \boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi \\
&- 2(\boldsymbol{\mu}_\varphi {}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes (\boldsymbol{\nu}_\varphi {}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes \boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi\big\} \\
= {}&\tfrac{1}{2}a_0 \mathbf{x}_\varphi^\top(\boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1) \otimes (\boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1) \otimes \boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi + \tfrac{1}{2} z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} \mathbf{x}_\varphi^\top({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes ({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes {}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_t})) \\
&\times (\tilde{\mathbf{a}}_{1\varphi} \otimes (\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}))\mathbf{x}_\varphi + \tfrac{1}{2} z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} \mathbf{u}_\varphi^\top({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes ({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes {}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_t}))(\tilde{\mathbf{a}}_{1\varphi} \otimes (\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}))\mathbf{u}_\varphi \\
&+ c + \tfrac{1}{2}a_4 z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y}\big\{\mathbf{x}_\varphi^\top({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes ({}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes \boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi \\
&- 2(\boldsymbol{\mu}_\varphi {}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_z} \otimes (\boldsymbol{\nu}_\varphi {}_0^1\boldsymbol{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes \boldsymbol{\varphi}^\top(1)))\mathbf{x}_\varphi\big\}. \qquad (3.7)
\end{aligned}
$$

Rearranging (3.7) gives

$$
J(x, u, \bar{\alpha}_z, \bar{\alpha}_y, \bar{\alpha}_t) = c + \tfrac{1}{2}\begin{bmatrix}\mathbf{x}_\varphi \\ \mathbf{u}_\varphi\end{bmatrix}^\top \begin{bmatrix}\mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_3 & \mathbf{H}_4\end{bmatrix} \begin{bmatrix}\mathbf{x}_\varphi \\ \mathbf{u}_\varphi\end{bmatrix} + \mathbf{f}_\varphi^\top \begin{bmatrix}\mathbf{x}_\varphi \\ \mathbf{u}_\varphi\end{bmatrix}, \qquad (3.8)
$$

where $c = \frac{1}{2} a_4 {}^{RL}_{0}I^{\bar{\alpha}_y}_{y_f} {}^{RL}_{0}I^{\bar{\alpha}_z}_{z_f} \left[x_d^2(z,y)\right]$ is obtained by integration and has no effect on the QP problem, and the elements of J in (3.8) are given by

$$\begin{aligned}\mathbf{H}_1 =& a_0(\boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1)\otimes(\boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1)\otimes\boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1))) + z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} ({}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_z}_{\varphi}\otimes({}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_y}_{\varphi}\otimes{}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_t}_{\varphi}))\\ &\times(\tilde{\mathbf{a}}_{1\varphi}\otimes(\tilde{\mathbf{a}}_{2\varphi}\otimes\tilde{\mathbf{a}}_{3\varphi})) + a_4 z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y}({}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_z}_{\varphi}\otimes({}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_y}_{\varphi}\otimes\boldsymbol{\varphi}(1)\boldsymbol{\varphi}^\top(1))),\end{aligned} \tag{3.9}$$

$$\mathbf{H}_2 = \mathbf{H}_3 = \mathbf{0}_{M_1M_2M_3\times M_1M_2M_3}, \tag{3.10}$$

$$\mathbf{H}_4 = z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} ({}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_z}_{\varphi}\otimes({}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_y}_{\varphi}\otimes{}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_t}_{\varphi}))(\tilde{\mathbf{a}}_{1\varphi}\otimes(\tilde{\mathbf{a}}_{2\varphi}\otimes\tilde{\mathbf{a}}_{3\varphi})), \tag{3.11}$$

$$\mathfrak{f}_{\varphi}^\top = \begin{bmatrix} -a_4 z_f^{\bar{\alpha}_z} y_f^{\bar{\alpha}_y}(\boldsymbol{\mu}_{\varphi}{}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_z}_{\varphi}\otimes(\boldsymbol{\nu}_{\varphi}{}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_y}_{\varphi}\otimes\boldsymbol{\varphi}^\top(1))) & \mathbf{0}\end{bmatrix}. \tag{3.12}$$

The procedure of modeling (1.2) is similar to that presented in [7], hence, we just present the results. Using (3.5), (3.6) and (3.2),

$$\begin{aligned}&{}^C_0D^{\alpha_t}_t(\boldsymbol{\varphi}^\top(z)\otimes(\boldsymbol{\varphi}^\top(y)\otimes\boldsymbol{\varphi}^\top(t)))\mathbf{x}_{\varphi} = b(z){}^C_0D^{\alpha_z}_z(\boldsymbol{\varphi}^\top(z)\otimes(\boldsymbol{\varphi}^\top(y)\otimes\boldsymbol{\varphi}^\top(t)))\mathbf{x}_{\varphi}\\ &\quad + c(y){}^C_0D^{\alpha_y}_y(\boldsymbol{\varphi}^\top(z)\otimes(\boldsymbol{\varphi}^\top(y)\otimes\boldsymbol{\varphi}^\top(t)))\mathbf{x}_{\varphi} + d(z)e(y)f_t(t)(\boldsymbol{\varphi}^\top(z)\otimes(\boldsymbol{\varphi}^\top(y)\otimes\boldsymbol{\varphi}^\top(t)))\mathbf{u}_{\varphi}\\ &\quad + g_z(z)g_y(y)g_t(t).\end{aligned} \tag{3.13}$$

Now by (3.1)–(3.3), it follows from (3.13) that

$$\begin{aligned}&t_f^{-\alpha_t}(\boldsymbol{\varphi}^\top(z)\otimes(\boldsymbol{\varphi}^\top(y)\otimes\boldsymbol{\varphi}^\top(t)))(\mathbf{I}_{M_1}\otimes(\mathbf{I}_{M_2}\otimes\mathfrak{D}^{\alpha_t\top}_{\varphi}))\mathbf{x}_{\varphi}\\ &= z_f^{-\alpha_z}(\boldsymbol{\varphi}^\top(z)\otimes(\boldsymbol{\varphi}^\top(y)\otimes\boldsymbol{\varphi}^\top(t)))(\tilde{\mathbf{b}}_{\varphi}\mathfrak{D}^{\alpha_z\top}_{\varphi}\otimes(\mathbf{I}_{M_2}\otimes\mathbf{I}_{M_3}))\mathbf{x}_{\varphi}\\ &\quad + y_f^{-\alpha_y}(\boldsymbol{\varphi}^\top(z)\otimes(\boldsymbol{\varphi}^\top(y)\otimes\boldsymbol{\varphi}^\top(t)))(\mathbf{I}_{M_1}\otimes(\tilde{\mathbf{c}}_{\varphi}\mathfrak{D}^{\alpha_y\top}_{\varphi}\otimes\mathbf{I}_{M_3}))\mathbf{x}_{\varphi}\\ &\quad + (\boldsymbol{\varphi}^\top(z)\otimes(\boldsymbol{\varphi}^\top(y)\otimes\boldsymbol{\varphi}^\top(t)))(\tilde{\mathbf{d}}_{\varphi}\otimes(\tilde{\mathbf{e}}_{\varphi}\otimes\tilde{\mathbf{f}}_{t\varphi}))\mathbf{u}_{\varphi}\\ &\quad + (\boldsymbol{\varphi}^\top(z)\otimes(\boldsymbol{\varphi}^\top(y)\otimes\boldsymbol{\varphi}^\top(t)))(\mathbf{g}_{z\varphi}\otimes(\mathbf{g}_{y\varphi}\otimes\mathbf{g}_{t\varphi})).\end{aligned} \tag{3.14}$$

From (3.14), we derive the static model of the fractional partial differential equation with a three-dimensional control (1.2) as

$$\begin{aligned}&[z_f^{-\alpha_z}(\tilde{\mathbf{b}}_{\varphi}\mathfrak{D}^{\alpha_z\top}_{\varphi}\otimes(\mathbf{I}_{M_2}\otimes\mathbf{I}_{M_3})) + y_f^{-\alpha_y}(\mathbf{I}_{M_1}\otimes(\tilde{\mathbf{c}}_{\varphi}\mathfrak{D}^{\alpha_y\top}_{\varphi}\otimes\mathbf{I}_{M_3}) - t_f^{-\alpha_t}(\mathbf{I}_{M_1}\otimes(\mathbf{I}_{M_2}\otimes\mathfrak{D}^{\alpha_t\top}_{\varphi}))]\mathbf{x}_{\varphi}\\ &+ (\tilde{\mathbf{d}}_{\varphi}\otimes(\tilde{\mathbf{e}}_{\varphi}\otimes\tilde{\mathbf{f}}_{t\varphi}))\mathbf{u}_{\varphi} = -(\mathbf{g}_{z\varphi}\otimes(\mathbf{g}_{y\varphi}\otimes\mathbf{g}_{t\varphi})).\end{aligned} \tag{3.15}$$

By

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\\ \vdots & \vdots\\ \mathbf{A}_{l1} & \mathbf{A}_{l2}\end{bmatrix}, \mathbf{b} = \begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \vdots\\ \mathbf{b}_l\end{bmatrix}, \tag{3.16}$$

the optimal PDE control problem with l constraints and conditions is modeled as

$$\begin{aligned}&\underset{\boldsymbol{\chi}_{\varphi}}{\text{minimize}}\ \ \tfrac{1}{2}\boldsymbol{\chi}_{\varphi}^\top\mathbf{H}_{\varphi}\boldsymbol{\chi}_{\varphi} + \mathfrak{f}_{\varphi}^\top\boldsymbol{\chi}_{\varphi}\\ &\text{subject to}\ \ \mathbf{A}\boldsymbol{\chi}_{\varphi} = \mathbf{b}_{\varphi},\end{aligned}$$

where $\boldsymbol{\chi}_{\varphi} = [\mathbf{x}_{\varphi};\mathbf{u}_{\varphi}]$, $\mathbf{H}_{\varphi}$ and $\mathfrak{f}_{\varphi}^\top$ are given in (3.9)–(3.12), and $\mathbf{A}_{\cdot}$ and $\mathbf{b}_{\cdot}$ in (3.16) are derived from the problem and its conditions and/or constraints. For example, from (3.15), we have

$$\mathbf{A}_{11} = z_f^{-\alpha_z}(\tilde{\mathbf{b}}_{\varphi}\mathfrak{D}^{\alpha_z\top}_{\varphi}\otimes(\mathbf{I}_{M_2}\otimes\mathbf{I}_{M_3})) + y_f^{-\alpha_y}(\mathbf{I}_{M_1}\otimes(\tilde{\mathbf{c}}_{\varphi}\mathfrak{D}^{\alpha_y\top}_{\varphi}\otimes\mathbf{I}_{M_3}) - t_f^{-\alpha_t}(\mathbf{I}_{M_1}\otimes(\mathbf{I}_{M_2}\otimes\mathfrak{D}^{\alpha_t\top}_{\varphi})), \tag{3.17}$$

$$\mathbf{A}_{12} = \tilde{\mathbf{d}}_{\varphi} \otimes (\tilde{\mathbf{e}}_{\varphi} \otimes \tilde{\mathbf{f}}_{t\varphi}), \tag{3.18}$$

$$\mathbf{b}_1 = -\mathbf{g}_{z\varphi} \otimes (\mathbf{g}_{y\varphi} \otimes \mathbf{g}_{t\varphi}). \tag{3.19}$$

Since the elements of (1.2) are different for each problem, some changes in (3.17)–(3.19) will be required.

Remark 3.2. We presented the procedure for modeling the Riemann–Liouville performance index and there may be the Riemann–Liouville constraint in the problem like that given in (1.14); by using the procedure, we can model it.

4. Modeling process for optimization of nonlinear PDEs

Now, we are going to develop our method for the systems with nonlinear partial differential equations. We present here the following lemma without proof, by extending the similar lemma, which was presented for one dimensional optimal control problems in [44].

Lemma 4.1. *Assume the fractional partial differential equation is in the form* (1.8)*, where $\Im$ is continuous and satisfies local Lipschitz conditions and the state variables are separable. Also, we have a performance index as* (1.7)*. The nonlinear optimal control problem of minimizing* (1.7) *for a nonlinear PDE* (1.8) *can be replaced by the following sequence of linear optimal control problems, which this sequence converges to a solution:*
for $i \geq 1$, minimize

$$\begin{aligned} J^{[i]}(x,u,\bar{\alpha}_t,\bar{\alpha}_y) &= \tfrac{1}{2}a_0 {x^{[i]}}^2(y_f,t_f) + \tfrac{1}{2}\,{}^{RL}_{\;0}I^{\bar{\alpha}_t}_{t_f}\,{}^{RL}_{\;0}I^{\bar{\alpha}_y}_{y_f}\left[a_2(y)a_3(t)\left\{{x^{[i]}}^2(y,t)+{u^{[i]}}^2(y,t)\right\}\right] \\ &\quad + \tfrac{1}{2}a_4\,{}^{RL}_{\;0}I^{\bar{\alpha}_y}_{y_f}\left[x^{[i]}(y,t_f)-x_d(y)\right]^2 \end{aligned} \tag{4.1}$$

subject to

$$\frac{\partial^{\alpha_t}x(y,t)^{[i]}}{\partial y^{\alpha_t}} = b^{[i-1]}(y,t)\frac{\partial^{\alpha_y}x(y,t)}{\partial t^{\alpha_y}} + c^{[i-1]}(y,t)x^{[i]}(y,t) + d^{[i-1]}(y,t)u^{[i]}(y,t) + e^{[i-1]}(y,t), \tag{4.2}$$

where $x^{[0]}(y,t) = x(0,0)$, $u^{[0]}(y,t) = 0$ and $b^{[i-1]}$, $c^{[i-1]}$ and $d^{[i-1]}$ are derived by linearizing (1.8).

Theorem 4.2. *Assume that $\boldsymbol{\varphi}(y)$ is of size $M_1 \times 1$ and $\boldsymbol{\varphi}(t)$ is of size $M_2 \times 1$. The optimal control of the nonlinear optimization problem for the fractional nonlinear partial differential equation* (1.8) *can be obtained by solving the following quadratic programming*

$$\text{for } i \geq 1, \underset{\boldsymbol{x}_{\varphi}}{minimize}\;\; \tfrac{1}{2}\begin{bmatrix}\mathbf{x}^{[i]}_{\varphi}\\ \mathbf{u}^{[i]}_{\varphi}\end{bmatrix}^{\top}\begin{bmatrix}\mathbf{H}_1 & \mathbf{H}_2\\ \mathbf{H}_3 & \mathbf{H}_4\end{bmatrix}\begin{bmatrix}\mathbf{x}^{[i]}_{\varphi}\\ \mathbf{u}^{[i]}_{\varphi}\end{bmatrix} + \mathbf{f}^{\top}_{\varphi}\begin{bmatrix}\mathbf{x}^{[i]}_{\varphi}\\ \mathbf{u}^{[i]}_{\varphi}\end{bmatrix} \tag{4.3}$$

$$\text{subject to}\;\; \begin{bmatrix}\mathbf{A}^{[i-1]}_1 \quad \mathbf{A}^{[i-1]}_2\\ \mathbf{A}_c\end{bmatrix}\begin{bmatrix}\mathbf{x}^{[i]}_{\varphi}\\ \mathbf{u}^{[i]}_{\varphi}\end{bmatrix} = \begin{bmatrix}\mathbf{b}^{[i-1]}_1\\ \mathbf{b}_c\end{bmatrix}, \tag{4.4}$$

until the conditions given in [44] is reached, in which

$$\mathbf{H}_1 = a_0(\boldsymbol{\varphi}(1)\boldsymbol{\varphi}^{\top}(1) \otimes \boldsymbol{\varphi}(1)\boldsymbol{\varphi}^{\top}(1)) + y_f^{\bar{\alpha}_y}t_f^{\bar{\alpha}_t}({}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_y}_{\varphi} \otimes {}^1_0\boldsymbol{\Gamma}^{\bar{\alpha}_t}_{\varphi}))(\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}), \tag{4.5}$$

$$\mathbf{H}_2 = \mathbf{0}_{M_1M_2 \times M_1M_2}, \tag{4.6}$$

$$\mathbf{H}_3 = \mathbf{0}_{M_1M_2 \times M_1M_2}, \tag{4.7}$$

(4.8) $$\mathbf{H}_4 = y_f^{\bar{\alpha}_y} t_f^{\bar{\alpha}_t} ({}_0^1\mathbf{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes {}_0^1\mathbf{\Gamma}_\varphi^{\bar{\alpha}_t})(\tilde{\mathbf{a}}_{2\varphi} \otimes \tilde{\mathbf{a}}_{3\varphi}),$$

(4.9) $$\mathbf{f}_\varphi^\top = \left[\begin{array}{cc} -a_4 y_f^{\bar{\alpha}_y} (\boldsymbol{\nu}_\varphi {}_0^1\mathbf{\Gamma}_\varphi^{\bar{\alpha}_y} \otimes \boldsymbol{\varphi}^\top(1)) & \mathbf{0} \end{array} \right],$$

(4.10) $$\mathbf{A}_1^{[i-1]} = t_f^{-\alpha_y} (\tilde{\mathbf{b}}_{y,\varphi}^{[i-1]} \otimes \tilde{\mathbf{b}}_{t,\varphi}^{[i-1]})(\mathbf{I}_{M_1} \otimes \mathfrak{D}_\varphi^{\alpha_y \top}) + (\tilde{\mathbf{c}}_{y,\varphi}^{[i-1]} \otimes \tilde{\mathbf{c}}_{t,\varphi}^{[i-1]}) - y_f^{-\alpha_t} (\mathfrak{D}_\varphi^{\alpha_t \top} \otimes \mathbf{I}_{M_2}),$$

(4.11) $$\mathbf{A}_2^{[i-1]} = \tilde{\mathbf{d}}_{y,\varphi}^{[i-1]} \otimes \tilde{\mathbf{d}}_{t,\varphi}^{[i-1]},$$

(4.12) $$\mathbf{b}_1^{[i-1]} = \mathbf{e}_{y,\varphi}^{[i-1]} \otimes \mathbf{e}_{t,\varphi}^{[i-1]},$$

and $\mathbf{A}_c$ and $\mathbf{b}_c$ are constructed based on the given initial and boundary conditions.

Proof. From Lemma 4.1, first we form (4.2) and (4.1); then, we take $b^{[i-1]}(y,t) = b_y^{[i-1]}(y) b_t^{[i-1]}(t)$, $c^{[i-1]}(y,t) = c_y^{[i-1]}(y) c_t^{[i-1]}(t)$, and $d^{[i-1]}(y,t) = d_y^{[i-1]}(y) d_t^{[i-1]}(t)$. Using (2.8), we can expand each term, for example, $b_y^{[i-1]}(y) = \boldsymbol{\varphi}(y)\mathbf{b}_{y,\varphi}^{[i-1]}$, $b_t^{[i-1]}(t) = \boldsymbol{\varphi}(t)\mathbf{b}_{t,\varphi}^{[i-1]}$. Similarly, we take $e^{[i-1]}(y,t) = \boldsymbol{\varphi}^\top(y)\mathbf{e}_{y,\varphi}^{[i-1]} \boldsymbol{\varphi}^\top(t)\mathbf{e}_{t,\varphi}^{[i-1]}$. Now by using the procedure given in Section 3, we can reach the given sequence of quadratic programming programs given in (4.3) and (4.4) with the elements given in (4.5)–(4.12). □

The main task in the above model is to determine the coefficients of each vector.

5. Modeling process by using wavelets for optimization of delay PDEs

Consider the optimal control problem of delay PDE, which the delayed plant is described by the delay PDE in (1.12) and the performance index or the cost function is given in (1.7). Here, we use the wavelets to model this optimal control problem. We express the state and the control of the system as products of separable functions of y and t such that

$$x(y,t) = f(y)g(t).$$

Using (2.8), we expand the state in terms of the desired wavelets as

(5.1) $$x(y,t) = (\mathfrak{w}^\top(y) \otimes \mathfrak{w}^\top(t))\mathbf{x}_\mathrm{w},$$

where $\mathbf{x}_\mathrm{w}$ is $M_1 M2 N_1 N_2 \times 1$ vector of unknown parameters. From this for the control, we have

(5.2) $$u(y,t) = (\mathfrak{w}^\top(y) \otimes \mathfrak{w}^\top(t))\mathbf{u}_\mathrm{w},$$

where $\mathbf{u}_\mathrm{w}$ is $M_1 M2 N_1 N_2 \times 1$ vector of unknown parameters.

For the initial function or the history function (1.13), when $0 < t < \tau$, we see that $-\tau < t - \tau < 0$, thus we find that

(5.3) $$x(y, t-\tau) = \begin{cases} \zeta(y)\theta(t-\tau), & 0 < t < \tau \\ x(y,t-\tau), & \tau \le t < t_f \end{cases}, 0 < y < y_f.$$

Similarly,

(5.4) $$x(y, t-\tau(t)) = \begin{cases} \zeta(y)\theta(t-\tau(t)), & 0 < t < \tau(t) \\ x(y,t-\tau(t)), & \tau(t) \le t < t_f \end{cases}, 0 < y < y_f.$$

After rescaling,

(5.5) $$\zeta(y) = \mathfrak{w}^\top(y)\boldsymbol{\zeta}_\mathrm{w},$$

$$\theta(t-\tau) = \mathfrak{w}^\top(t)\boldsymbol{\theta}_{\mathrm{w}}, \tag{5.6}$$

$$\theta(t-\tau(t)) = \mathfrak{w}^\top(t)\boldsymbol{\theta}^t_{\mathrm{w}}. \tag{5.7}$$

The procedures of expanding $\theta(t-\tau)$ and $\theta(t-\tau(t))$ by the wavelets are given in [41] and [40].

Inserting the results in (5.1)–(5.7) into (1.12) and using (2.21)-(2.24) yield

$$\begin{aligned} &t_f^{-\alpha_t}(\mathbf{I}\otimes\mathfrak{D}_{\mathrm{w}}^{\alpha_t\top})\mathbf{x}_{\mathrm{w}} = y_f^{-\alpha_y}(\tilde{\mathbf{b}}_{1\mathrm{w}}\mathfrak{D}_{\mathrm{w}}^{\alpha_y\top}\otimes\tilde{\mathbf{b}}_{2\mathrm{w}})\mathbf{x}_{\mathrm{w}} + (\mathbf{I}\otimes\tilde{\mathbf{c}}_{\mathrm{w}}\mathbf{D}^\top)\mathbf{x}_{\mathrm{w}} + (\boldsymbol{\zeta}_{\mathrm{w}}\otimes\tilde{\mathbf{c}}_{\mathrm{w}}\boldsymbol{\theta}_{\mathrm{w}}) \\ &+ (\mathbf{I}\otimes\tilde{\mathbf{d}}_{\mathrm{w}}\mathbf{D}^{t\top})\mathbf{x}_{\mathrm{w}} + (\boldsymbol{\zeta}_{\mathrm{w}}\otimes\tilde{\mathbf{d}}_{\mathrm{w}}\boldsymbol{\theta}^t_{\mathrm{w}}) + (\mathbf{I}\otimes\tilde{\mathbf{e}}_{\mathrm{w}}\mathbf{S}^\top)\mathbf{x}_{\mathrm{w}} + (\tilde{\mathbf{f}}_{1\mathrm{w}}\otimes\tilde{\mathbf{f}}_{2\mathrm{w}})\mathbf{u}_{\mathrm{w}} + (\mathbf{g}_{1\mathrm{w}}\otimes\mathbf{g}_{2\mathrm{w}}). \end{aligned} \tag{5.8}$$

Factoring and reformulating (5.8), we obtain

$$\begin{aligned} &[-t_f^{-\alpha_t}(\mathbf{I}\otimes\mathfrak{D}_{\mathrm{w}}^{\alpha_t\top}) + y_f^{-\alpha_y}(\tilde{\mathbf{b}}_{1\mathrm{w}}\mathfrak{D}_{\mathrm{w}}^{\alpha_y\top}\otimes\tilde{\mathbf{b}}_{2\mathrm{w}}) + (\mathbf{I}\otimes\tilde{\mathbf{c}}_{\mathrm{w}}\mathbf{D}^\top) + (\mathbf{I}\otimes\tilde{\mathbf{d}}_{\mathrm{w}}\mathbf{D}^{t\top}) + (\mathbf{I}\otimes\tilde{\mathbf{e}}_{\mathrm{w}}\mathbf{S}^\top)]\mathbf{x}_{\mathrm{w}} \\ &+ (\tilde{\mathbf{f}}_{1\mathrm{w}}\otimes\tilde{\mathbf{f}}_{2\mathrm{w}})\mathbf{u}_{\mathrm{w}} = -(\boldsymbol{\zeta}_{\mathrm{w}}\otimes\tilde{\mathbf{c}}_{\mathrm{w}}\boldsymbol{\theta}_{\mathrm{w}}) - (\boldsymbol{\zeta}_{\mathrm{w}}\otimes\tilde{\mathbf{d}}_{\mathrm{w}}\boldsymbol{\theta}^t_{\mathrm{w}}) - (\mathbf{g}_{1\mathrm{w}}\otimes\mathbf{g}_{2\mathrm{w}}). \end{aligned} \tag{5.9}$$

So we find the equvalent model of (1.12) as a static equation (5.9).

We must model the Riemann–Liouville performance index by the wavelets. The procedure is similar to that the procedure given for (1.1) but by using (2.20), hence we do not present it.

Now, we model the initial condition (1.9). From (5.1), we have

$$x(0,t) = (\mathfrak{w}^\top(0)\otimes\mathfrak{w}^\top(t))\mathbf{x}_{\mathrm{w}}$$

and

$$x(y,0) = (\mathfrak{w}^\top(y)\otimes\mathfrak{w}^\top(0))\mathbf{x}_{\mathrm{w}}.$$

Knowing $\mathbf{x}_{\mathrm{w}} = [x_1, x_2, \ldots, x_{M^2N^2}]$ for $M_1 = M_2 = M$ and $N_1 = N_2 = N$, expanding (after rescaling)

$$h(t) = \mathfrak{w}^\top(t)\mathbf{h}^\top_{\mathrm{w}},$$

taking $w_{10}(0) = w^0_0$, $w_{11}(0) = w^0_1$, $w_{1M-1}(0) = w^0_{M-1}$, $M_1 = M_2 = M$ and $N_1 = N_2 = N$, by defining

$$\mathbf{W}^0 = \begin{bmatrix} w^0_0 & \mathbf{0} & w^0_1 & \mathbf{0} & w^0_2 & \cdots & w^0_{M-1} & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & w^0_0 & \mathbf{0} & w^0_1 & \mathbf{0} & w^0_2 & \cdots & w^0_{M-1} & 0 & 0 & 0 & \ldots & 0 \\ & & & & & & \vdots & & & & & & \\ 0 & 0 & 0 & \ldots & 0 & 0 & w^0_0 & \mathbf{0} & w^0_1 & \mathbf{0} & w^0_2 & \cdots & w^0_{M-1} \end{bmatrix}, \tag{5.10}$$

and

$$\mathbf{W}_c = \begin{bmatrix} \mathbf{W}^0 & \mathbf{0}_{MN\times(M^2N^2-M^2N)} \end{bmatrix}, \tag{5.11}$$

we model the first condition in (1.9) as

$$\mathbf{W}_c\mathbf{x}_{\mathrm{w}} = \mathbf{h}^\top_{\mathrm{w}}. \tag{5.12}$$

We can use (5.12) with the elements given in (5.10), (5.11) as a model of the given condition. For the second condition in (1.9), after rescaling by

$$i(y) = \mathfrak{w}^\top(y)\mathbf{i}^\top_{\mathrm{w}},$$

we have

$$\begin{aligned} x(y,0) &= \mathfrak{w}^\top(y)(\mathbf{I}\otimes\mathfrak{w}^\top(0))\mathbf{x}_{\mathrm{w}} \\ &= \mathfrak{w}^\top(y)\mathbf{i}^\top_{\mathrm{w}}. \end{aligned} \tag{5.13}$$

Factoring the wavelets vector in (5.13), yields

$$(\mathbf{I} \otimes \mathfrak{w}^\top(0))\mathbf{x}_{\mathrm{w}} = \mathbf{i}_{\mathrm{w}}^\top. \tag{5.14}$$

Hence, we modeled the second condition in (1.9) as (5.14). For modeling the boundary conditions in (1.10), after rescaling,

$$x(1,t) = (\mathfrak{w}^\top(1) \otimes \mathfrak{w}^\top(t))\mathbf{x}_{\mathrm{w}} = h(t)$$

and

$$x(y,1) = (\mathfrak{w}^\top(y) \otimes \mathfrak{w}^\top(1))\mathbf{x}_{\mathrm{w}} = i(y).$$

When $y = 1$, by setting $w_{10}(1) = w_0^1$, $w_{11}(1) = w_1^1$, $w_{1M-1}(1) = w_{M-1}^1$, and

$$\mathbf{W}^1 = \begin{bmatrix} w_0^1 & \mathbf{0} & w_1^1 & \mathbf{0} & w_2^1 & \cdots & w_{M-1}^1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & w_0^1 & \mathbf{0} & w_1^1 & \mathbf{0} & w_2^1 & \cdots & w_{M-1}^1 & 0 & 0 & 0 & \dots & 0 \\ & & & & & \vdots & & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & w_0^1 & \mathbf{0} & w_1^1 & \mathbf{0} & w_2^1 & \cdots & w_{M-1}^1 \end{bmatrix}, \tag{5.15}$$

we find

$$\mathbf{W}_c = \begin{bmatrix} \mathbf{0}_{MN\times(M^2N^2-M^2N)} & \mathbf{W}^1 \end{bmatrix}. \tag{5.16}$$

We can use (5.12) with the elements given in (5.15), (5.16) as a model of the given condition. In a similar manner, when $t = 1$,

$$\begin{aligned} x(y,1) &= \mathfrak{w}^\top(y)(\mathbf{I} \otimes \mathfrak{w}^\top(1))\mathbf{x}_{\mathrm{w}} \\ &= \mathfrak{w}^\top(y)\mathbf{i}_{\mathrm{w}}^\top. \end{aligned} \tag{5.17}$$

From (5.17),

$$(\mathbf{I} \otimes \mathfrak{w}^\top(1))\mathbf{x}_{\mathrm{w}} = \mathbf{i}_{\mathrm{w}}^\top. \tag{5.18}$$

We modeled the second condition in (1.10) as (5.18). Taking the given equations together, we reach the model. Using a similar procedure and the given properties, we can model fractional/integer Robin conditions given in (1.11). Hence, the optimal control problem of the delayed PDE is modeled as

$$\begin{aligned} &\underset{\boldsymbol{\chi}_{\mathrm{w}}}{\text{minimize}} \;\; \tfrac{1}{2}\boldsymbol{\chi}_{\mathrm{w}}^\top \mathbf{H}_{\mathrm{w}}\boldsymbol{\chi}_{\mathrm{w}} + \mathfrak{f}_{\mathrm{w}}^\top\boldsymbol{\chi}_{\mathrm{w}} \\ &\text{subject to} \;\; \mathbf{A}\boldsymbol{\chi}_{\mathrm{w}} = \mathbf{b}_{\mathrm{w}}, \end{aligned}$$

After using the QP solver from a software package to find the solutions, we find the solutions from the wavelets expansions (5.1) and (5.2). In the given QP problems, $J = c + J_{QP}$.

Proposition 1. From the definitions of the wavelets, we must add the $2N(N-1)$ following constraint to the model of the problem for the wavelets vectors defined with the same N and M

$$\mathbf{C}_{lcc}\mathbf{x}_{\mathrm{w}} = \mathbf{0}, \tag{5.19}$$

where

$$\mathbf{C}_{lcc} = \left[\mathbf{C}_{1y\nearrow 1t}; \mathbf{C}_{2y\nearrow 1t}; \dots; \mathbf{C}_{Ny\nearrow(N-1)t}; \mathbf{C}_{1t\nearrow 1y}; \mathbf{C}_{2t\nearrow 1y}; \dots; \mathbf{C}_{Nt\nearrow(N-1)y}\right]. \tag{5.20}$$

By $\mathfrak{w}(t) := [w_{10}(t), w_{11}(t), \dots, w_{1M-1}(t)]$, the blocks of this matrix can be found as follows

$$\begin{aligned} \mathbf{C}_{1y\nearrow 1t} &= [\boldsymbol{\Omega}_1 \otimes \boldsymbol{\rho}_1^f - \boldsymbol{\Omega}_1 \otimes \boldsymbol{\rho}_1^i], \\ \boldsymbol{\Omega}_1 &= \left[[\mathfrak{w}(y_s)], \mathbf{0}_{\mathrm{numRow}([\mathfrak{w}^\top(y_s)])\times M(N-1)}\right], \\ \boldsymbol{\rho}_1^f &= \left[\mathfrak{w}(\tfrac{1}{N}), \mathbf{0}_{1\times M(N-1)}\right], \end{aligned}$$

$$\boldsymbol{\rho}_1^i = \left[\mathbf{0}_{1\times M}, \mathfrak{w}(0), \mathbf{0}_{1\times M(N-2)}\right],$$
$$\mathbf{C}_{2y\nearrow 1t} = [\boldsymbol{\Omega}_2 \otimes \boldsymbol{\rho}_1^f - \boldsymbol{\Omega}_2 \otimes \boldsymbol{\rho}_1^i],$$
$$\boldsymbol{\Omega}_2 = \left[\mathbf{0}_{\mathrm{numRow}([\mathfrak{w}(y_s)])\times M}, [\mathfrak{w}(y_s)], \mathbf{0}_{\mathrm{numRow}([\mathfrak{w}(y_s)])\times M(N-2)}\right],$$
$$\mathbf{C}_{1y\nearrow 2t} = [\boldsymbol{\Omega}_1 \otimes \boldsymbol{\rho}_2^f - \boldsymbol{\Omega}_1 \otimes \boldsymbol{\rho}_2^i],$$
$$\boldsymbol{\rho}_2^f = \left[\mathbf{0}_{1\times M}, \mathfrak{w}(\tfrac{1}{N}), \mathbf{0}_{1\times M(N-1)}\right],$$
$$\boldsymbol{\rho}_2^i = \left[\mathbf{0}_{1\times 2M}, \mathfrak{w}(0), \mathbf{0}_{1\times M(N-3)}\right],$$
$$\mathbf{C}_{1t\nearrow y} = [\boldsymbol{\rho}_1^f \otimes \boldsymbol{\Omega}_1 - \boldsymbol{\rho}_1^i \otimes \boldsymbol{\Omega}_1].$$

Proof. From the difintion of the wavelets, we have some intersection, by setting $I_n = [(n-1)/N, n/N]$, we see that there exist some points as t_c, where $\{t_c\} = I_n \cap I_{n+1}$. By looking at Figure 1, we see that there are red lines at which we must to ensure the continuity of the obtained state. We must have $\mathbf{x}(t_c^-) = \mathbf{x}(t_c^+)$ at $\{t_c\}$, which by considering another dimensional of the state is called the lines compatibility constraint. Using the lines compatibility constraint at sample points y_s, we reach the form given in (5.19) and (5.20). □

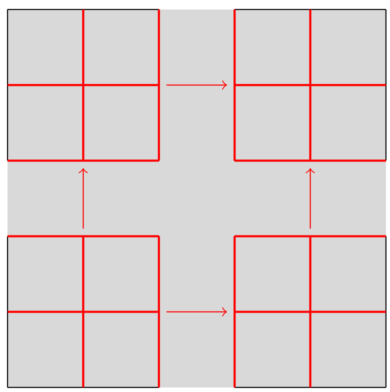

FIGURE 1. Lines of compatibility constraint.

6. CONVERGENCE

In this section, we discuss the convergence of the methods. Since several theorems are given for the polynomials in [7], we just present one theorem for them; then we present some theorems for the wavelets.

Theorem 6.1. *Let $f(t)$ be a twice differentiable function on $[0,1]$ with bounded second derivative l, $l = \max_{0\le t\le 1} |f''(t)|$. For $M > 2$, the expansion $f(t) = \mathbf{f}_\varphi \boldsymbol{\varphi}(t)$ and the desired polynomial as $f_{M,\varphi}(t) = \mathbf{f}_\varphi \boldsymbol{\varphi}(t)$, where $\mathbf{f}_\varphi$ and $\boldsymbol{\varphi}(t)$ are given by*

$$\mathbf{f}_\varphi := [f_0^\varphi, f_1^\varphi, f_2^\varphi, \ldots, f_{M-2}^\varphi, f_{M-1}^\varphi],$$
$$\boldsymbol{\varphi}(t) := [\varphi_0(t), \varphi_1(t), \varphi_2(t), \ldots, \varphi_{M-1}(t)]^\top,$$

and the constant coefficients $\{f_m^\varphi\}$ are obtained from

$$f_m^\varphi = \int_0^1 f(t)\varphi_m(t) w_\varphi(t)\, dt, \tag{6.1}$$

we have the following error estimates

$$\|f-f_{M,SL}\|_{w_{\bar{P}_m}} \le l\sqrt{\frac{1}{512}\ln\frac{2M+3}{2M-5}+\frac{1}{256}\ln\frac{2M-3}{2M+1}},$$

$$\|f-f_{M,SC}\|_{w_{\bar{T}_m}} \le l\sqrt{\frac{\pi}{16}\left(\frac{M-1}{2M(M-2)}+\frac{1}{4}\ln\frac{M-2}{M}\right)}.$$

As mentioned in [39], the difference between these values is due to the different processes of approximating $f_{M,SL}$ and $f_{M,SC}$ and it does not prove anything.

Proof. For both polynomials, we can write

$$\begin{aligned}\|f-f_{M,\varphi}\|^2_{w_\varphi} &= \Big\|\sum_{m=0}^{\infty} f_m^\varphi \varphi_m(t) - \sum_{m=0}^{M-1} f_m^\varphi \varphi_m(t)\Big\|^2_{w_\varphi}\\ &= \Big\|\sum_{m=M}^{\infty} f_m^\varphi \varphi_m(t)\Big\|^2_{w_\varphi}\\ &= \sum_{m=M}^{\infty} \left(f_m^\varphi\right)^2. \end{aligned} \tag{6.2}$$

For the shifted Legendre polynomials, "SLPs", it follows from (6.1) that

$$f_m^{SL} = \frac{c_m}{2}\int_{-1}^{1} f\Big(\frac{z+1}{2}\Big)\, P_m(z)\, dz. \tag{6.3}$$

By applying integration by parts, for $m \ge 2$ in (6.3),

$$\begin{aligned} f_m^{SL} &= \frac{c_m}{2}\int_{-1}^{1} f(\frac{z+1}{2})P_m(z)\,dz = \frac{c_m}{4}\frac{1}{2m+1}\Big[f(\frac{z+1}{2})\left\{P_{m+1}(z)-P_{m-1}(z)\right\}\Big]_{z=-1}^{z=1}\\ &\quad - \frac{c_m}{4}\frac{1}{2m+1}\int_{-1}^{1} f'(\frac{z+1}{2})\Big\{P_{m+1}(z)-P_{m-1}(z)]\Big\}\,dz\\ &= -\frac{c_m}{4}\frac{1}{2m+1}\int_{-1}^{1} f'(\frac{z+1}{2})\Big\{\frac{1}{2m+3}(P'_{m+2}(z)-P'_m(z)) - \frac{1}{2m-1}(P'_m(z)-P'_{m-2}(z))\Big\}\,dz\\ &= \frac{c_m}{8\sqrt{2m+1}}\int_{-1}^{1} f''(\frac{z+1}{2})\Big\{\frac{P_{m+2}(z)}{2m+3} - \frac{2(2m+1)P_m(z)}{(2m-1)(2m+3)} + \frac{P_{m-2}(z)}{2m-1}\Big\}\,dz. \end{aligned}$$

Thus we can get an upper bound as

$$\left|f_m^{SL}\right| \le \frac{\sqrt{6}l}{4\sqrt{(2m-3)(2m-1)(2m+3)(2m+5)}}.$$

By setting in (6.2), we get

$$\begin{aligned}\|f-f_{M,SL}\|^2_{w_{\bar{T}_m}} &\le \frac{6l^2}{16}\sum_{m=M}^{\infty}\frac{1}{(2m-3)(2m-1)(2m+3)(2m+5)}\\ &\le \frac{6l^2}{16}\int_{M-1}^{\infty}\frac{1}{(2y-3)(2y-1)(2y+3)(2y+5)}dy\\ &\le \frac{l^2}{512}\ln\frac{2M+3}{2M-5}+\frac{l^2}{256}\ln\frac{2M-3}{2M+1}.\end{aligned}$$

For the shifted Chebyshev polynomials, "SCPs", from (6.1)

$$f_m^{SC} = \frac{c_m}{\sqrt{2}}\int_{0}^{\pi} f\Big(\frac{\cos(\theta)+1}{2}\Big)\,\cos m\theta\, d\theta. \tag{6.4}$$

Hence for $m \geq 2$, by applying integration by parts in (6.4),

$$\left|f_m^{SC}\right| \leq \frac{\sqrt{\pi} l}{4(m^2-1)}.$$

Setting in (6.2) yields

$$\begin{aligned}\|f - f_{M,SC}\|^2_{w_{\bar{T}_m}} &\leq \frac{\pi l^2}{16} \sum_{m=M}^{\infty} \frac{1}{(m^2-1)^2} \\ &\leq \frac{\pi l^2}{16} \int_{M-1}^{\infty} \frac{1}{(y^2-1)^2} dy \\ &\leq \frac{\pi l^2}{16} \Big(\frac{\frac{1}{2}(M-1)}{M(M-2)} + \frac{1}{4} \ln \frac{M-2}{M} \Big).\end{aligned}$$

□

Theorem 6.2. *Assume that $f(y)$ and $g(t)$ are Lipschitz functions on $y \in [0,1]$ and $t \in [0,1]$. Then, a two-dimensional function $x(y,t) = f(y)g(t)$ can be expanded as*

$$x(y,t) = \sum_{n=1}^{N} \sum_{m=0}^{\infty} f_{nm} w_{nm}(y) \sum_{n=1}^{N} \sum_{m=0}^{\infty} g_{nm} w_{nm}(t),$$

where this series converges uniformly to $x(y,t)$.

Proof. Since $f(y)$ and $g(t)$ are Lipschitz in y and t, we can get $|f(y)|, |f'(y)|, |f''(y)| \leq k_f$ and $|g(t)|, |g'(t)|, |g''(t)| \leq k_g$, where k_f and k_g are constants. Then from (2.7), we can find upper bound for $|f_{ni}|$ and $|g_{ni}|$, where $i = 0, 1, 2$. Now we can write

$$\begin{aligned}|x(y,t)| &= \left| \sum_{n=1}^{N} \sum_{m=0}^{\infty} f_{nm} w_{nm}(y) \sum_{n=1}^{N} \sum_{m=0}^{\infty} g_{nm} w_{nm}(t) \right| \\ &= \left| \sum_{n=1}^{N} \sum_{m=0}^{\infty} f_{nm} w_{nm}(y) \right| \left| \sum_{n=1}^{N} \sum_{m=0}^{\infty} g_{nm} w_{nm}(t) \right| \\ &= \left| \sum_{n=1}^{N} f_{n0} w_{n0}(y) + f_{n1} w_{n1}(y) + \sum_{m=2}^{\infty} f_{nm} w_{nm}(y) \right| \left| \sum_{n=1}^{N} g_{n0} w_{n0}(t) + g_{n1} w_{n1}(t) + \sum_{m=2}^{\infty} g_{nm} w_{nm}(t) \right| \\ &= \begin{cases} |f_{10} w_{n0}(y) + f_{11} w_{11}(y) + \sum_{m=2}^{\infty} f_{1m} w_{1m}(y)| & \text{if} \quad 0 \leq y \leq \frac{1}{N} \\ \vdots \\ |f_{n0} w_{n0}(y) + f_{n1} w_{n1}(y) + \sum_{m=2}^{\infty} f_{nm} w_{nm}(y)| & \text{if} \quad \frac{(n-1)}{N} \leq y \leq \frac{n}{N} \\ \vdots \\ |f_{N0} w_{N0}(y) + f_{N1} w_{N1}(y) + \sum_{m=2}^{\infty} f_{Nm} w_{Nm}(y)| & \text{if} \quad \frac{(N-1)}{N} \leq y \leq 1 \end{cases} \\ &\quad \times \begin{cases} |g_{10} w_{n0}(t) + g_{11} w_{11}(t) + \sum_{m=2}^{\infty} g_{1m} w_{1m}(t)| & \text{if} \quad 0 \leq t \leq \frac{1}{N} \\ \vdots \\ |g_{n0} w_{n0}(t) + g_{n1} w_{n1}(t) + \sum_{m=2}^{\infty} g_{nm} w_{nm}(t)| & \text{if} \quad \frac{(n-1)}{N} \leq t \leq \frac{n}{N} \\ \vdots \\ |g_{N0} w_{N0}(t) + g_{N1} w_{N1}(t) + \sum_{m=2}^{\infty} g_{Nm} w_{Nm}(t)| & \text{if} \quad \frac{(N-1)}{N} \leq t \leq 1 \end{cases} \\ &\leq \sum_{n=1}^{N} \delta_{y,\mathrm{w}} \delta_{n,t,\mathrm{w}} \leq \delta,\end{aligned}$$

where $\delta_{y,\mathrm{w}} \in \mathbb{R}$ is the upper bound obtained for the first series by using $|f_{ni}|$ for the n-th subinterval in which we have the maximum error, for example, see [41], , $\delta_{n,t,\mathrm{w}}$ for the second series by using $|g_{ni}|$, and $\delta \in \mathbb{R}$. Hence the given series is absolutely convergent and the expansion converges to $x(y,t)$ uniformly. □

Lemma 6.3. *For a twice differentiable function $f(y)$ defined on $[0,1]$, with bounded second derivatives $|f''(y)| \leq \delta_{2f}$, we have the following norm in $\mathcal{L}_2$ with respect to the weight function $\varsigma_{n\mathrm{w}}(y)$,*

$$\|f(y) - \mathbf{f}_{\mathrm{w}}\boldsymbol{\mathfrak{w}}(y)\|_{\varsigma_{n\mathrm{w}}(y)} \leq h_{\mathrm{w}}(\delta_{2f}, m, N),$$

where $h_{\mathrm{w}}(\delta_{2f}, m, N)$ is a constant and obtained according to the relative wavelets and it tends to zero as $M \to \infty$.

Lemma 6.4. *For a function $f(y)$ defined in Lemma 6.3, we have the following inequality norm in $\mathcal{L}_2$ with respect to the weight function $\varsigma_{n\mathrm{w}}(y)$,*

$$\|\mathbf{f}_{\mathrm{w}}\boldsymbol{\mathfrak{w}}(y)\|_{\varsigma_{n\mathrm{w}}(y)} \leq \|f(y)\|_{\varsigma_{n\mathrm{w}}(y)}.$$

Theorem 6.5. *Assume that a function $f(y)$ is a differentiable function $f(y)$, defined on $[0,1]$, with bounded derivatives as $|{}_0^C D_y^{2+\alpha} f(y)| \leq \delta_f$. Then the following norm in $\mathcal{L}_2$ with respect to $\varsigma_{n\mathrm{w}}(y)$*

$$\left\|{}_0^C D_y^{\alpha} f(y) - \mathbf{f}_{\mathrm{w}}\boldsymbol{\mathfrak{D}}_{\mathrm{w}}^{\alpha}\boldsymbol{\mathfrak{w}}(y)\right\|_{\varsigma_{n\mathrm{w}}(y)}$$

tends to zero as $M \to \infty$, where $\boldsymbol{\mathfrak{D}}_{\mathrm{w}}^{\alpha}$ is Caputo derivative operational matrix of the desired wavelet.

Proof. See [43]. □

Corollary 1. For a differentiable bounded function defined as $f(y)g(t)$, by using the given expansions,

$$\left\|\frac{\partial^{\alpha_t}[f(y)g(t)]}{\partial t^{\alpha_t}} - (\boldsymbol{\mathfrak{w}}^{\top}(y) \otimes \boldsymbol{\mathfrak{w}}^{\top}(t))(\mathbf{I} \otimes \tilde{\mathbf{b}}_{\mathrm{w}}\boldsymbol{\mathfrak{D}}_{\mathrm{w}}^{\alpha_t\top})(\mathbf{f}_{\mathrm{w}}^{\top} \otimes \mathbf{g}_{\mathrm{w}}^{\top})\right\|_{\varsigma_{n\mathrm{w}}(y)\varsigma_{n\mathrm{w}}(t)}$$

tends to zero as $M \to \infty$.

Theorem 6.6. *Suppose the assumptions of Theorems 6.5 are satisfied for $f(y)$ and $g(t)$, $x(y,t) = f(y)g(t)$, $\|f(y)\|_{\varsigma_{n\mathrm{w}}(y)} \leq \mu_f$, and $\|g(t)\|_{\varsigma_{n\mathrm{w}}(y)} \leq \mu_g$. Then*

$$\|f(y)g(t) - (\boldsymbol{\mathfrak{w}}^{\top}(y) \otimes \boldsymbol{\mathfrak{w}}^{\top}(t))\mathbf{x}_{\mathrm{w}}\|_{\varsigma_{n\mathrm{w}}(y)\varsigma_{n\mathrm{w}}(t)} \leq \mu_g h_{\mathrm{w}}(\delta_{2f}, m, N) + \mu_f h_{\mathrm{w}}(\delta_{2g}, m, N),$$

where $h_{\mathrm{w}}(\delta_f, m, N)$ and $h_{\mathrm{w}}(\delta_f, m, N)$ are constants and obtained according to the relative wavelets. It means if $M \to \infty$, then the given norm for the expansion of two-dimensional function tends to zero.

Proof. From Lemmas 6.3 and 6.4, we can write

$$\begin{aligned}
&\left\|f(y)g(t) - (\boldsymbol{\mathfrak{w}}^{\top}(y) \otimes \boldsymbol{\mathfrak{w}}^{\top}(t))\mathbf{x}_{\mathrm{w}}\right\|_{\varsigma_{n\mathrm{w}}(y)\varsigma_{n\mathrm{w}}(t)} \\
&\leq \left\|g(t)f(y) - \boldsymbol{\mathfrak{w}}^{\top}(y)\mathbf{f}_{\mathrm{w}}^{\top}\right\|_{\varsigma_{n\mathrm{w}}(y)\varsigma_{n\mathrm{w}}(t)} + \left\|f(y)g(t) - \boldsymbol{\mathfrak{w}}^{\top}(t)\mathbf{g}_{\mathrm{w}}^{\top}\right\|_{\varsigma_{n\mathrm{w}}(y)\varsigma_{n\mathrm{w}}(t)} \\
&\leq \mu_g h_{\mathrm{w}}(\delta_{2f}, m, N) + \mu_f h_{\mathrm{w}}(\delta_{2g}, m, N).
\end{aligned}$$

From the relations of the wavelets, we can find $h_{\mathrm{w}}(\delta_f, m, N)$ and $h_{\mathrm{w}}(\delta_f, m, N)$ and we see that if $M \to \infty$, then the given norm for the expansion of two-dimensional function tends to zero. □

7. Modeling Results for PDE optimization examples

In this section, we apply the methods to some optimal control problems of PDEs that have applications in engineering and natural sciences. The fractional PDEs of these problems are governed from the well-known equations. The given results are generated by MATLAB R2013b. Based on the given ideas, we model each of them to obtain their optimal solutions. We consider complicated conditions or constraints to show the advantage of the method. In a traditional method, one should resolve the problem and repeat the solution processes for a new scenario while in our method there is no need to do these tasks. Knowing the fact that solving the integer order of any of the following problems is not an easy task, one realizes that solving the fractional order is much more complicated, especially with the new given scenarios.

Remark 7.1. As can easily be seen in some texts, one can define countless trivial problems with exact solutions. First, a problem for $X(y,t) = 0$ and $U(y,t) = 0$ is defined as $0 = f(X(y,t), U(y,t))$, then by setting $0 = X(y,t) = x(y,t) - x(y,t) = x(y,t) - f_1(y)g_1(t)$ and $0 = U(y,t) = u(y,t) - u(y,t) = u(y,t) - f_2(y)g_2(t)$, the transformed problem is defined as a new problem with exact solutions as $x(y,t) = f_1(y)g_1(t)$ and $u(y,t) = f_2(y)g_2(t)$. The initial and boundary conditions can be easily derived, for example, $x(y,0) = f_1(y)g_1(0)$. Similar procedure can be applied, for example, just setting $U(y,t) = 0$ or setting $0 = (f_1(y)g_1(t)x(y,t) - f_2(y)g_2(t)u(y,t))$. But the resulted problem is not an optimal control problem and it contains several unfixable fundamental flaws; for instance, for infinite performance indices such as indices with $(x(y,t) - f_1(y)g_1(t))^{n_1}$ and/or $(u(y,t) - f_2(y)g_2(t))^{n_2}$ for $n_1, n_2 = 1, 2, \ldots$, the solutions remain the same, while this is definitely impossible in the optimal control theory.

7.1. **Example 1, a system with a nonlinear PDE.** This interesting optimal control problem is adapted from [45]. Consider new fractional types of one dimensional heat equation with a nonlinearity of Schlögl type studied in [45] with a performance index as

$$J = \tfrac{1}{2}\int_0^{t_f}\int_0^{y_f}\left\{x^2(y,t) + u^2(y,t)\right\} dy\, dt,$$

where $t_f = 2$ and $y_f = 1$. Now, if we change the order of the PDE,

• Case 1:

$$\begin{aligned}
&\frac{\partial^\alpha x(y,t)}{\partial t^\alpha} = \frac{\partial^2 x(y,t)}{\partial y^2} + 15(x(y,t) - x^3(y,t)) + u(y,t),\\
&\alpha = 0.91,\\
&x(0,t) = 0, x(1,t) = 0,\\
&x(y,0) = 0.2\sin(\pi y).
\end{aligned}$$

Ref. [45] used the model predictive control combination method for solving the integer version of this problem, that is, $\alpha = 1$ in Case 1. Here, we use the proposed method and the optimal solutions for the problems is presented in Figure 2. Then, as more complicated situations, we consider different scenarios. As another case, we change the conditions and use the given state equation in which

• Case 2:

$$\begin{aligned}
&\alpha = 0.4,\\
&x(0,t) = -0.15t,\\
&x(1,t) = 0.1t,\\
&x(y,0) = 0.1 - 0.1\cos(2\pi y).
\end{aligned}$$

The results of our method for this fractional nonlinear optimization problem are shown in Figure 3. Now, we again change the order of the PDE,

• Case 3:

$$\begin{aligned}
&\frac{\partial^\alpha x(y,t)}{\partial t^\alpha} = \frac{\partial^{\alpha+1} x(y,t)}{\partial y^{\alpha+1}} + 15(x(y,t) - x^3(y,t)) + u(y,t),\\
&\alpha = 0.4,\\
&x(0,t) = -0.15t,\\
&x(1,t) = 0.1t,\\
&x(y,0) = 0.1 - 0.1\cos(2\pi y).
\end{aligned}$$

The optimal solutions are shown in Figure 4, which one can see the differences due to changing a derivative order of the PDE, that is, changing $\partial^2 x(y,t)/\partial y^2$ to $\partial^{\alpha+1} x(y,t)/\partial y^{\alpha+1}$.

Now as another case, assume that there is a temperature constraint in Case 3 as a plate that the temperature

must be less than or equal to temperature of this plate as
• Case 4: the given equations in Case 3 hold and also, we must have

$$x(y,t) \leq 0.11 \quad \text{when} \quad 0.3 \leq y \leq 0.8 \quad \text{and} \quad 1.3 \leq t \leq 2.$$

This plate is shown schematically in the graph of optimal state in Figure 5. It is obvious from the previous solutions that the solutions of the system for Case 3 cannot satisfy the state constraint. As an advantage of the proposed QP modeling, we solve the problem with this temperature constraint and the optimal solutions are presented in Figure 5; we can see the constraint is satisfied.
Now as new constraint, we have a fractional isoperimetric constraint in Case 3 as
• Case 5: the given equations in Case 3 hold and also we must have

$${}_0^{RL}I_{t_f}^{\bar{\alpha}_{i,t}}\, {}_0^{RL}I_{y_f}^{\bar{\alpha}_{i,y}}[15x(y,t) - u(y,t)] \leq 0.5,$$

where $\bar{\alpha}_{i,t} = 0.8$, $\bar{\alpha}_{i,y} = 0.7$. We have the two-dimensional Riemann–Liouville isoperimetric constraint in the new problem. The solutions of Case 3 cannot satisfy the fractional isoperimetric constraint. As an advantage of the QP method, we can solve this problem; by doing so, the optimal solutions are presented in Figure 6.

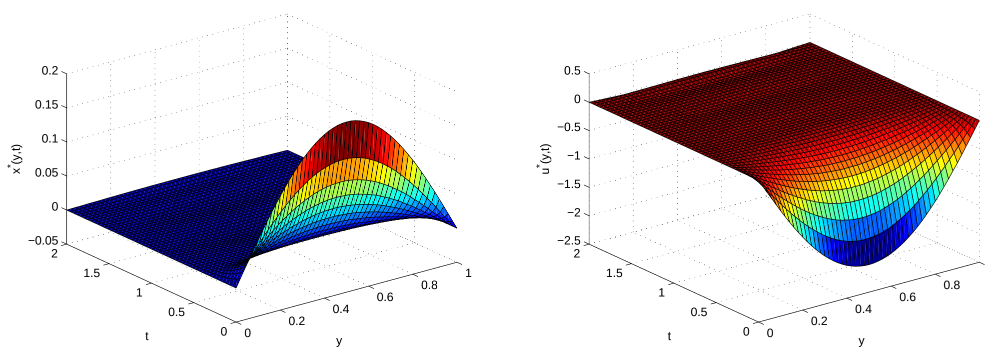

FIGURE 2. Optimal solutions for Case 1 of Example 1.

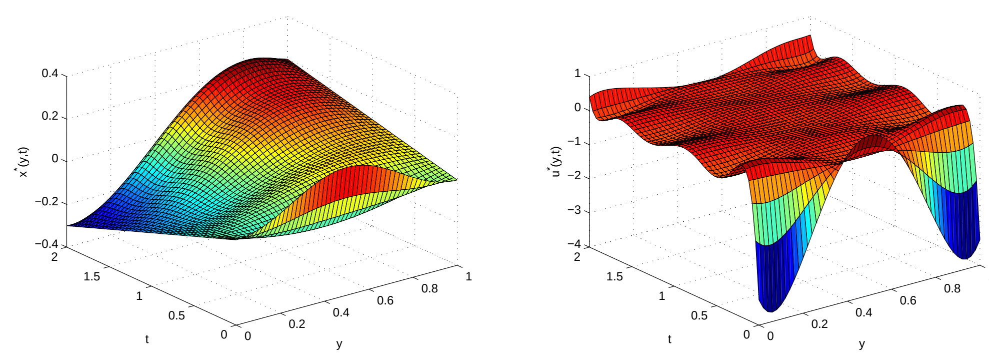

FIGURE 3. Optimal solutions for Case 2 of Example 1.

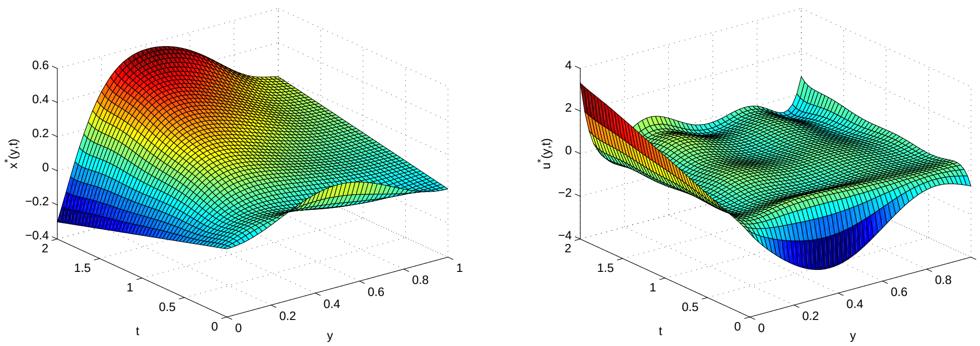

FIGURE 4. Optimal solutions for Case 3 of Example 1.

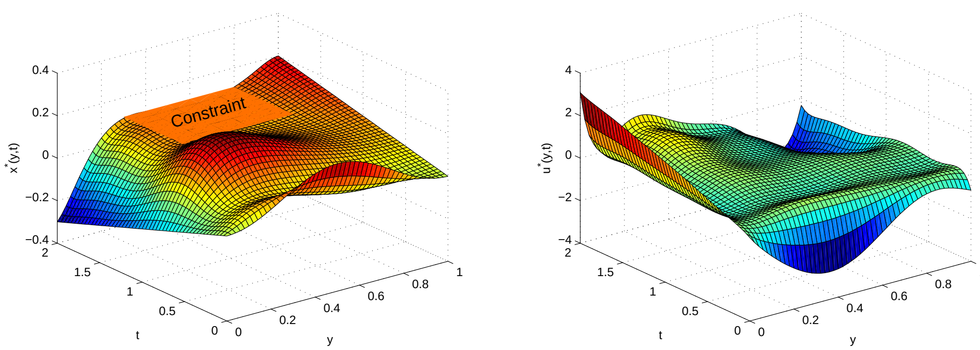

FIGURE 5. Optimal solutions for Case 4 of Example 1.

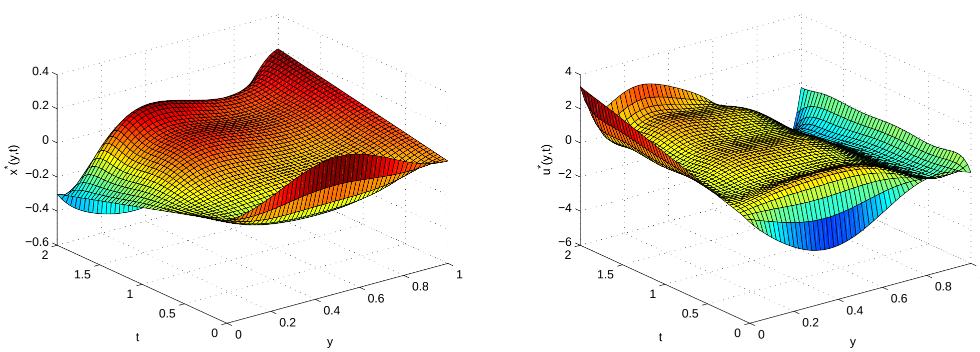

FIGURE 6. Optimal solutions for Case 5 of Example 1.

7.2. **Example 2, a system with a nonlinear delay/stretch PDE.** This interesting optimal control problem is adapted from [28]. Consider a new pure fractional version of a reaction diffusion model presented in [28]. By taking $x(y,t)$ as the population density at time t and position y, where $0 \leq t \leq 5$ and $0 \leq y \leq \pi$,

$$\frac{\partial^{\alpha_t} x(y,t)}{\partial t^{\alpha_t}} = \frac{\partial^{\alpha_y} x(y,t)}{\partial y^{\alpha_y}} - 0.5x(y,\omega(t))[1 + x(y,t)] + u(y,t)$$

with the boundary conditions

$$\left.\frac{\partial x(y,t)}{\partial y}\right|_{y=0} = 0, \left.\frac{\partial x(y,t)}{\partial y}\right|_{y=\pi} = 0,$$

$$x(y,0) = 1 - \cos(2y).$$

The goal is to minimize the new, combined, Riemann–Liouville performance indices as

$$J = 0.1\,{}_{0}^{RL}I_{5}^{\bar{\alpha}_t}\,{}_{0}^{RL}I_{\pi}^{\bar{\alpha}_y}\left[u^2(y,t)\right] + {}_{0}^{RL}I_{\pi}^{\bar{\alpha}_y}\left[x(y,5) - 6 + 3\cos(3y)\right]^2$$

subject to the given PDE, where $0 < \alpha_t \leq 1$, $1 < \alpha_y \leq 2$, $\bar{\alpha}_t = \alpha_t$ and $\bar{\alpha}_y = \alpha_y - 1$. Also:

• Case 1: $\omega(t)$ is a stretched argument, which is also called a proportional delay, scaled delay or pantograph-type delay in some texts like [46], defined as

$$\omega(t) = \frac{t}{2}.$$

• Case 2: $\omega(t)$ is a time-delay argument defined with a history function as

$$\omega(t) = t - 0.5, \quad \text{and} \quad x(y,t) = 1 - \cos(2y) \quad \text{when} \quad -0.5 \leq t \leq 0.$$

We solve this problem for $\alpha_t = 1$, $\alpha_y = 2$ and $\alpha_t = 0.9$, $\alpha_y = 1.7$ for the both cases. The results are show in Figures 7–10.

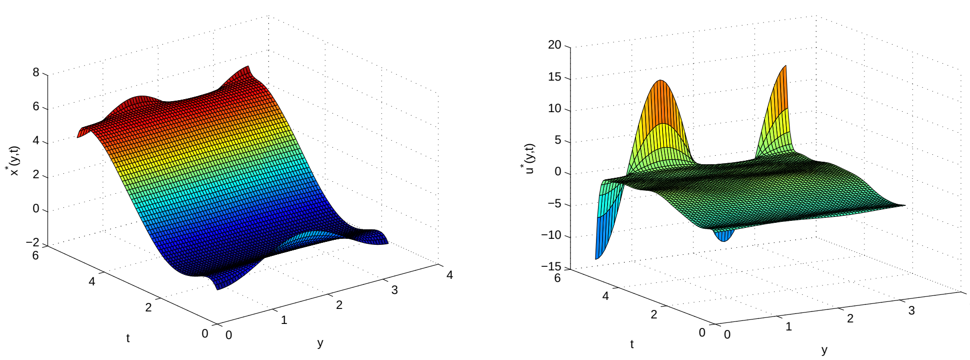

FIGURE 7. Optimal solutions for Case 1 of Example 2, $\alpha_t = 1$, $\alpha_y = 2$.

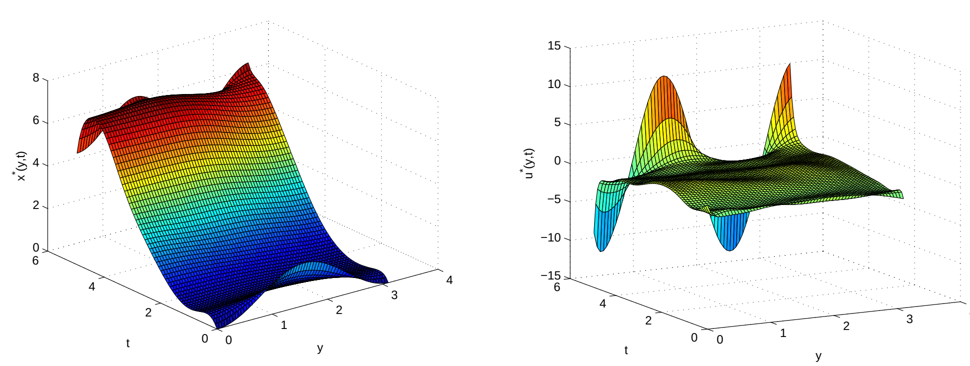

FIGURE 8. Optimal solutions for Case 1 of Example 2, $\alpha_t = 0.9$, $\alpha_y = 1.7$.

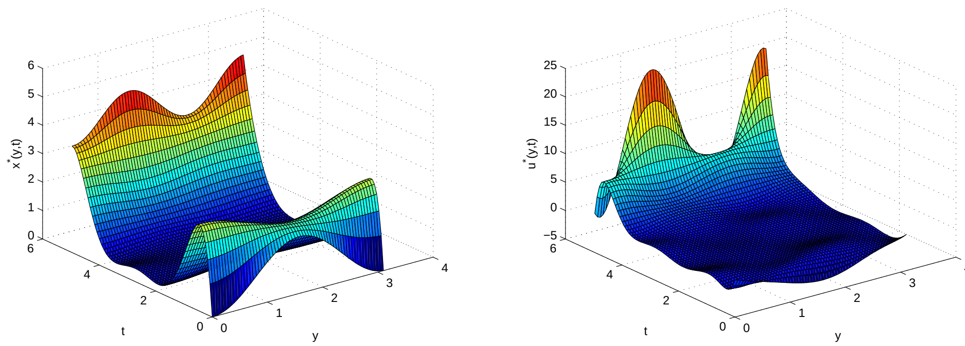

FIGURE 9. Optimal solutions for Case 2 of Example 2, $\alpha_t = 1$, $\alpha_y = 2$.

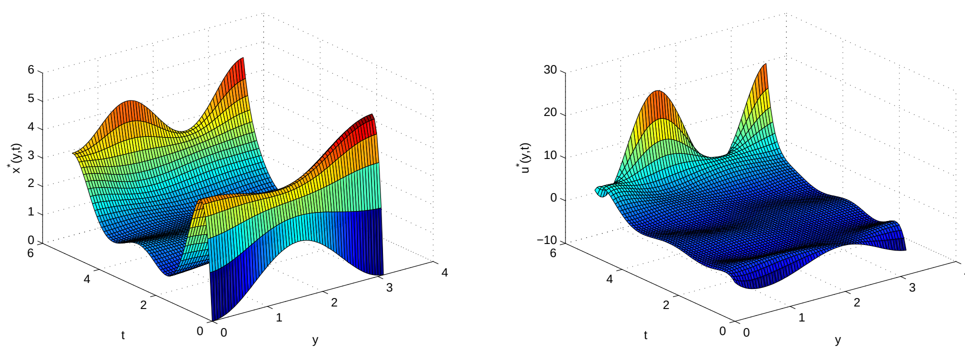

FIGURE 10. Optimal solutions for Case 2 of Example 2, $\alpha_t = 0.9$, $\alpha_y = 1.7$.

7.3. **Example 3, a system with a two-dimensional PDE.** Consider the problem of controlling two-dimensional wave equation in Cartesian zy-coordinate system as

$$\frac{\partial^{\alpha_t} x(z,y,t)}{\partial t^{\alpha_t}} = \frac{\partial^{\alpha_z} x(z,y,t)}{\partial z^{\alpha_z}} + \frac{\partial^{\alpha_y} x(z,y,t)}{\partial y^{\alpha_y}} + (z+1)(y+1)u(z,y,t),$$

where t is the time, $t_f = 2$, $z_f = 1$, $y_f = 1$, $\alpha_t = 1.9$, $\alpha_z = 1.8$ and $\alpha_y = 1.6$; the performance index and the boundary conditions are given as the following scenarios:

- Case 1:

$$J = \tfrac{1}{2}\,{}^{RL}_{\;0}I^{\bar{\alpha}_t}_{t_f}\,{}^{RL}_{\;0}I^{\bar{\alpha}_y}_{y_f}\,{}^{RL}_{\;0}I^{\bar{\alpha}_z}_{z_f}\left[x^2(z,y,t) + u^2(z,y,t)\right],$$
$$x(0,y,t) = 0, \quad x(1,y,t) = 100zy.$$

- Case 2: the performance index is the same as before, and we only have one condition as

$$x(0,y,t) - x(1,y,t) = 10.$$

- Case 3:

$$\begin{aligned} J =& \tfrac{1}{2}x^2(z_f,y_f,t_f) + \tfrac{1}{2}\,{}^{RL}_{\;0}I^{\bar{\alpha}_t}_{t_f}\,{}^{RL}_{\;0}I^{\bar{\alpha}_y}_{y_f}\,{}^{RL}_{\;0}I^{\bar{\alpha}_z}_{z_f}\left[x^2(z,y,t) + u^2(z,y,t)\right] \\ &+ \tfrac{1}{2}\,{}^{RL}_{\;0}I^{\bar{\alpha}_y}_{y_f}\,{}^{RL}_{\;0}I^{\bar{\alpha}_z}_{z_f}\left[10\left(x(z,y,t_f) - x_d(z,y)\right)^2\right], \end{aligned}$$
$$x(0,y,t) = 10, \quad x_d(z,y) = 100 - 100\cos(z)\cos(y).$$

For Case 1, the optimal state and control for $z = 0.5$ are shown in Figure 11. For Case 2, the optimal states are shown at $z = 0$ and $z = 1$ in Figure 12. And for Case 3, the desired reference and the optimal state and control at $t = t_f$ are shown in Figure 13.

Some of important PDEs are given by two-dimensional PDEs. Hence, we need to study systems described by two-dimensional PDEs in the optimal control systems. Here, we present this example as a two-dimensional PDE optimal control problem to see applicability of the proposed method. Some scenarios have been chosen and new scenarios such as a three-dimensional Riemann–Liouville isoperimetric constraint can be investigated for two-dimensional PDE optimal control problems such as this.

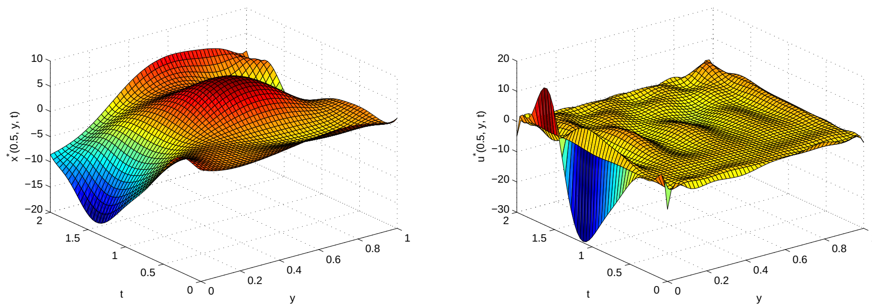

FIGURE 11. Optimal solutions for Case 1 of Example 3.

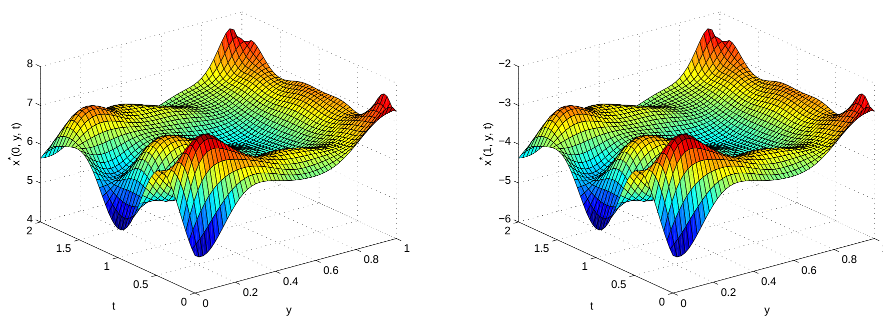

FIGURE 12. Optimal solutions for Case 2 of Example 3.

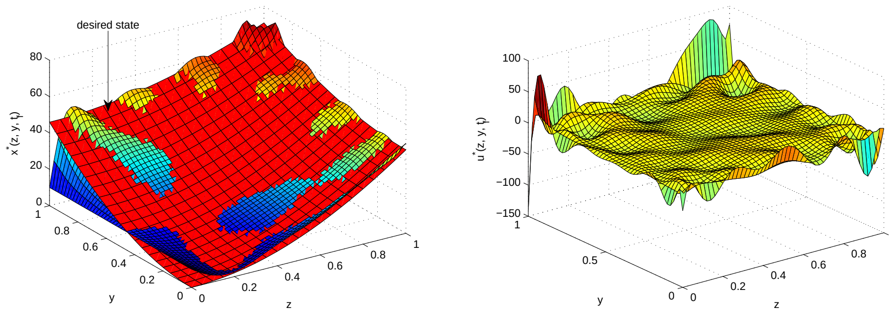

FIGURE 13. Optimal solutions for Case 3 of Example 3.

Conclusion

We extended the ideas presented in the previous works to the pure fractional optimal control of nonlinear, delay and two-dimensional PDEs and we presented the new methods. The methods can be used to solve such problems with different kinds of the conditions and/or constraints. We saw that there is no need to obtain the optimality conditions in these methods. Some problems with fractional PDEs and the Riemann–Liouville performance indices have been solved to show their applicability. The Riemann–Liouville isoperimetric constraints for PDEs has been introduced. The formulations have been presented in general forms with terminal, minimum total energy and tracking performance indices for different purposes of PDE optimization.

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References

[1] E. Kreyszig, H. Kreyszig, E.J. Norminton, Advanced Engineering Mathematics, John Wiley & Sons, 2011.

[2] D.G. Zill, W.S. Wright, Advanced Engineering Mathematics, Jones & Bartlett Learning, 2014.

[3] A.P.S. Selvadurai, Partial Differential Equations in Mechanics 2: The Biharmonic Equation, Poisson's Equation, Vol. 2, Springer Science & Business Media, 2000.

[4] T. Kaczorek, Two-Dimensional Linear Systems, Berlin, Springer, Heidelberg, 1985.

[5] R.L. Herman, Introduction to Partial Differential Equations, R.L. Herman, North Carolina, 2015.

[6] E. Madenci, I. Guven, The Finite Element Method and Applications in Engineering Using ANSYS, Springer, 2015. https://doi.org/10.1007/978-1-4899-7550-8.

[7] I. Malmir, Aerospace and Mechanical Engineering Group, Ronin Institute, Montclair, NJ 07043, USA, Simulating Two-Dimensional Optimal Control Problem of Fractional Partial Differential Equations, Adv. Comput. Sci. Eng. 1 (2023), 271–297. https://doi.org/10.3934/acse.2023012.

[8] I. Malmir, New Pure Multi-Order Fractional Optimal Control Problems with Constraints: QP and LP Methods, ISA Trans. 153 (2024), 155–190. https://doi.org/10.1016/j.isatra.2024.08.003.

[9] R. Luus, Iterative Dynamic Programming, CRC Press, 2019.

[10] Y. Chen, Y. Nan, X. Sun, T. Tan, Superior Control of Spacecraft Re-Entry Trajectory, Appl. Sci. 14 (2024), 10585. https://doi.org/10.3390/app142210585.

[11] V. Ghaffari, S. Mobayen, S. Ud Din, T. Rojsiraphisal, M.T. Vu, Robust Tracking Composite Nonlinear Feedback Controller Design for Time-Delay Uncertain Systems in the Presence of Input Saturation, ISA Trans. 129 (2022), 88–99. https://doi.org/10.1016/j.isatra.2022.02.029.

[12] V. Ghaffari, A Robust Predictive Control Scheme for Uncertain Continuous-time Delayed Systems under Actuator Saturation, IET Control Theory Appl. 18 (2024), 1779–1790. https://doi.org/10.1049/cth2.12725.

[13] J. He, Construction and Simulation of a Joint Scale Model for Power Electronic Converters Based on Wavelet Decomposition and Reconstruction Algorithms, PLOS ONE 19 (2024), e0298590. https://doi.org/10.1371/journal.pone.0298590.

[14] B. Chachuat, Nonlinear and Dynamic Optimization: From Theory to Practice, LA-TEACHING-2007-001, École Polytechnique Fédeérale de Lausanne, 2007.

[15] M. Krstić, A. Smyshlyaev, Boundary Control of PDEs: A Course on Backstepping Designs, SIAM, Philadelphia, 2008.

[16] P.K.C. Wang, Asymptotic Stability of a Time-Delayed Diffusion System, J. Appl. Mech. 30 (1963), 500–504. https://doi.org/10.1115/1.3636609.

[17] A. Bressan, Viscosity Solutions of Hamilton-Jacobi Equations and Optimal Control Problems, Lecture Notes, Penn State University, USA, 2019.

[18] M. Blatt, K. Schittkowski, Optimal Control of One-Dimensional Partial Differential Algebraic Equations With Applications, Ann. Oper. Res. 98 (2000), 45–64. https://doi.org/10.1023/A:1019231718807.

[19] W. Marszalek, Two-Dimensional State-Space Discrete Models for Hyperbolic Partial Differential Equations, Appl. Math. Model. 8 (1984), 11–14. https://doi.org/10.1016/0307-904X(84)90170-7.

[20] M. Gugat, M. Schuster, E. Zuazua, Optimal Control for ODEs and PDEs: The Turnpike Phenomenon, in: The 65th Joint Conference on Automatic Control, Utsunomiya, 2022.

[21] R. Tanase, C. Trenchea, I. Yotov, Optimal Control of Systems Governed by PDEs with Random Parameter Fields, Technical Report, University of Pittsburgh, 2016.

[22] R. Altmann, C. Zimmer, On the Smoothing Property of Linear Delay Partial Differential Equations, J. Math. Anal. Appl. 467 (2018), 916–934. https://doi.org/10.1016/j.jmaa.2018.07.049.

[23] F. Sami Khana, M. Sultanab, M. Khalid, Numerical Solution of Time Fractional Delay Partial Differential Equationsby Perturbation Iteration Algorithm, Punjab Univ. J. Math. 53 (2021), 557–573. https://doi.org/10.52280/pujm.2021.530803.

[24] S.I. Solodushkin, I.F. Yumanova, R.H. De Staelen, First Order Partial Differential Equations with Time Delay and Retardation of a State Variable, J. Comput. Appl. Math. 289 (2015), 322–330. https://doi.org/10.1016/j.cam.2014.12.032.

[25] M.M. Hasan, X.W. Tangpong, O.P. Agrawal, Fractional Optimal Control of Distributed Systems in Spherical and Cylindrical Coordinates, J. Vib. Control 18 (2012), 1506–1525. https://doi.org/10.1177/1077546311408471.

[26] S. Li, Z. Zhou, Legendre Pseudo-Spectral Method for Optimal Control Problem Governed by a Time-Fractional Diffusion Equation, Int. J. Comput. Math. 95 (2018), 1308–1325. https://doi.org/10.1080/00207160.2017.1417591.

[27] J. Zhou, Optimal Control of a Stochastic Delay Partial Differential Equation with Boundary-Noise and Boundary-Control, J. Dyn. Control Syst. 20 (2014), 503–522. https://doi.org/10.1007/s10883-014-9228-5.

[28] J.T. Betts, S.L. Campbell, K.C. Thompson, Chapter 10: Optimal Control of a Delay PDE, in: L.T. Biegler, S.L. Campbell, V. Mehrmann (Eds.), Control and Optimization with Differential-Algebraic Constraints, SIAM, Philadelphia, PA, 2012: pp. 213–231. https://doi.org/10.1137/9781611972252.ch10.

[29] L. Huang, H. Zhao, T. Sun, An Iterative Proper Orthogonal Decomposition Method for a Parabolic Optimal Control Problem, J. Appl. Math. Comput. 70 (2024), 47–72. https://doi.org/10.1007/s12190-023-01961-w.

[30] S. Mastoi, A.H. Ganie, A.M. Saeed, U. Ali, U.A. Rajput, W.A. Mior Othman, Numerical Solution for Two-Dimensional Partial Differential Equations Using SM's Method, Open Phys. 20 (2022), 142–154. https://doi.org/10.1515/phys-2022-0015.

[31] J. Carlsson, Optimal Control of Partial Differential Equations in Optimal Design, PhD Thesis, KTH Royal Institute of Technology, 2008.

[32] C.I. Bajer, A. Myslinski, A. Zochowski, B. Dyniewicz, D. Pisarski, Solving optimal control problems described by PDEs, in: Interactive Computer Environment for Solving Optimal Problems-IDOS, 2012, 347–394.

[33] P. Philip, Optimal Control of Partial Differential Equations, Lecture Notes, Ludwig-Maximilians-Universität, Germany, 2009.

[34] E.H. Zerrik, A. Ait Aadi, M. Ouhafsa, On the Regional Tracking Problem of the Bilinear Wave Equation Subject to Bounded Controls, Eur. J. Control 76 (2024), 100943. https://doi.org/10.1016/j.ejcon.2023.100943.

[35] M. Dymkov, Linear-Quadratic Optimization Problems for Generalized 2D Continuous Control Systems, J. Math. Sci. 286 (2024), 508–525. https://doi.org/10.1007/s10958-024-07524-z.

[36] T.M. Atanacković, S. Pilipović, B. Stanković, D. Zorica, Fractional Calculus With Applications in Mechanics: Vibrations and Diffusion Processes, John Wiley & Sons, 2014.

[37] I. Malmir, An Efficient Method for a Variety of Fractional Time-Delay Optimal Control Problems with Fractional Performance Indices, Int. J. Dyn. Control 11 (2023), 2886–2910. https://doi.org/10.1007/s40435-023-01113-9.

[38] I. Malmir, A New Fractional Integration Operational Matrix of Chebyshev Wavelets in Fractional Delay Systems, Fractal Fract. 3 (2019), 46. https://doi.org/10.3390/fractalfract3030046.

[39] I. Malmir, Novel Closed-Loop Controllers for Fractional Linear Quadratic Time-Varying Systems, Numer. Algebra Control Optim. 14 (2024), 366–403. https://doi.org/10.3934/naco.2022032.

[40] I. Malmir, Legendre Wavelets with Scaling in Time-Delay Systems, Statistics, Optim. Inf. Comput. 7 (2019), 235–253. https://doi.org/10.19139/soic.v7i1.460.

[41] I. Malmir, Novel Chebyshev Wavelets Algorithms for Optimal Control and Analysis of General Linear Delay Models, Appl. Math. Model. 69 (2019), 621–647. https://doi.org/10.1016/j.apm.2018.12.009.

[42] I. Malmir, A Novel Wavelet-Based Optimal Linear Quadratic Tracker for Time-Varying Systems with Multiple Delays, Statistics, Optim. Inf. Comput. 9 (2021), 418–434. https://doi.org/10.19139/soic-2310-5070-1228.

[43] I. Malmir, Caputo Fractional Derivative Operational Matrices of Legendre and Chebyshev Wavelets in Fractional Delay Optimal Control, Numer. Algebra Control Optim. 12 (2022), 395–426. https://doi.org/10.3934/naco.2021013.

[44] I. Malmir, A General Framework for Optimal Control of Fractional Nonlinear Delay Systems by Wavelets, Stat. Optim. Inf. Comput. 8 (2020), 858–875. https://doi.org/10.19139/soic-2310-5070-939.

[45] N. Altmüller, Model Predictive Control for Partial Differential Equations, PhD Thesis, Universitaet Bayreuth, Germany, 2014.

[46] X. Wang, J. Liu, H. Peng, X. Zhao, An Iterative Framework to Solve Nonlinear Optimal Control with Proportional Delay Using Successive Convexification and Symplectic Multi-Interval Pseudospectral Scheme, Appl. Math. Comput. 435 (2022), 127448. https://doi.org/10.1016/j.amc.2022.127448.