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APPLICATIONS OF UJLAYAN AND DIXIT FRACTIONAL UNIFORM PROBABILITY DISTRIBUTION

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ABSTRACT. In this work, we consider the Ujlayan-Dixit (UD) fractional derivative and integral to present the fractional probability density function for the Continuous Uniform Distribution (CUD). Some applications for this distribution using the UD approach are developed by introduce new fractional notions on the probability theory, which involve cumulative distribution, survival and hazard functions. For continuous random variables, various additional concepts and applications are established, including the fractional expectation, the fractional variance and the fractional moments. Finally, the UD fractional standard deviation and the UD fractional Shannon entropy are provided.

1. INTRODUCTION

In the last few decades, fractional calculus has emerged as a focus of research and development related to the modeling of practical problems. Over time, various definitions of fractional derivatives and integrals have been proposed, with possible modifications applied as needed (e.g. Riemann-Liouville [1], Caputo [2], Caputo-Hadamard [3] and Conformable derivative [4]).

This field of research has many applications across different scientific and engineering domains. It plays a crucial role in disciplines such as physics, computer science, biology and economics, contributing to advancements in each; for further information, refer to sources [5–9]. Within this wide range of uses, our interest has focused on probability theory, where it aids in understanding complex stochastic processes, modeling uncertainty, and solving problems related to random variables and distributions.

In fact, numerous published articles have specifically explored the intersection of probability theory and fractional calculus. To cite few, in [10], a new concepts on probability theory using fractional integration in the RL-sense has introduced and some applications on expectations, variances and moments of continuous random variables having probability density functions (p.d.f.) defined on some bonded real lines are presented in [11]. Then, in [12], the authors explored several applications of the continuous uniform distribution and the Beta probability distribution using the fractional normalized concepts on continuous random variables. Recently, the authors in [13] have used fractional differential equations (FDE) to produce

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the probability density functions of random variables. For more details about other works in relation to the fractional calculus and some of its applications, the reader may refer to references [14–24].

Dixit and Ujlayan, in [25] and [26], defined the Dixit-Ujlayan (UD) fractional derivative, a new approach that transforms the fractional derivative into a convex combination of the function and its ordinary derivative, without introducing extra variables. This simplification reduces the equation to a classical differential form, allowing it to be solved using standard, well-known techniques. From there, it started with several applications in the fields of science, engineering, and finance. Then, in [27], Alhribat et al. utilized UD fractional differential equations to create new fractional distributions based on existing probability distributions, establishing UD fractional probability distribution functions for the exponential, Pareto, Levy, and Lomax distributions.

In this paper, UD fractional derivative and integral are applied to derive the fractional probability density function for the Continuous Uniform Distribution (CUD) and some applications for this distribution are developed, which involve cumulative distribution, survival and hazard functions with some graphical representations. Additionally, new concepts and applications, such as fractional expectation, fractional variance, and fractional moments, are introduced. Finally, the UD fractional standard deviation and the UD fractional Shannon entropy are presented.

2. Some Preliminary Results

The purpose of this section is to provide clear information regarding the basic definitions and properties of the UD derivative. For additional details, we motivate readers to consult these references [25,26].

Definition 2.1 (UD derivative)**.** [26] For a function $g : [0, +\infty) \to \mathbb{R}$ and $\alpha \in [0, 1]$, we define the UD derivative of order α of g by

$$D^{\alpha} g(x) = \lim_{\varepsilon \to 0} \frac{e^{\varepsilon(1-\alpha)} g\left(x e^{\frac{\varepsilon \alpha}{x}}\right) - g(x)}{\varepsilon}, \tag{2.1}$$

if limit exists. Also, if g is UD differentiable in the interval $(0, x)$ and for $x > 0$ and $\alpha \in [0, 1]$ such that $\lim_{x \to 0^+} D^{\alpha} g(x)$ exist, then

$$D^{\alpha} g(0) = \lim_{t \to 0^+} D^{\alpha} g(x).$$

We take into account that,

$$D^{\alpha} g(x) = \frac{d^{\alpha} g(x)}{dx^{\alpha}}.$$

Theorem 2.2. *[25] Let $g : [0, +\infty) \to \mathbb{R}$ be a differentiable function and $\alpha \in [0, 1]$. Then, g is UD differentiable, and*

$$D^{\alpha} g(x) = (1 - \alpha) g(x) + \alpha g'(x). \tag{2.2}$$

Note that, if $\alpha = 0$ we have $D^0 g(x) = g(x)$, and for $\alpha = 1$ we have $D^1 g(x) = g'(x)$.

Let us now state some properties of the UD derivative.

Property 1. [25] Let $0 \le \alpha, \beta \le 1$. If g_1 and g_2 are UD differentiable, then

1. The UD derivative is a linear operator, i.e.,

$$D^{\alpha}(\mu_1 g_1 + \mu_2 g_2) = \mu_1 D^{\alpha} g_1 + \mu_2 D^{\alpha} g_2, \quad \text{for all } \mu_1, \mu_2 \in \mathbb{R}.$$

2. The UD derivative is a commutative operator, i.e.,

$$D^{\alpha}(D^{\beta} g_1) = D^{\beta}(D^{\alpha} g_1).$$

3. The UD derivative satisfies the following product rule

$$D^{\alpha}(g_1.g_2) = (D^{\alpha}g_1)g_2 + \alpha(D^{\alpha}g_2)g_1.$$

So, Leibnitz's rule for fractional derivatives $D^{\alpha}(g_1.g_2) \neq g_2 D^{\alpha}g_1 + g_1 D^{\alpha}g_2$, does not satisfied here.

4. The quotient rule is satisfied by the UD derivative, i.e.,

$$D^{\alpha}\left(\frac{g_1}{g_2}\right) = \frac{(D^{\alpha}g_1)g_2 - \alpha(D^{\alpha}g_2)g_1}{(g_2)^2}, \quad \text{with } g_2(t) \neq 0.$$

5. The semi-group property is not satisfied by the UD derivative, i.e.,

$$D^{\alpha}(D^{\beta}g_1) \neq D^{\alpha+\beta}g_1.$$

6. In particular case, for $\alpha \in [0,1]$ and $x \geq 0$, The UD derivatives of some elementary real-valued differentiable functions are given by
   * $D^{\alpha}(\mathcal{C}) = (1-\alpha)\mathcal{C}$, where $\mathcal{C} \in \mathbb{R}$.
   * $D^{\alpha}((ax+b)^n) = (1-\alpha)(ax+b)^n + an\alpha(ax+b)^{n-1}$, for each $a, b \in \mathbb{R}$ and $n \in \mathbb{N}^*$.

Definition 2.3 (UD integral). [25] Given a continuous function $g : [a,b] \to \mathbb{R}$. The UD integral operator of order α where $\alpha \in (0,1]$, is defined as

$$I_a^{\alpha} g(x) = \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}x} \int_a^x e^{\frac{(1-\alpha)}{\alpha}t} g(t)dt.$$

The following property contains some fundamental characteristics of the UD integral.

Property 2. [25] If g_1 and g_2 are continuous functions then the operator I_a^{α} has the following properties:

1. $I_a^{\alpha}(\mu_1 g_1 + \mu_2 g_2) = \mu_1 I_a^{\alpha} g_1 + \mu_2 I_a^{\alpha} g_2, \quad \text{for all } \mu_1, \mu_2 \in \mathbb{R}.$
2. $I_a^{\alpha}(I_a^{\beta} g_1) = I_a^{\beta}(I_a^{\alpha} g_1), \quad \alpha, \beta \in (0,1].$
3. $I_a^{\alpha}(I_a^{\alpha} g_1) \neq I_a^{2\alpha} g_1.$
4. $D^{\alpha}(I_a^{\alpha} g_1) = g_1.$

3. Main Results

We begin our main Results by some definitions and properties.

3.1. Expectation, Variance, and Moments.

Definition 3.1. The UD Fractional expectation of order $\alpha \in (0,1]$, for a random variable X with a fractional $p.d.f.$ f_{α} defined on $[a,b]$ is given by

$$E_{\alpha}(X) = \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}x} \int_a^x e^{\frac{(1-\alpha)}{\alpha}t} t f_{\alpha}(t)dt \ , \ a \leq x \leq b.$$

Definition 3.2. The UD fractional variance of order $\alpha \in (0,1]$, for a random variable X having a fractional $p.d.f.$ f_{α} on $[a,b]$ is defined as

$$\sigma_{\alpha}^2(X) = V_{\alpha}(X) := \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}x} \int_a^x e^{\frac{(1-\alpha)}{\alpha}t} (t - E_{\alpha}(X))^2 f_{\alpha}(t)dt.$$

Definition 3.3. The UD fractional moment of orders $r > 0$, $\alpha \in (0,1]$, for a continuous random variable X having a fractional $p.d.f.$ f_{α} on $[a,b]$ is defined by

$$E_{\alpha}(X^r) := \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}x} \int_a^x e^{\frac{(1-\alpha)}{\alpha}t} t^r f_{\alpha}(t)dt \ , \ a \leq x \leq b.$$

Proposition 1. Based on the above definitions and the fact that $I_a^{\alpha} f_{\alpha}(b) = 1$, we represent some properties. For $\alpha \in (0,1]$, we have

1*: For any real number C, we have

$$E_\alpha(C) = C.$$

2*:

$$E_\alpha(E_\alpha(X)) = E_\alpha(X).$$

3*:

$$Var_\alpha(X) = E_\alpha(X^2) - E_\alpha^2(X).$$

Proof. **1*:** For the first property, we have

$$\begin{aligned} E_\alpha(C) &= \frac{C}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} f_\alpha(t) dt \\ &= C I_a^\alpha f(b), \\ &= C. \end{aligned}$$

2*: For the second property, we have

$$\begin{aligned} E_\alpha(E_\alpha(X)) &= \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} (E_\alpha(X)) f_\alpha(t) dt, \\ &= E_\alpha(X) \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} f_\alpha(t) dt, \\ &= E_\alpha(X) I_a^\alpha f(b), \\ &= E_\alpha(X). \end{aligned}$$

3*: Indeed, by definition

$$\begin{aligned} Var_\alpha(X) &= \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} (t - E_\alpha(X))^2 f_\alpha(t) dt \\ &= \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} t^2 f_\alpha(t) dt + E_\alpha^2(X) \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} f_\alpha(t) dt \\ &- 2E_\alpha(X) \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} t f_\alpha(t) dt \\ &= E_\alpha(X^2) + I_a^\alpha f_\alpha(b) E_\alpha^2(X) - 2E_\alpha(X) E_\alpha(X) \\ &= Var_\alpha(X) = E_\alpha(X^2) - E_\alpha^2(X). \end{aligned}$$

Using (3.5) we get

$$\begin{aligned} Var_\alpha(X) &= E_\alpha(X^2) + E_\alpha^2(X) - 2E_\alpha^2(X), \\ &= E_\alpha(X^2) - E_\alpha^2(X). \end{aligned}$$

□

3.2. **The UD Fractional Continuous Uniform Distribution (CUD).** The probability density function for the continuous uniform distribution (CUD) for any $x \in [a, b]$, where $b > a$, is given by

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b. \tag{3.1}$$

Let us take $y = \frac{1}{b-a}$, then the first derivative of y can be written as follows:

$$y' = 0, \tag{3.2}$$

where the obtained equation (3.2) is a first-order ordinary differential equation. Now, consider the α-order differential equation with regard to the UD derivative as follows:

$$\begin{aligned} D^{\alpha}y &= 0, \\ (1-\alpha)y + \alpha y' &= 0, \\ \alpha y' + (1-\alpha)y &= 0. \end{aligned} \tag{3.3}$$

Thus, the equation (3.3) is a linear differential equation of first order with an integrating factor

$$\begin{aligned} \varpi(x) &= e^{\int \frac{(1-\alpha)}{\alpha} dx}, \\ &= e^{\frac{(1-\alpha)}{\alpha} x}. \end{aligned}$$

So, the following represents the general solution of the equation (3.3):

$$\begin{aligned} y &= \frac{\mathcal{K}}{\varpi(x)}, \\ &= \mathcal{K} e^{\frac{(\alpha-1)}{\alpha} x}. \end{aligned}$$

Consequently, the new probability distribution will be

$$f_{\alpha}(x) = \mathcal{K} e^{\frac{(\alpha-1)}{\alpha} x}. \tag{3.4}$$

To determine the normalizing constant $\mathcal{K}$, we need to propose the following condition

$$I_a^{\alpha} f_{\alpha}(b) = 1, \tag{3.5}$$

which implies that

$$\begin{aligned} \frac{\mathcal{K}}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} f_{\alpha}(t) dt &= 1, \\ \frac{\mathcal{K}}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} e^{\frac{(\alpha-1)}{\alpha} t} dt &= 1, \\ \frac{\mathcal{K}}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b 1 dt &= 1. \end{aligned}$$

As a result, the normalizing constant $\mathcal{K}$ is presented by

$$\mathcal{K} = \frac{\alpha e^{\frac{(1-\alpha)}{\alpha} b}}{b-a}. \tag{3.6}$$

Therefore, the UD fractional probability density function (UDFPDF) of the α-continuous uniform distribution can be written as,

$$f_{\alpha}(x) = \frac{\alpha e^{\frac{(1-\alpha)}{\alpha} b}}{b-a} e^{\frac{(\alpha-1)}{\alpha} x}, \quad a \le x \le b,\ 0 < \alpha < 1. \tag{3.7}$$

We take into account that

$$\lim_{\alpha \to 1^-} f_{\alpha}(x) = \frac{1}{b-a} = f(x). \tag{3.8}$$

In order to plot the graph of the UD fractional probability density function (UDPDF) of α-continuous uniform distribution (α-CUD) according to different values of α, we take the data $a = 5$ and $b = 10$, then the result is shown in Figure 1.

We note that The UD fractional probability density function is a constant function when $\alpha = 1$, which means that it is graphically identical to the classical case of the probability density function for the continuous uniform distribution.

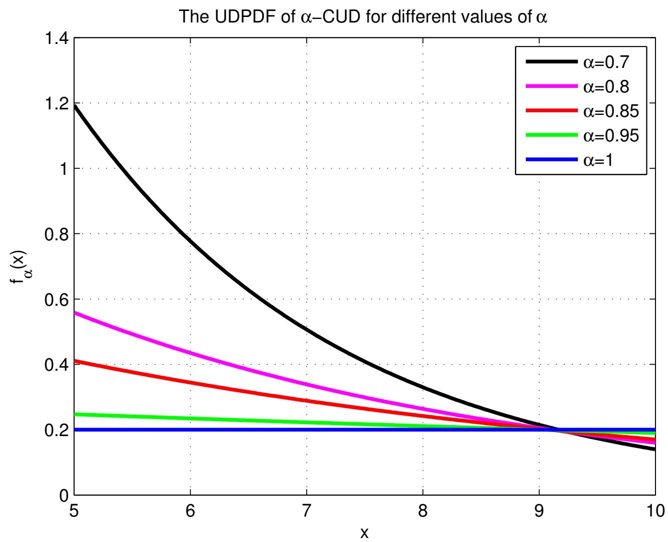

FIGURE 1. The UD fractional probability density function of α-CUD for different values of α.

3.3. **Applications of the UD Fractional Continuous Uniform Distribution (CUD).** In this part, we establish some new applications of fractional calculus on probabilistic random variables, and accordingly we decided to address the UD fractional probability for the α-CUD.

3.3.1. *The UD fractional cumulative distribution function.* For the α-CUD, the UD fractional cumulative distribution function can be obtained by

$$\mathcal{F}_\alpha(x) = e^{\frac{(1-\alpha)}{\alpha}b} e^{\frac{(\alpha-1)}{\alpha}x} \left[\frac{x-a}{b-a}\right]. \tag{3.9}$$

In fact, we have

$$\begin{aligned}
\mathcal{F}_\alpha(x) &= \mathbb{P}_\alpha(X \leq x), \\
&= I_a^\alpha f_\alpha(x), \\
&= \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}x} \int_a^x e^{\frac{(1-\alpha)}{\alpha}t} f_\alpha(t) dt, \\
&= \frac{\alpha e^{\frac{(1-\alpha)}{\alpha}b}}{b-a} \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}x} \int_a^x e^{\frac{(1-\alpha)}{\alpha}t} e^{\frac{(\alpha-1)}{\alpha}t} dt, \\
&= \frac{e^{\frac{(1-\alpha)}{\alpha}b} e^{\frac{(\alpha-1)}{\alpha}x}}{b-a} \int_a^x 1 dt, \\
&= e^{\frac{(1-\alpha)}{\alpha}b} e^{\frac{(\alpha-1)}{\alpha}x} \left[\frac{x-a}{b-a}\right].
\end{aligned}$$

Note that

$$\lim_{\alpha \to 1^-} \mathcal{F}_\alpha(x) = \frac{x-a}{b-a} = \mathcal{F}(x), \tag{3.10}$$

where $\mathcal{F}$ is the classical cumulative distribution function for CUD. Figure 3.3.1 show the UD fractional cumulative distribution function (UDFCDF) of α-CUD under different values of α, and Figure 3.3.1 show the comparison between the UD fractional cumulative distribution function for $\alpha = 1$, and the classical case of the cumulative distribution function for the continuous uniform distribution.

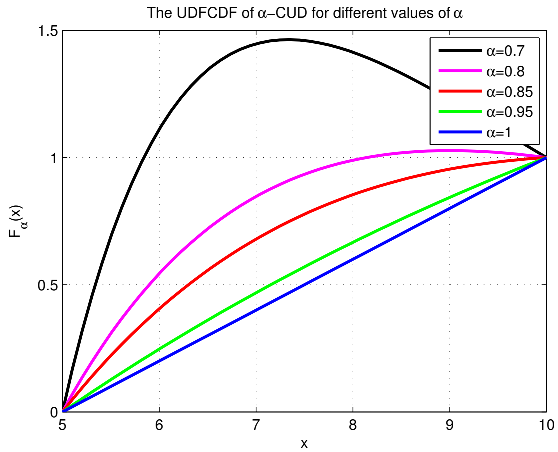

FIGURE 2. The UDFCDF of α-CUD according to different values of α.

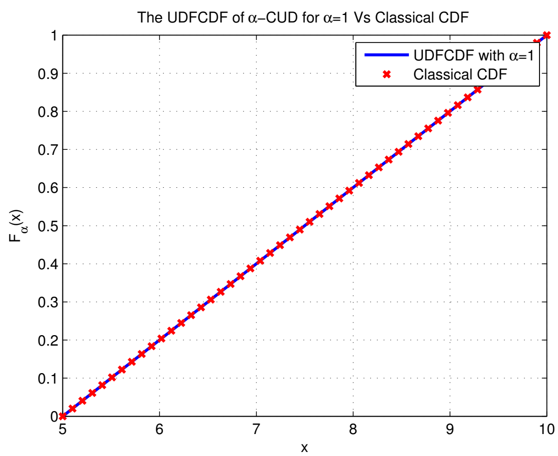

FIGURE 3. A graphical comparison between the UDFCDF of α-CUD for $\alpha = 1$ and the classical CDF.

3.3.2. *The UD fractional survival function.* The UD fractional survival distribution function of the random variable X is given by

$$\begin{aligned} \mathcal{S}_\alpha(x) &= 1 - \mathcal{F}_\alpha(x), \\ &= 1 - e^{\frac{(1-\alpha)}{\alpha}b} e^{\frac{(\alpha-1)}{\alpha}x} \left[\frac{x-a}{b-a}\right]. \\ &= \frac{b-a-e^{\frac{(1-\alpha)}{\alpha}b} e^{\frac{(\alpha-1)}{\alpha}x}\,[x-a]}{b-a}. \end{aligned} \tag{3.11}$$

In particular case, we can have

$$\lim_{\alpha \to 1^-} \mathcal{S}_\alpha(x) = \frac{b-x}{b-a} = \mathcal{S}(x), \tag{3.12}$$

where $\mathcal{S}$ is the classical survival function for CUD.

The fractional survival distribution function UDFSDF for α-CUD can also be plotted by taking different values of α, as illustrated in Figure 4. Then, a comparison between the UD fractional survival distribution

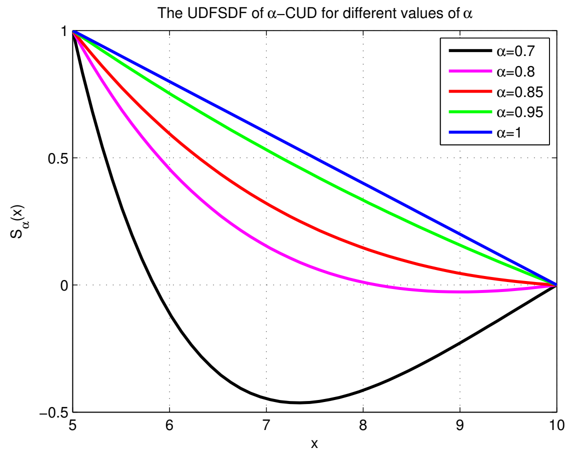

FIGURE 4. The UDFSDF of α-CUD under different values of α.

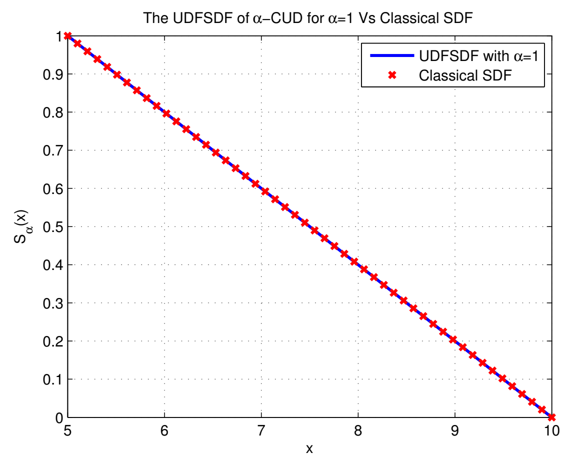

FIGURE 5. A graphical comparison between the UDFSDF of α-CUD for $\alpha = 1$ and the classical SDF.

function for $\alpha = 1$, and the classical case of the survival distribution function for the continuous uniform distribution, is then shown graphically in Figure 5.

3.3.3. *The UD fractional hazard function.* The UD fractional hazard distribution function of X is defined by

$$\begin{aligned} \mathcal{H}_\alpha(x) &= \frac{f_\alpha(x)}{\mathcal{S}_\alpha(x)}, \\ &= \frac{\frac{\alpha e^{\frac{(1-\alpha)}{\alpha}b}}{b-a} e^{\frac{(\alpha-1)}{\alpha}x}}{\frac{b-a-e^{\frac{(1-\alpha)}{\alpha}b} e^{\frac{(\alpha-1)}{\alpha}x}[x-a]}{b-a}}, \\ &= \frac{\alpha e^{\frac{(1-\alpha)}{\alpha}b} e^{\frac{(\alpha-1)}{\alpha}x}}{b-a-e^{\frac{(1-\alpha)}{\alpha}b} e^{\frac{(\alpha-1)}{\alpha}x}\,[x-a]}. \end{aligned} \tag{3.13}$$

If, $\alpha \to 1^-$ in the above formula, then we get the classical hazard function for CUD

$$\lim_{\alpha \to 1^-} \mathcal{H}_\alpha(x) = \frac{1}{b-x} = \mathcal{H}(x). \tag{3.14}$$

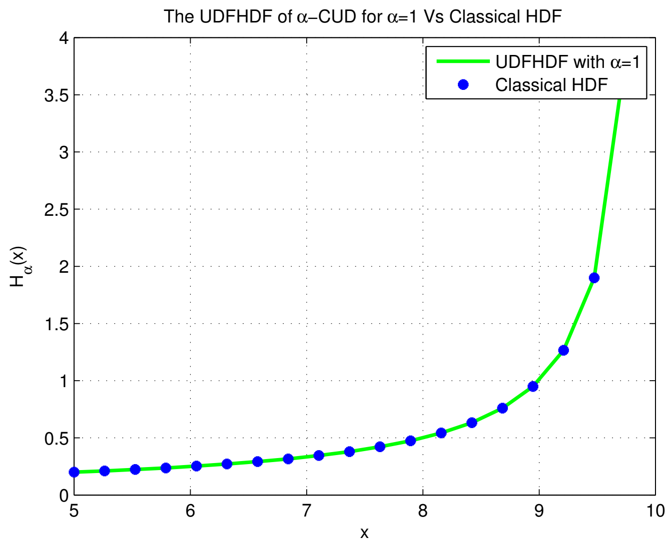

FIGURE 6. A comparison between the UDFHDF of α-CUD for $\alpha = 1$ and the classical HDF

Similarly, a graphical comparison between the UD fractional hazard distribution function UDFHDF for $\alpha = 1$, and the classical case of the hazard distribution function for the continuous uniform distribution, is then illustrated in Figure 6.

3.3.4. *The UD fractional expectation.* According to the Definition 3.1, The UD fractional expectation E_α of a continuous random variable X whose UDFPDF $f_\alpha(x)$ is given by

$$\begin{aligned}
E_\alpha[X] &= I_a^\alpha x f_\alpha(x)|_{x=b}, \\
&= \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}b} \int_a^b e^{\frac{(1-\alpha)}{\alpha}t} t f_\alpha(t)dt, \\
&= \frac{\alpha e^{\frac{(1-\alpha)}{\alpha}b}}{b-a} \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}b} \int_a^b t e^{\frac{(1-\alpha)}{\alpha}t} e^{\frac{(\alpha-1)}{\alpha}t} dt, \\
&= \frac{e^{\frac{(1-\alpha)}{\alpha}b} e^{\frac{(\alpha-1)}{\alpha}b}}{b-a} \int_a^b t dt, \\
&= \frac{1}{b-a} \left[\frac{t^2}{2}\right]_a^b, \\
&= \left[\frac{b^2-a^2}{2(b-a)}\right], \\
&= \left[\frac{b+a}{2}\right] = E(X). \qquad (3.15)
\end{aligned}$$

3.3.5. *The UD fractional moment of orders* (r, α). The UD fractional moment of orders (r, α) denoted by $E_\alpha[X^r]$ of a continuous random variable X whose UDFPDF $f_\alpha(x)$ is given by

$$\begin{aligned}
E_\alpha[X^r] &= I_a^\alpha x^r f_\alpha(x)|_{x=b}, \\
&= \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}b} \int_a^b e^{\frac{(1-\alpha)}{\alpha}t} t^r f_\alpha(t)dt, \\
&= \frac{\alpha e^{\frac{(1-\alpha)}{\alpha}b}}{b-a} \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}b} \int_a^b t^r e^{\frac{(1-\alpha)}{\alpha}t} e^{\frac{(\alpha-1)}{\alpha}t} dt,
\end{aligned}$$

$$= \frac{e^{\frac{(1-\alpha)}{\alpha}b}e^{\frac{(\alpha-1)}{\alpha}b}}{b-a}\int_a^b t^r dt,$$

$$= \frac{1}{b-a}\left[\frac{t^{r+1}}{r+1}\right]_a^b,$$

$$= \left[\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right] = E(X^r). \tag{3.16}$$

For $r = 2$, we have

$$E_\alpha(X^2) = \left[\frac{b^3-a^3}{3(b-a)}\right],$$

$$= \left[\frac{a^2+b^2+ab}{3}\right] = E(X^2). \tag{3.17}$$

3.3.6. *The r^{th}-UD fractional central moment $E_\alpha(X-\mu)^r$.* First, we put

$$\mu = E_\alpha(X) = \left[\frac{b+a}{2}\right].$$

Then, the r^{th} UD fractional central moment $E_\alpha(X-\mu)^r$ of X is defined by

$$E_\alpha(X-\mu)^r = I_a^\alpha (x-\mu)^r f_\alpha(x)|_{x=b}. \tag{3.18}$$

Using the formula (3.18), we can determine the following list of central moments:

1. First central moment

$$E_\alpha(X-\mu) = 0. \tag{3.19}$$

   This is always zero since it represents the mean of the deviations from the mean itself.

2. Second central moment

$$E_\alpha(X-\mu)^2 = \frac{(b-a)^2}{12}. \tag{3.20}$$

3.3.7. *The UD fractional variance.* The UD fractional variance Var_α or σ_α^2 of X is defined by

$$Var_\alpha(X) = E_\alpha(X^2) - E_\alpha^2(X),$$

$$= \left[\frac{a^2+b^2+ab}{3}\right] - \left[\frac{b+a}{2}\right]^2,$$

$$= \frac{(b-a)^2}{12}. \tag{3.21}$$

3.3.8. *The UD fractional standard deviation σ_α.* The UD fractional standard deviation σ_α of X for a α-CUD is given by

$$\sigma_\alpha = \sqrt{Var_\alpha(X)},$$

$$= \sqrt{\frac{(b-a)^2}{12}},$$

$$= \frac{\sqrt{3}(b-a)}{6}. \tag{3.22}$$

3.3.9. *The UD fractional Shannon entropy αH.* For a α-CUD, the UD fractional Shannon entropy αH of X is given by

$$\alpha H(X) = -\frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} f_\alpha(t) \log(f_\alpha(t)) dt. \tag{3.23}$$

First, we calculate the the quantity $\log(f_\alpha(t))$, we have

$$\begin{aligned}
\log(f_\alpha(t)) &= \log\left(\frac{\alpha e^{\frac{(1-\alpha)}{\alpha} b}}{b-a} e^{\frac{(\alpha-1)}{\alpha} t}\right), \\
&= \log\left(\frac{\alpha}{b-a}\right) + \log\left(e^{\frac{(1-\alpha)}{\alpha} b}\right) + \log\left(e^{\frac{(\alpha-1)}{\alpha} t}\right), \\
&= \log\left(\frac{\alpha}{b-a}\right) + \frac{(1-\alpha)}{\alpha} b + \frac{(\alpha-1)}{\alpha} t.
\end{aligned}$$

Next, we calculate $f_\alpha(t)\log(f_\alpha(t))$, we get

$$\begin{aligned}
f_\alpha(t)\log(f_\alpha(t)) &= \left(\frac{\alpha e^{\frac{(1-\alpha)}{\alpha} b}}{b-a} e^{\frac{(\alpha-1)}{\alpha} t}\right)\left[\log\left(\frac{\alpha}{b-a}\right) + \frac{(1-\alpha)}{\alpha} b + \frac{(\alpha-1)}{\alpha} t\right], \\
&= \left[\log\left(\frac{\alpha}{b-a}\right) \frac{\alpha e^{\frac{(1-\alpha)}{\alpha} b}}{b-a} e^{\frac{(\alpha-1)}{\alpha} t}\right] \\
&+ \left[\frac{(1-\alpha)}{\alpha} b \frac{\alpha e^{\frac{(1-\alpha)}{\alpha} b}}{b-a} e^{\frac{(\alpha-1)}{\alpha} t}\right] + \left[\frac{(\alpha-1)}{\alpha} \frac{\alpha e^{\frac{(1-\alpha)}{\alpha} b}}{b-a} t e^{\frac{(\alpha-1)}{\alpha} t}\right].
\end{aligned}$$

Therefore, the UD fractional Shannon entropy αH of X becomes

$$\begin{aligned}
\alpha H(X) &= -\frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha} b} \int_a^b e^{\frac{(1-\alpha)}{\alpha} t} f_\alpha(t) \log(f_\alpha(t)) dt, \\
&= -\log\left(\frac{\alpha}{b-a}\right) - \left(\frac{1-\alpha}{\alpha}\right) b - \left(\frac{\alpha-1}{\alpha}\right)\left(\frac{b+a}{2}\right). \qquad (3.24)
\end{aligned}$$

Note that

$$\begin{aligned}
\lim_{\alpha \to 1^-} \alpha H(X) &= -\log\left(\frac{1}{b-a}\right), \\
&= -\log(1) + \log(b-a), \\
&= \log(b-a) = H(X),
\end{aligned}$$

where H is the classical Shannon entropy of X for the continuous uniform distribution (CUD).

4. Conclusion

In this paper, we have presented some original results on the probability theory. Cumulative distribution, survival, and hazard functions with some graphical representations represent some of the applications for the Continuous Uniform Distribution (CUD) that are developed using the UD fractional derivative and integral to derive the fractional probability density function. Novel notions are also presented, including UD fractional moments, UD fractional variance, and UD fractional expectation. The UD fractional standard deviation and UD fractional Shannon entropy are finally provided.

Competing interests. The authors declare no competing interests.

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