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# BI-STATISTICAL $\gamma$-COVERS CONTROLLED BY A PAIR OF WEIGHT FUNCTIONS IN TOPOLOGY

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ABSTRACT. In this current framework, a new notion called double weighted statistical $\gamma$-cover is being introduced within the context of topological space. This notion extends the idea of a traditional open cover by incorporating two weight functions $g$ and $h$ which allocate non-negative real numbers to element of the cover. This approach allows for more versatile and nuanced coverings, providing a powerful tool for analyzing the structure and properties of topological space, particularly in cases where double statistical $\gamma$ covers may be too restrictive or fail to capture aspects of the space complexity. Furthermore, the application of denseness for two weight functions has been investigated thoroughly. Additionally, explored the concept of product space of double statistical $\gamma$-cover equipped with two weight functions $g$ and $h$.

## 1. INTRODUCTION

The concept of statistical convergence was first encompass by Fast in 1951 that represents a generalization of classical convergence by considering the maximum number of terms in a sequence rather than all of them. In addition to exploring the concept as a summability approach, Schoenberg [21] and Fast [16] provided some fundamental features of statistical convergence for sequences of real numbers. This technique operates under the idea that a sequence converges statistically to a point if the terms that deviate from the point form a set of natural density zero. For any subset $L \subseteq N$, the natural density or asymptotic density of the set $L$ will be represented by

$$\delta(L) = \lim_{n\to\infty} \frac{|\{k \leq n : k \in L\}|}{n}.$$

In 2008, Maio and Kočinac [18] extended this concept of summability technique of statistical convergence in topological spaces. In a topological space $X$, a sequence $\{p_n : n \in \mathbb{N}\}$ is said to be statistically convergent to $p \in X$ if the natural density of the set $\{n \in \mathbb{N} : p_n \notin \mathbb{N}\}$ is zero for every open neighborhood $U$ of zero. This concept bridges statistical convergence with the topological notion of proximity to a point. Maio and Kočinac also introduced statistical $\gamma$-covers, which generalize certain covering properties in topology. More related work on the concept of covering properties can be found in [2,3,6,7,9–11,13,14,17,20].

Alfred Pringsheim, a German mathematician, initially proposed the concept of the convergence of double sequences in 1900. In essence, a double sequence is a function $a : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ (or to another set) that, for every pair of natural numbers $(m, n)$, assigns a real number $a_{mn}$. He specifically investigated the behavior of such double sequences when both indices $m$ and $n$ tend to infinity, as well as the limits of sequences of

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sequences. According to Pringsheim, a double sequence $a_{mn}$ converges to a limit $L$ if, for each $\epsilon > 0$, there exists an integer $n_0$ such that, for every $m, n \geq n_0$, the inequality $|a_{mn} - L| < \epsilon$ holds.

The idea of statistical convergence was expanded by Mursaleen [19] to include double sequences. Statistical convergence was initially presented for single sequences but was later extended to double sequences. It generalizes the concept of convergence by using density limits instead of pointwise limits. For double sequences $x_{mn}$, where $m, n$ are indices from $\mathbb{N} \times \mathbb{N}$, Mursaleen expanded on this concept which proposed a double density method of statistical convergence for double sequences, which requires that the density of the set of pairs $(m, n)$ where $|x_{mn} - L| \geq \epsilon$ is 0. Because it enables a more comprehensive knowledge of convergence behavior in several dimensions without depending on rigid pointwise constraints, this generalization has found important applications in summability theory, approximation theory, and functional analysis.

In 2012, Bhunia et.al [8] introduced a refinement based on an upper bound on the asymptotic density of the exceptional set of indices, strengthening the idea of statistical convergence (also known as s-convergence) for real sequences by a parameter $\alpha$ where $0 < \alpha < 1$. The same is true for $s_g$-convergence, which Das et al. [4] developed using weight function $g$. This is a generalization of statistical convergence, adapted to fit more complex structures like topological spaces. Two different functions are included in this concept, acting as weights that regulates from natural numeber to non-negative real numbers. The weights $g$ and $h$ in $s_{g,h}$ -convergence enable a more adaptable measure of convergence, where the functions can alter how we handle various segments of a sequence or space.

Following the work of [1, 2, 4, 6, 12, 19] in this study, we will present the concept of $s_{g,h}$-$\gamma$ cover by integrating the ideas of double statistical $\gamma$-cover and $s_{g,h}$ convergence.

## 2. PRELIMINARIES

The fundamental ideas and symbols needed to comprehend double weighted statistical $\gamma$-cover in topology are presented in this section. In this work, no separation axiom has been taken for granted unless otherwise stated. We provide an overview of the necessary topological terms, symbols, and fundamental concepts for reader's convenience. For a detailed discussion of common symbols and concepts, we refer readers to [15].

**Definition 2.1.** [17] A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is called $\gamma$ cover if for every $x \in X$, $\{n \in \mathbb{N} : x \notin U_n\}$ is finite .

**Definition 2.2.** [18] A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is called an statistical $\gamma$ cover if for each $x \in X$, the set $\{n \in \mathbb{N} : x \notin U_n\}$ has natural density 0.

**Definition 2.3.** [5] A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of $X$ is called $s^\alpha$-$\gamma$ cover if for every $x \in X$, $\delta^\alpha(\{n \in \mathbb{N} : x \notin U_n\})$=0.

**Definition 2.4.** [4] A countable open cover $\mathcal{W} = \{w_n : n \in \mathbb{N}\}$ of a topological space $X$ is said to be $s_g$-$\gamma$ cover if for each $x \in X$, the set $\{n \in \mathbb{N} : x \notin W_n\}$ has $g$-weighted density 0 i.e. $\delta_g(\{n \in \mathbb{N} : x \notin W_n\}) = 0$.

**Definition 2.5.** [5] A subset $\mathcal{V}$ of a cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a space $X$ is $s^\alpha$ dense in $\mathcal{U}$ if the set of indices of elements of $\mathcal{V}$ has $s^\alpha$ density infinite($\infty$).

**Definition 2.6.** [12] A subset $\mathcal{W} = \{V_{n_k} : k \in \mathbb{N}\}$ of an open cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ of a topological space $(X, \tau)$ is said to be $s_g$-dense in $\mathcal{V}$ if the set $\{n_k \leq n : V_{n_k} \in \mathcal{W}\}$ has g-weighted density non-zero such that

$$\lim_{n\to\infty} \frac{g(|\{n_k \leq n : V_{n_k} \in \mathcal{W}\}|)}{g(n)} \neq 0.$$

The weight functions $g, h : \mathbb{N} \longrightarrow [0, \infty)$ defined as $g(m) = \ln(1 + m)$ and $g(m) = m^{\alpha}, 0 < \alpha < 1$; and $h(n) = \ln(1 + n)$ and $h(n) = n^{\alpha}, 0 < \alpha < 1$, are readily verifiable. This research will generate a number of several examples using these weight functions.

## 3. $s_{g,h}$-$\gamma$ COVER

In this study, we attempt to demonstrate a new countable covering notion characterized by two weight functions, $g$ and $h$ that lies between double $\gamma$ covers and double $s$-$\gamma$ covers. Thus, we examine the following covering qualities and incorporate the idea of double weighted density.
**Notations:** $s_{g,h}$-$\Gamma$ denotes the collection of all $s_{g,h}$-$\gamma$ cover of a topological space $(X, \tau)$.

**Definition 3.1.** In a topological space $(X, \tau)$, a countable open cover $\mathcal{W} = \{W_{p,q} : (p, q) \in \mathbb{N} \times \mathbb{N}\}$ will be called double weighted statistical $\gamma$ cover (or shortly $s_{g,h}$-$\gamma$ cover) if for every $x \in X$, $\delta_{g,h}(\{(p, q) \in \mathbb{N} \times \mathbb{N} : x \notin W_{p,q}\}) = 0$.

**Theorem 3.2.** $\Gamma \subseteq s_{g,h} - \Gamma \subseteq s - \Gamma$.

*Proof.* Let $\mathcal{W} = \{W_{p,q} : (p, q) \in \mathbb{N} \times \mathbb{N}\} \in \Gamma$. Therefore $\delta_{g,h}(\{(p, q) \in \mathbb{N} \times \mathbb{N} : x \notin W_{p,q}\}) = 0$ for all $x \in X$. Since for every $x \in X$ the set $\{(p, q) \in \mathbb{N} \times \mathbb{N} : x \notin W_{p,q}\}$ is finite that implies $\mathcal{W} \in s_{g,h} - \Gamma$. Thus $\Gamma \subseteq s_{g,h} - \Gamma$. Here, $\mathcal{W} = \{W_{p,q} : (p, q) \in \mathbb{N} \times \mathbb{N}\} \in s_{g,h} - \Gamma$ with $\delta_{g,h}(\{(p, q) \in \mathbb{N} \times \mathbb{N} : x \notin W_{p,q}\}) = 0$ for all $x \in X$. Clearly, that implies $\delta(\{(p, q) \in \mathbb{N} \times \mathbb{N} : x \notin W_{p,q}\}) = 0$ for all $x \in X$. i.e., $\mathcal{W} \in s - \Gamma$. Thus $s_{g,h} - \Gamma \subseteq s - \Gamma$. Hence the theorem. □

**Example 3.3.** But the converse may not guaranteed to be true. There exist an $s - \gamma$ cover that is not an $s_{g,h} - \gamma$ cover. Also there exist an $s_{g,h} - \gamma$ cover that is not a $\gamma$ cover.
Let $(X, \tau)$ be a topological space where $X = \{p, q, r\}$ and $\tau = \{\emptyset, X, \{p, q\}, \{r\}\}$. Considering two weight functions $g(m) = \ln(1 + m)$ and $h(n) = \{ln(1 + n)\}$ and the cover $\mathcal{W} = \{W_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ such that

$$W_{m,n} = \begin{cases} \{q, r\}, & \text{if } m \in \{i^3 : i \in \mathbb{N}\} \text{ and } n \in \{j^4 : j \in \mathbb{N}\} \\ X, & \text{otherwise} \end{cases}$$

For every $x \in X$, the double density of the set $\{(m, n) \in \mathbb{N} \times \mathbb{N} : x \notin W_{m,n}\}$ is equal to zero. i.e., $\delta(\{(m, n) \in \mathbb{N} \times \mathbb{N} : x \notin W_{m,n}\}) = 0$. Therefore, $\mathcal{W}$ is double $s - \gamma$ cover.
But for $p \in X$,

$$\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : p \notin W_{m,n}\}) = \delta_{g,h}(\{(1, 1), (8, 16), (27, 81), (64, 256), \cdots\})$$

$$= \lim_{m,n \to \infty} \frac{mn}{\ln(1 + m^3) \ln(1 + n^4)} \neq 0.$$

So for every $x \in X$, $\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : x \notin W_{m,n}\}) \neq 0$. Therefore $\mathcal{W}$ is not an $s_{g,h} - \gamma$ cover.
Consider another cover $\mathcal{V} = \{V_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ such that

$$V_{m,n} = \begin{cases} \{p, q\}, & \text{if } m \in \{i^i : i \in \mathbb{N}\} \text{ and } n \in \{j^j : j \in \mathbb{N}\} \\ X, & \text{otherwise} \end{cases}$$

For every $x \in X$, $\delta_{g.h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : x \notin V_{m,n}\}) = 0$. Therefore, $\mathcal{V}$ is an $s_{g,h} - \gamma$ cover. But the set $\{(m, n) \in \mathbb{N} \times \mathbb{N} : x \notin V_{m,n}\} = \{(1, 1), (4, 4), (27, 27), (256, 256), \cdots\}$ is not finite. Thus $\mathcal{V}$ is not an double $\gamma$ cover.
Hence, $\Gamma \not\supseteq s_{g,h} - \Gamma \not\supseteq s - \Gamma$.

**Example 3.4.** Sub cover of an $s_{g,h}-\gamma$ cover may not be an $s_{g,h}-\gamma$ cover.
Let $X=\{0,1\}$ and $\tau=\{\emptyset, X, \{1\}\}$. Then $(X,\tau)$ is a topological space. Considering weight functions $g(m)=\ln(1+m)$ and $h(n)=\ln(1+n)$ and the cover $\mathcal{W}=\{W_{m,n}:(m,n)\in\mathbb{N}\times\mathbb{N}\}$ where,

$$W_{m,n}=\begin{cases}\{1\}, & \text{if } \{m=i^i : i\in\mathbb{N}\} \text{ and } \{n=j^j : j\in\mathbb{N}\}\\ X, & \text{otherwise}\end{cases}$$

Here for every $x\in X$,

$$\delta_{g,h}(\{(m,n)\in\mathbb{N}\times\mathbb{N}: x\notin W_{m,n}\})=\delta_{g,h}(\{(1,1),(4,4),(27,27),(256,256),\cdots\})$$
$$=\lim_{m,n\to\infty}=\frac{mn}{\ln(1+m^m)\ln(1+n^n)}=0.$$

Therefore $\mathcal{W}$ is an $s_{g,h}$-$\gamma$ cover.
Consider the sub cover $\mathcal{V}=\{V_{m,n}:(m,n)\in\mathbb{N}\times\mathbb{N}\}$ of $\mathcal{W}=\{W_{m,n}:(m,n)\in\mathbb{N}\times\mathbb{N}\}$ such that

$$V_{m,n}=\begin{cases}W_{2,3}, & \text{if } m=2, n=3\\ W_{m^m,n^n}, & \text{otherwise}\end{cases}$$

Here for $0\in X$, $\delta_{g,h}(\{(m,n)\in\mathbb{N}\times\mathbb{N}: 0\notin V_{m,n}\})=\delta_{g,h}(\{(1,1),(256,256),\cdots\})\neq 0$.
Therefore, $\mathcal{V}$ is not an $s_{g,h}$-$\gamma$ cover.

**Theorem 3.5.** *In a topological space $(X,\tau)$, union of two $s_{g,h}$-$\gamma$ covers is also an $s_{g,h}$-$\gamma$ cover.*

*Proof.* Let $\mathcal{A}=\{A_{m,n}:(m,n)\in\mathbb{N}\times\mathbb{N}\}$ and $\mathcal{B}=\{B_{m,n}:(m,n)\in\mathbb{N}\times\mathbb{N}\}$ be two $s_{g,h}$-$\gamma$ covers of topological space $(X,\tau)$. Therefore for every $x\in X$, $\delta_{g,h}(\{(m,n)\in\mathbb{N}\times\mathbb{N}: x\notin A_{m,n}\})=0$ and $\delta_{g,h}(\{(m,n)\in\mathbb{N}\times\mathbb{N}: x\notin B_{m,n}\})=0$.
But $A_{m,n}\cup B_{m,n}\supseteq A_{m,n}$ for all $(m,n)\in\mathbb{N}\times\mathbb{N}$.
Now for every $x\in X$, $\{(m,n)\in\mathbb{N}\times\mathbb{N}: x\notin A_{m,n}\cup B_{m,n}\}\subseteq\{(m,n)\in\mathbb{N}\times\mathbb{N}: x\notin A_{m,n}\}$
that implies

$$\delta_{g,h}(\{(m,n)\in\mathbb{N}\times\mathbb{N}: x\notin A_{m,n}\cup B_{m,n}\})\leq\delta_{g,h}(\{(m,n)\in\mathbb{N}\times\mathbb{N}: x\notin A_{m,n}\})=0$$

for every $x\in X$.
Thus $\delta_{g,h}(\{(m,n)\in\mathbb{N}\times\mathbb{N}: x\notin A_{m,n}\cup B_{m,n}\})=0$ for every $x\in X$.
Hence $\mathcal{A}\sqcup\mathcal{B}=\{A_{m,n}\cup B_{m,n}: A_{m,n}\in\mathcal{A} \text{ and } B_{m,n}\in\mathcal{B} \text{ and } (m,n)\in\mathbb{N}\times\mathbb{N}\}$ is also an $s_{g,h}$-$\gamma$ cover of $(X,\tau)$. □

**Theorem 3.6.** *If $(B,\tau_B)$ is a subspace topological space of $(X,\tau)$ and $\mathcal{W}=\{W_{p,q}:(p,q)\in\mathbb{N}\times\mathbb{N}\}$ is an $s_{g,h}$-$\gamma$ cover of $(X,\tau)$. Then $\mathcal{W}_B=\{B\cap W_{p,q}: W_{p,q}\in\mathcal{W}$ and $(p,q)\in\mathbb{N}\times\mathbb{N}\}$ is an $s_{g,h}$-$\gamma$ cover of $(B,\tau_B)$.*

*Proof.* Arbitrarily, let $x\in B\subseteq X$ and $\delta_{g,h}(\{(p,q)\in\mathbb{N}\times\mathbb{N}: x\notin W_{p,q}\})=0$.
Now, $(m,n)\in\{(p,q)\in\mathbb{N}\times\mathbb{N}: x\notin W_{p,q}\}$ if and only if $x\notin W_{m,n}$ if and only if $x\notin W_{m,n}\cap B$ if and only if $(m,n)\in\{(p,q)\in\mathbb{N}\times\mathbb{N}: x\notin W_{p,q}\cap B\}$. Therefore for all $x\in B$, $\{(p,q)\in\mathbb{N}\times\mathbb{N}: x\notin W_{p,q}\}=\{(p,q)\in\mathbb{N}\times\mathbb{N}: x\notin W_{p,q}\cap B\}$. i.e.,

$$0=\delta_{g,h}(\{(p,q)\in\mathbb{N}\times\mathbb{N}: x\notin W_{p,q}\})=\delta_{g,h}(\{(p,q)\in\mathbb{N}\times\mathbb{N}: x\notin W_{p,q}\cap B\})$$

for all $x\in B$.
Thus $\delta_{g,h}(\{(p,q)\in\mathbb{N}\times\mathbb{N}: x\notin W_{p,q}\cap B\})=0$ for all $x\in B$.
Hence $\mathcal{W}_B=\{B\cap W_{p,q}: W_{p,q}\in\mathcal{W}$ and $(p,q)\in\mathbb{N}\times\mathbb{N}\}$ is also an $s_{g,h}$-$\gamma$ cover of $(B,\tau_B)$. □

**Theorem 3.7.** *Let $(X_m, \tau_{X_m})$ be finite topological sub spaces of $(X, \tau)$ with $X = X_1 \cup X_2 \cup \cdots \cup X_k$ and $\mathcal{P} = \{P_{i,j} : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ be a countable open cover of $(X, \tau)$. If $\mathcal{P}_{X_m} = \{X_m \cap P_{i,j} : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ is an $s_{g,h}$-$\gamma$ cover of $(X_m, \tau_{X_m})$ for $m = 1, 2, \cdots, k$. Then $\mathcal{P}$ is also an $s_{g,h}$-$\gamma$ cover of $(X, \tau)$.*

*Proof.* Let $s$ be any arbitrary element of $X$. Therefore it is obvious that $s \in \{X_1 \cup X_2 \cup \cdots \cup X_k\}$, so there exists one $k_0 \in \{1, 2, \cdots, k\}$ such that $s \in X_{k_0}$. For $m = 1, 2, \cdots, k$, $\mathcal{P}_{X_m}$ is an $s_{g,h}$-$\gamma$ cover of $(X_m, \tau_{X_m})$.
Therefore, $\delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : s \notin X_{k_0} \cap P_{i,j}\}) = 0$ for all $s \in X$. But $X_{k_0} \cap P_{i,j} \subseteq P_{i,j}$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$.
So, that implies $\{(i,j) \in \mathbb{N} \times \mathbb{N} : s \notin X_{k_0} \cap P_{i,j}\} \supseteq \{(i,j) \in \mathbb{N} \times \mathbb{N} : s \notin P_{i,j}\}$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. i.e.,

$$0 = \delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : s \notin X_{k_0} \cap P_{i,j}\}) \geq \delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : s \notin P_{i,j}\})$$

for all $s \in X$.
Thus $\delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : s \notin P_{i,j}\}) = 0$ for all $s \in X$.
Hence $\mathcal{P}$ is also an $s_{g,h}$-$\gamma$ cover of $(X, \tau)$. □

**Theorem 3.8.** *Let $\mathcal{W} = \{W_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$ be an $s_{g,h}$-$\gamma$ cover of a topological space $(X, \tau)$ if and only if for every $D \subseteq X$, $D \setminus W_{i,j}$ is finite for every $(i,j) \in \{(m,n) \in \mathbb{N} \times \mathbb{N} : D \nsubseteq W_{m,n}\}$ that implies $\delta_{g,h}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : D \nsubseteq W_{m,n}\}) = 0$.*

*Proof.* Let $\mathcal{W} = \{W_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$ be an $s_{g,h}$-$\gamma$ cover of $(X, \tau)$ and for every $D \subseteq X$, $D \setminus W_{i,j}$ is finite for every $(i,j) \in \{(m,n) \in \mathbb{N} \times \mathbb{N} : D \nsubseteq W_{m,n}\}$.
Also, let $(p,q) \in \{(m,n) \in \mathbb{N} \times \mathbb{N} : D \nsubseteq W_{m,n}\}$ then $D \subseteq W_{p,q}$ i.e., $D \setminus W_{p,q} \neq \emptyset$.
Let $x \in D \setminus W_{p,q}$ so, $x \in D$ and $x \notin W_{p,q}$ which implies $(p,q) \in \{(m,n) \in \mathbb{N} \times \mathbb{N} : x \notin W_{m,n}\}$ for all $x \in D \setminus W_{p,q}$.
Now, $\{(m,n) \in \mathbb{N} \times \mathbb{N} : D \nsubseteq W_{m,n}\} \subseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} : x \notin W_{m,n}\}$ for all $x \in D \setminus W_{p,q}$. i.e.,

$$\delta_{g,h}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : D \nsubseteq W_{m,n}\}) \leq \delta_{g,h}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : x \notin W_{m,n}\}) = 0$$

for all $x \in D \setminus W_{p,q}$.
Thus $\delta_{g,h}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : D \nsubseteq W_{m,n}\}) = 0$.
Conversely, let for every $D \subseteq X$, $D \setminus W_{i,j}$ is finite for every $(i,j) \in \{(m,n) \in \mathbb{N} \times \mathbb{N} : D \nsubseteq W_{m,n}\}$ implies that $\delta_{g,h}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : D \nsubseteq W_{m,n}\}) = 0$.
Now, let $x \in X$, then for every $\{x\} \subseteq X$, $\{x\} \setminus W_{i,j}$ is finite for all $(i,j) \in \{(m,n) \in \mathbb{N} \times \mathbb{N} : \{x\} \nsubseteq W_{m,n}\}$.
Clearly, $\delta_{g,h}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \{x\} \nsubseteq W_{m,n}\}) = 0$ i.e., $\delta_{g,h}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : x \notin W_{m,n}\}) = 0$ for all $x \in X$.
Hence $\mathcal{W}$ is an $s_{g,h}$-$\gamma$ cover of $(X, \tau)$. □

**Example 3.9.** Infinite subset of an $s_{g,h}$-$\gamma$ cover may not be an $s_{g,h}$-$\gamma$ cover.
Let $X = [0, \infty)$ be the set induced with topology $\tau = \{[0, n) : n \in \mathbb{N}\} \cup \{\emptyset, X\}$. Then $(X, \tau)$ is a topological space.
Let $\mathcal{A} = \{A_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ be a countable open cover of $(X, \tau)$ such that

$$A_{p,q} = \begin{cases} [p+q, \infty), & \text{if } \{p = i^i : i \in \mathbb{N}\} \text{ and } \{q = j^j : j \in \mathbb{N}\} \\ X, & \text{if otherwise} \end{cases}$$

Arbitrarily, let $x \in X$, then $M = \{(p,q) \in \mathbb{N} \times \mathbb{N} : x \notin A_{p,q}\}$ and $N = \{(i^i, j^j) : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ so, it is obvious that for any $x \in X$, $M \subseteq N$. As $\delta_{g,h}(N) = 0$ for all $x \in X$ so, $\delta_{g,h}(M) = 0$ for all $x \in X$. Thus $\mathcal{A}$ is an $s_{g,h}$-$\gamma$ cover.
Now considering an infinite sub sequence $\mathcal{B} = \{B_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ of $\mathcal{A}$ such that

$$B_{p,q} = \begin{cases} A_{2,3}, & \text{if } p = q = 1 \\ A_{p^p, q^q} = [p^p + q^q, \infty), & \text{otherwise} \end{cases}$$

Now for $0 \in X$, $\{(p,q) \in \mathbb{N}\times\mathbb{N} : 0 \notin A_{p,q}\} = \{(i^i, j^j) : (i,j) \in \mathbb{N}\times\mathbb{N}\} \setminus \{1,1\} = D$(say). But $\delta_{g,h}(D) \neq 0$. So, $\mathcal{B}$ is not an $s_{g,h}$-$\gamma$ cover.

**Definition 3.10.** A subset $\mathcal{S} = \{V_{m_i,n_j} : (i,j) \in \mathbb{N}\times\mathbb{N}\}$ of an open cover $\mathcal{V} = \{V_{m,n} : (m,n) \in \mathbb{N}\times\mathbb{N}\}$ of a topological space $(X,\tau)$ is said to be $s_{g,h}$-dense in $\mathcal{V}$ if the index set $\{(m_i,n_j) \leq (m,n) : V_{m_i,n_j} \in \mathcal{S}\}$ has double weighted density non-zero such that

$$\lim_{m,n\to\infty} \frac{g(|\{m_i \leq m : V_{m_i,n_j} \in \mathcal{S}\}|)h(|\{n_j \leq n : V_{m_i,n_j} \in \mathcal{S}\}|)}{g(m)h(n)} = pq(\neq 0)$$

provided that the limit exists.

**Theorem 3.11.** *$s_{g,h}$-dense subset of an $s_{g,h}$-$\gamma$ cover is an $s_{g,h}$-$\gamma$ cover.*

*Proof.* Let $\mathcal{S} = \{V_{m_i,n_j} : (i,j) \in \mathbb{N}\times\mathbb{N}\}$ be an $s_{g,h}$-dense subset of an $s_{g,h}$-$\gamma$ cover $\mathcal{V} = \{V_{m,n} : (m,n) \in \mathbb{N}\times\mathbb{N}\}$. Then for any point $t \in X$, $\delta_{g,h}(\{(p,q) \in \mathbb{N}\times\mathbb{N} : t \notin V_{m,n}\}) = 0$ which can be written as

$$\lim_{p,q\to\infty} \frac{|\{(p,q) \leq (k,l) : t \notin V_{p,q}\}|}{g(p)h(q)} = 0.$$

Also,

$$\lim_{m,n\to\infty} \frac{g(|\{m_i \leq m : V_{m_i,n_j} \in \mathcal{S}\}|)h(|\{n_j \leq n : V_{m_i,n_j} \in \mathcal{S}\}|)}{g(m)h(n)} = pq(\neq 0).$$

$$\begin{aligned}
\text{Now, } \lim_{p,q\to\infty} \frac{|\{(p,q) \leq (k,l) : t \notin V_{p,q}\}|}{g(p)h(q)} &\leq \lim_{p,q\to\infty} \frac{|\{(p,q) \leq (m_i,n_j) : t \notin V_{m_i,n_j}\}|}{g(p)h(q)} \\
&= \lim_{p,q\to\infty} \frac{\frac{|\{(p,q)\leq(m_i,n_j):t\notin V_{m_i,n_j}\}|}{g(m)h(n)}}{\frac{g(p)h(q)}{g(m)h(n)}} \\
&= \lim_{p,q\to\infty} \frac{\frac{|\{(p,q)\leq(m_i,n_j):t\notin V_{m_i,n_j}\}|}{g(m)h(n)}}{\frac{g(|\{m_i\leq m:V_{m_i,n_j}\in\mathcal{S}\}|)h(|\{n_j\leq n:V_{m_i,n_j}\in\mathcal{S}\}|)}{g(m)h(n)}} \\
&= \frac{0}{pq(\neq 0)} = 0.
\end{aligned}$$

Thus, $\delta_{g,h}(\{(p,q) \in \mathbb{N}\times\mathbb{N} : t \notin V_{m_i,n_j}\}) = 0$.
Hence $\mathcal{S}$ is also an $s_{g,h}$-$\gamma$ cover. □

**Theorem 3.12.** *If $\mathcal{V} = \{W_p \times S_q : W_p \in \mathcal{W};\ S_q \in \mathcal{S} \text{ and } (p,q) \in \mathbb{N}\times\mathbb{N}\}$ is an $s_{g,h}$-$\gamma$ cover in the product space $X \times Y$. Then either $\mathcal{W} = \{W_p : p \in \mathbb{N}\}$ is $s_g$-$\gamma$ cover or $\mathcal{S} = \{S_q : q \in \mathbb{N}\}$ is $s_h$-$\gamma$ cover in their respective topological spaces.*

*Proof.* Let $(x,y)$ be any arbitrary element in the product space $X \times Y$ and consider $(r,s) \in \{p \in \mathbb{N} : x \notin W_p\} \times \{q \in \mathbb{N} : y \notin S_q\}$ which implies $x \notin W_r$ and $y \notin S_s$.
Therefore, $(x,y) \in W_r \times S_s$ which will give $(r,s) \in \{(p,q) \in \mathbb{N}\times\mathbb{N} : (x,y) \notin W_p \times S_q\}$
Now, $\{p \in \mathbb{N} : x \notin W_p\} \times \{q \in \mathbb{N} : y \notin S_q\} \subseteq \{(p,q) \in \mathbb{N}\times\mathbb{N} : (x,y) \notin W_p \times S_q\}$ that implies

$$\begin{aligned}
&\lim_{p,q\to\infty} \frac{|\{p \in \mathbb{N} : x \notin W_p\}| \times |\{q \in \mathbb{N} : y \notin S_q\}|}{g(p)h(q)} \\
&\leq \lim_{p,q\to\infty} \frac{|\{(p,q) \in \mathbb{N}\times\mathbb{N} : (x,y) \notin W_p \times S_q\}|}{g(p)h(q)} = 0 \\
&\leq \lim_{p\to\infty} \frac{|\{p \in \mathbb{N} : x \notin W_p\}|}{g(p)} \times \lim_{q\to\infty} \frac{|\{q \in \mathbb{N} : y \notin S_q\}|}{h(q)} = 0 \\
&\leq \delta_g(W_p) \times \delta_h(S_q) = 0.
\end{aligned}$$

So, either $\delta_g(W_p) = 0$ or $\delta_h(S_q) = 0$.
Hence the theorem. □

## 4. CONCLUSION

This concept of double weighted statistical $\gamma$-cover in topology extends the idea of statistical convergence to cover functions with respect to a double sequence of the collection of open covers under two weight functions $g$ and $h$. This work examines the interrelationship among several variations of $\gamma$-covers; finite union of $s_{g,h}$-$\gamma$ cover also results in another $s_{g,h}$-$\gamma$ cover; an $s_{g,h}$–$\gamma$ covering property is preserved under the sub-space topology; and also examine the idea of product space. The application of a $\gamma$-cover adds further flexibility, enabling the examination of localized or specific topological behaviors influenced by the weighting scheme.

**Competing interests.** The authors declare no competing interests.

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