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MIXED C-COSINE OF CONTINUOUS LINEAR OPERATORS ON COMPLEX p-BANACH SPACES

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ABSTRACT. In the present work, we define and examine the mixed C-cosine families of continuous linear operators on complex p-Banach spaces where $0 < p < 1$. Finally, we will demonstrate several results on the mixed C-cosine families of continuous linear operators on complex p-Banach spaces where $0 < p < 1$. Finally, we give an application related to the second order abstract Cauchy problem.

1. INTRODUCTION AND PRELIMINARIES

S. G. Gal and J. A. Goldstein [3] started the study of semigroups of continuous linear operators on complex p-Banach spaces where $0 < p < 1$. They presented several applications of the results in [3]. There are numerous studies on p-Banach spaces, see, e.g., [1,–9, 11].

In this article, we define and study the mixed C-cosine families of continuous linear operators on p-Banach spaces where p is a real number such that $0 < p < 1$. We prove several results related to this class of operators. We recall a few preliminaries.

Definition 1.1. [1, 7, 11] Let $\mathcal{F}$ be a vector space over $\mathbb{K}$ (=$\mathbb{R}$ or $\mathbb{C}$) and $0 < q \leq 1$ be a fixed real number. A q-norm on $\mathcal{F}$ is a function $\|\cdot\|_q : \mathcal{F} \to \mathbb{R}_+$ such that

(i) For all $u \in \mathcal{F}$, $\|u\|_q = 0$ if and only if $u = 0$;
(ii) For any $u \in \mathcal{F}$ and $b \in \mathbb{K}$, $\|bu\|_q = |b|^q \|u\|_q$;
(iii) For each $u, w \in \mathcal{F}$, $\|u + w\|_q \leq \|u\|_q + \|w\|_q$.

The pair $(\mathcal{F}, \|\cdot\|_q)$ is called q-normed space.

Definition 1.2. [1] A complete q-normed space will be called a q-Banach space.

Definition 1.3. [1] Let $(\mathcal{F}, \|\cdot\|_q)$ be a q-Banach space.

(i) A sequence $(u_n) \subset \mathcal{F}$ is called converge to $u \in \mathcal{F}$, designated by $\lim_{n\to\infty} u_n = u$, if $\|u_n - u\|_q \to 0$ as $n \to \infty$.
(ii) A sequence $(u_n) \subset \mathcal{F}$ is called Cauchy sequence, if $\|u_n - u_m\|_q \to 0$ as $m, n \to \infty$.
(iii) A subset $M \subseteq \mathcal{F}$ is called complete if all Cauchy sequence in M converges in M.
(iv) A subset $M \subseteq \mathcal{F}$ is called bounded if there exists a real number $k > 0$ such that $\|u\|_q \leq k$ for any $u \in M$.

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Definition 1.4. [8] Let $(\mathcal{F}, \|\cdot\|_q)$ be a q-normed space and let S be a linear operator on $\mathcal{F}$. The operator S is continuous if and only if there exists a constant $M > 0$ such that

$$\|Su\|_q \leq M\|u\|_q, \tag{1.1}$$

for all $u \in \mathcal{F}$. The collection $\mathcal{L}(\mathcal{F})$ denotes the set of any continuous linear operators on $\mathcal{F}$.

Definition 1.5. [8] Let $(\mathcal{F}, \|\cdot\|_q)$ be a q-normed space. Let $S \in \mathcal{L}(\mathcal{F})$, we define

$$\|S\|_q = \sup_{u\in\mathcal{F}\setminus\{0\}} \frac{\|Su\|_q}{\|u\|_q}.$$

So, $(\mathcal{L}(\mathcal{F}), \|\cdot\|_q)$ is a quasi-normed space.

Remark 1.6. [8] Let $(\mathcal{F}, \|\cdot\|_q)$ be a q-normed space. If $S \in \mathcal{L}(\mathcal{F})$, then

$$\|S\|_q = \sup_{u\in\mathcal{F}:\|u\|_q=1} \|Su\|_q.$$

Definition 1.7. Let $(\mathcal{F}, \|\cdot\|_q)$ be a q-normed space. Let $S \in \mathcal{L}(\mathcal{F})$, we have

(i) S is called one-to-one if $N(S) = \{0\}$.
(ii) S is called onto if $R(S) = \mathcal{F}$.
(iii) S is called invertible if S is one-to-one and onto.

Definition 1.8. Let $(\mathcal{F}, \|\cdot\|_q)$ be a q-normed space and $S \in \mathcal{L}(\mathcal{F})$. If S is invertible, then there exists an unique continuous linear operator denoted $S^{-1} : \mathcal{F} \to \mathcal{F}$ called the inverse of S such that $SS^{-1} = S^{-1}S = I_{\mathcal{F}}$, where $I_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is the identity operator.

Proposition 1.9. *Let $(\mathcal{F}, \|\cdot\|_q)$ be a complex q-normed space.*

(i) *If $S, B \in \mathcal{L}(\mathcal{F})$ and $b \in \mathbb{C}$, then $S + bB \in \mathcal{L}(\mathcal{F})$.*
(ii) *If $(\mathcal{F}, \|\cdot\|_q)$ is complete, then $(\mathcal{L}(\mathcal{F}), \|\cdot\|_q)$ is a complete quasi-normed space.*

Definition 1.10. [3] Let $\mathcal{F}$ be a q-Banach space. A family $(J(t))_{t\geq 0}$ of continuous linear operators on $\mathcal{F}$ is said to be semigroup if

(i) $J(0) = I$,
(ii) For any $t, s \geq 0$, $J(t + s) = J(t)J(s)$.

The semigroup $(J(t))_{t\geq0}$ is said to C_0-semigroup if for any $x \in \mathcal{F}, \lim_{t\to0} \|J(t)x - x\| = 0$. The semigroup $(J(t))_{t\geq0}$ is said to be strongly continuous if for each $x \in \mathcal{F}, J(\cdot)x : \mathbb{R}_+ \longrightarrow \mathcal{F}$ is continuous.

A semigroup of continuous linear operators $(J(t))_{t\geq0}$ is uniformly continuous if $\lim_{t\to0} \|J(t) - I\| = 0$. The linear operator S defined by

$$D(S) = \{x \in \mathcal{F} : \lim_{t\to0} \frac{J(t)x - x}{t} \text{ exists}\}$$

and

$$Sx = \lim_{t\to0} \frac{J(t)x - x}{t}, \text{ for any } x \in D(S)$$

is called the infinitesimal generator of the semigroup $(J(t))_{t\geq0}$.

Theorem 1.11. *[3] Let $\mathcal{F}$ be a q-Banach space where $0 < q < 1$ and let $S \in \mathcal{L}(\mathcal{F})$. If we define $J_m(t)(x) = \sum_{k=0}^{n} \frac{t^k S^k(x)}{k!}, n \in \mathbb{N}, x \in X, t \in \mathbb{R}$, then $(J_m(t)(x))_m$ is a Cauchy sequence in X, that is convergent.*

Corollary 1.12. *[3] The sequence $J_m(t)(x) : \mathcal{F} \longrightarrow \mathcal{F}, J_m(t)(x) \in \mathcal{L}(\mathcal{F}), m \in \mathbb{N}$ is a Cauchy sequence and therefore it is convergent in $\mathcal{L}(\mathcal{F})$. Its limit is denoted by*

$$e^{tS}x = \sum_{k=0}^{\infty} \frac{t^k S^k(x)}{k!}.$$

Theorem 1.13. *[3] If we denote $J(t) = e^{tS}, t \in \mathbb{R}$, then the following properties hold:*

(i) *$J(t) \in \mathcal{L}(\mathcal{F})$, for any $t \in \mathbb{R}$ and $J(t+s) = J(t)J(s)$ for any $t, s \in \mathbb{R}$.*

(ii) *The limit*

$$\lim_{h\to 0} \| \frac{J(h)(x) - x}{h} - S(x)\| = 0$$

exists for any $x \in \mathcal{F}$.

(iii) *$J(t)$ is continuous as a function of $t \in \mathbb{R}$ to $\mathcal{L}(\mathcal{F})$ and $J(0) = I$. Also, $J(t)$ is differentiable and*

$$\frac{dJ(t)}{dt} x = SJ(t)x = J(t)Sx$$

for any $t \in \mathbb{R}$.

2. Main Results

Definition 2.1. Let $\mathcal{F}$ be a q-Banach space. A family $(J(t))_{t\in\mathbb{R}}$ of continuous linear operators on $\mathcal{F}$ is said to be C_0-cosine family if

(i) $J(0) = I$,

(ii) For any $t, s \in \mathbb{R}$, $J(t+s) + J(t-s) = 2J(t)J(s)$,

(iii) For each $x \in \mathcal{F}$, $J(\cdot)x : \mathbb{R} \longrightarrow \mathcal{F}$ is continuous.

The linear operator S defined by

$$D(S) = \{x \in \mathcal{F} : \lim_{t\to 0} \frac{J(t)x - x}{t^2} \text{ exists in } X\}$$

and

$$Sx = 2\lim_{t\to 0} \frac{J(t)x - x}{t^2}, \text{ for any } x \in D(S)$$

is called the infinitesimal generator of the C_0-cosine family $(J(t))_{t\in\mathbb{R}}$.

2.1. **Mixed C-cosine families of operators in p-Banach space.** In this paper, $C \in \mathcal{L}(\mathcal{F})$ is an injective bounded operator. We have:

Definition 2.2. A family $(L(t))_{t\in\mathbb{R}} \subset \mathcal{L}(\mathcal{F})$ on the p-Banach space $\mathcal{F}$ is called a C-cosine family if

(i) $L(0) = C$;

(ii) For all $t, s \in \mathbb{R}$, $C\Big((L(t+s) + L(t-s)\Big)) = 2L(t)L(s)$;

(iii) For each $x \in \mathcal{F}$, $\mathbb{R} \longrightarrow L(t)x$ is continuous on $\mathbb{R}$.

The linear operator L defined by

$$D(L) = \{x \in \mathcal{F} : \lim_{t\to 0} 2\frac{L(t)x - Cx}{t^2} \text{ exists in } R(C)\},$$

and

$$\text{for any } x \in D(L), Lx = C^{-1}\lim_{t\to 0} 2\frac{L(t)x - Cx}{t^2},$$

is called the infinitesimal generator of $(L(t))_{t\in\mathbb{R}}$.

Definition 2.3. A family $(J(t))_{t\in\mathbb{R}} \subset \mathcal{L}(\mathcal{F})$ is said to satisfy H-C-generalized cosine family on the p-Banach space $\mathcal{F}$ if

$$\text{for all } t,\ s \in \mathbb{R},\ C(J(s+t) + J(s-t)) = H\Big(J(s), J(t)\Big),$$

where $H : \mathcal{L}(\mathcal{F}) \times \mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{F})$ is a function.

Remark 2.4. If $H\Big(J(s), J(t)\Big) = 2J(s)J(t)$ and $J(0) = C$, $(J(t))_{t\in\mathbb{R}}$ is a C-cosine family of continuous linear operators on $\mathcal{F}$.

We have the following definition.

Definition 2.5. A family $(J(t))_{t\in\mathbb{R}} \subset \mathcal{L}(\mathcal{F})$ is called H-C-cosine family or generalized C-cosine family on $\mathcal{F}$ if

(i) $J(0) = C$,

(ii) For any $t, s \in \mathbb{R}$,

$$\begin{aligned} C(J(s+t) + J(s-t)) &= H\Big(J(s), J(t)\Big) \\ &= 2J(s)J(t) + 2D\Big(J(s) - L(s)\Big)\Big(J(t) - L(t)\Big) \end{aligned}$$

where $(L(t))_{t\in\mathbb{R}}$ is a C-cosine family of infinitesimal generator L_0 and $D \in \mathcal{L}(\mathcal{F})$.

(iii) For all $x \in \mathcal{F}$, $S(\cdot)x : \mathbb{R} \longrightarrow \mathcal{F}$ is continuous on $\mathbb{R}$.

The linear operator J defined by

$$D(J) = \{x \in \mathcal{F} : 2C^{-1} \lim_{t\to 0} \frac{J(t)x - Cx}{t^2} \text{ exists in } R(C)\}$$

and

$$\text{for any } x \in D(J), Jx = 2C^{-1} \lim_{t\to 0} \frac{J(t)x - Cx}{t^2},$$

is called the infinitesimal generator of the H-C-cosine family $(J(t))_{t\in\mathbb{R}}$.

Remark 2.6. Let $(J(t))_{t\in\mathbb{R}}$ be a generalized $C-$cosine family on $\mathcal{F}$, if $D = 0$, then $(J(t))_{t\in\mathbb{R}}$ is a C-cosine family on $\mathcal{F}$.

Utilizing Definition 2.5, when $D = \alpha I$ for $\alpha \in \mathbb{C}$, we have.

Definition 2.7. A family $(J(t))_{t\in\mathbb{R}}$ is called mixed C-cosine family on $\mathcal{F}$ if

(i) $J(0) = C$;

(ii) For any $t, s \in \mathbb{R}$,

$$\begin{aligned} C(J(s+t) + J(s-t)) &= H\Big(J(s), J(t)\Big) \\ &= 2J(s)J(t) + 2\alpha\Big(J(s) - L(s)\Big)\Big(J(t) - L(t)\Big) \end{aligned}$$

where $(L(t))_{t\in\mathbb{R}}$ is a C-cosine family with the infinitesimal generator L_0 and $\alpha \in \mathbb{C}$.

(iii) For each $x \in \mathcal{F}$, $J(\cdot)x : \mathbb{R} \longrightarrow \mathcal{F}$ is continuous on $\mathbb{R}$.

The linear operator J defined by

$$D(J) = \{x \in \mathcal{F} : 2C^{-1} \lim_{t\to 0} \frac{J(t)x - Cx}{t^2} \text{ exists}\}$$

and

$$\text{for any } x \in D(J), Jx = 2C^{-1} \lim_{t\to 0} \frac{J(t)x - Cx}{t^2},$$

is called the infinitesimal generator of the H-C-cosine family $(J(t))_{t\in\mathbb{R}}$.

2.2. **Question.** Can we characterize the infinitesimal generator of a mixed C-cosine family of continuous linear operators on complex p-Banach space?

Remark 2.8. Let $(J(t))_{t\in\mathbb{R}}$ be a mixed C-cosine family on $\mathcal{F}$, if $\alpha = 0$, then $(J(t))_{t\in\mathbb{R}}$ is a C-cosine family on $\mathcal{F}$.

Example 2.9. Let $J_0, J, C \in \mathcal{L}(\mathcal{F})$ such that $J_0J = JJ_0$, $J_0C = CJ_0$ and $CJ = JC$. Put

$$\text{for any } t \in \mathbb{R}, J(t) = Ccosh(tJ) + t(J_0 - J)Csinh(tJ),$$

where $cosh(tJ) = \sum_{n\in\mathbb{N}} \frac{t^{2n}}{(2n)!}J^{2n}$ and $sinh(tJ) = \sum_{n\in\mathbb{N}} \frac{t^{2n+1}}{(2n+1)!}J^{2n+1}$. It is easy to see that the following statements hold:

(i) If $\alpha = -1$, then $(J(t))_{t\in\mathbb{R}}$ is a mixed C-cosine family with $C(t) = Ccosh(tJ)$.
(ii) If $\alpha = -1$, then for each $t,\ s \in \mathbb{R}, J(s)J(t) = J(t)J(s)$.

We have the following lemma.

Lemma 2.10. *Let $(J(t))_{t\in\mathbb{R}}$ be an H-C-cosine family on $\mathcal{F}$, then for any $t \in \mathbb{R}, J(-t) = J(t)$.*

Proof. Obvious. □

The following proposition gives a condition for which an H-C-cosine family commute.

Proposition 2.11. *Let $(J(t))_{t\in\mathbb{R}}$ be an H-C-cosine family on $\mathcal{F}$ such that $I + D$ is inective and for any $t, s \in \mathbb{R}, L(s)J(t) = J(t)L(s)$, then for each $t, s \in \mathbb{R}, J(s)J(t) = J(t)J(s)$.*

Proof. Presume that $I + D$ is injective and for any $t, s \in \mathbb{R}, L(s)J(t) = J(t)L(s)$, then for any $t, s \in \mathbb{R}$,

$$\begin{aligned} 2J(s)J(t) + 2D\Big(J(s) - L(s)\Big)\Big(J(t) - L(t)\Big) &= C(J(s+t) + J(s-t)) \\ &= C(J(t+s) + J(t-s)) \\ &= 2J(t)J(s) \\ &\quad +2D\Big(J(t) - L(t)\Big) \\ &\quad \times\Big(J(s) - L(s)\Big). \end{aligned}$$

Hence $(I + D)\Big(J(t)J(s) - J(s)J(t)\Big) = 0$, thus for any $t, s \in \mathbb{R}, J(s)J(t) = J(t)J(s)$. □

We have the following theorem.

Proposition 2.12. *Let $(J(t))_{t\in\mathbb{R}}$ be an H-C-cosine commuting family on $\mathcal{F}$ of infinitesimal generator J with $(L(t))_{t\in\mathbb{R}}$ be a C-cosine family such that for any $t, s \in \mathbb{R}, L(s)J(t) = J(t)L(s)$. If $x \in D(J)$, then for any $t \in \mathbb{R}, J(t)x, L(t)x \in D(J)$ and $JJ(t)x = J(t)Jx$ and $JL(t)x = L(t)Jx$.*

Proof. Let $x \in D(J)$ and let $s \in \mathbb{R}^*$ and $t \in \mathbb{R}$. It is easy to see that

$$2\left(\frac{J(s)J(t)x - CJ(t)x}{s^2}\right) = 2J(t)\left(\frac{J(s)x - Cx}{s^2}\right) \to CJ(t)Jx \text{ as } s \to 0. \tag{2.1}$$

Consequently, for any $t \in \mathbb{R}, J(t)x \in D(J)$ and $JJ(t)x = J(t)Jx$.
Let $x \in D(J)$ and let $s \in \mathbb{R}^*$ and $t \in \mathbb{R}$. Then

$$2\left(\frac{J(s)L(t)x - CL(t)x}{s^2}\right) = 2L(t)\left(\frac{J(s)x - Cx}{s^2}\right) \to CL(t)Jx \text{ as } s \to 0. \tag{2.2}$$

Consequently, for any $t \in \mathbb{R},\ L(t)x \in D(J)$ and $JL(t)x = L(t)Jx$. □

For $\alpha \in \mathbb{C}\backslash\{-1\}$, put $J_1 = (1+\alpha)J - \alpha J_0$ where J_0 is the infinitesimal generator of the C-cosine family $(L(t))_{t\in\mathbb{R}}$ and J is the infinitesimal generator of the mixed C-cosine family $(J(t))_{t\in\mathbb{R}}$.

Theorem 2.13. *Let $(J(t))_{t\in\mathbb{R}}$ be a mixed C-cosine family of inifinitesimal generator J on $\mathcal{F}$ with $(L(t))_{t\in\mathbb{R}}$ is a C-cosine family of infinitesimal generator J_0 and $\alpha \in \mathbb{C}_p\backslash\{-1\}$. Set $L_1(t)x = (1+\alpha)J(t)x - \alpha L(t)x$, $x \in \mathcal{F}$, then $\{L_1(t)\}_{t\in\mathbb{R}}$ is a C-cosine family whose infinitesimal generator is an extension of J_1. Furthermore, for any $x \in \mathcal{F}$ and $t \in \mathbb{R}$,*

$$J(t)x = \frac{1}{1+\alpha}L_1(t)x + \frac{\alpha}{1+\alpha}L(t)x.$$

Proof.

(i) We get $L_1(0)x = (1+\alpha)S(0)x - \alpha C(0)x = C$.

(ii) For any $t, s \in \mathbb{R}, x \in \mathcal{F}$, we have

$$\begin{aligned}
CL_1(s+t)x + CL_1(s-t)x &= (1+\alpha)C\Big(J(s+t) + J(s-t)\Big)x \\
&\quad -\alpha C\Big(L(s+t) + L(s-t)\Big)x \\
&= (1+\alpha)\Big(2J(s)J(t) + 2\alpha(J(s) - L(s)) \times \\
&\quad (J(t) - L(t))\Big)x - 2\alpha L(s)L(t)x \\
&= 2(1+\alpha)J(s)J(t)x + 2\alpha(1+\alpha)J(s)J(t)x \\
&\quad -2\alpha(1+\alpha)J(s)L(t)x - 2\alpha(1+\alpha)L(s)J(t)x \\
&\quad +2\alpha(1+\alpha)L(s)L(t)x - 2\alpha L(s)L(t)x \\
&= 2(1+\alpha)^2J(s)J(t)x - 2\alpha(1+\alpha)J(s)L(t)x \\
&- \;\; 2\alpha(1+\alpha)L(s)J(t)x + 2\alpha(1+\alpha)L(s)L(t)x \\
&\quad -2\alpha L(s)L(t)x \\
&= 2\Big((1+\alpha)J(s) - \alpha L(s)\Big)\Big(1+\alpha)J(t) - \alpha L(t)\Big)x \\
&= 2L_1(s)L_2(t)x.
\end{aligned}$$

Moreover, $L_1(0)x = (1+\alpha)Cx - \alpha Cx = Cx$. Then $(L_1(t))_{t\in\mathbb{R}}$ is a C-cosine family on $\mathcal{F}$. Utilizing $(L(t))_{t\in\mathbb{R}}$ and $(J(t))_{t\in\mathbb{R}}$ are continuous, hence $(L_1(t))_{t\in\mathbb{R}}$ is continuous. So, $(L_1(t))_{t\in\mathbb{R}}$ is a C-cosine family on $\mathcal{F}$.

(iii) Now, we show an extension of J_1 is the infinitesimal generator of $\{L_1(t)\}_{t\in\mathbb{R}}$. Let B be the infinitesimal generator of $\{L_1(t)\}_{t\in\mathbb{R}}$. For $x \in D(L_1) = D(J) \cap D(J_0)$. Utilizing definition of $D(J)$ and $D(J_0)$, we get $2\lim_{t\to 0}\Big(\frac{J(t)x - Cx}{t^2}\Big) = CJx$ and $2\lim_{t\to 0}\Big(\frac{L(t)x - Cx}{t^2}\Big) = CJ_0x$. So

$$\begin{aligned}
2\lim_{t\to 0}\Big(\frac{L_1(t)x - Cx}{t^2}\Big) &= 2\lim_{t\to 0}\Big(\frac{(1+\alpha)J(t)x - \alpha L(t)x - Cx}{t^2}\Big) \\
&= 2(1+\alpha)\lim_{t\to 0}\Big(\frac{J(t)x - Cx}{t^2}\Big) - 2\alpha\lim_{t\to 0}\Big(\frac{L(t)x - Cx}{t^2}\Big)
\end{aligned}$$

exists in $R(C)$. It follows that $x \in D(B)$ and $J_1(x) = Bx$. Then an extension of J_1 is the infinitesimal generator of $(L_1(t))_{t\in\mathbb{R}}$. □

Proposition 2.14. *Let $(J(t))_{t\in\mathbb{R}}$ be a mixed C-cosine family on $\mathcal{F}$ with $\alpha \in \mathbb{C}\backslash\{-1\}$ such that for any $t, s \in \mathbb{R}, C(s)J(t) = J(t)C(s)$, then for all $t, s \in \mathbb{R}$, $J(s)J(t) = J(t)J(s)$.*

Proof. Presume that for any $t, s \in \mathbb{R}, C(s)J(t) = J(t)C(s)$, then for all $t, s \in \mathbb{R}$,

$$\begin{aligned}
2J(s)J(t) + 2\alpha(J(s) - C(s))(J(t) - C(t)) &= C(J(s+t) + J(s-t)) \\
&= J(t+s) + J(t-s) \\
&= 2J(t)J(s)
\end{aligned}$$

$$+2\alpha(J(t)-C(t))\times$$
$$(J(s)-C(s)).$$

Thus, $(1+\alpha)\Big(J(t)J(s)-J(s)J(t)\Big)=0$. Then, for all $t,s\in\mathbb{R},\ J(s)J(t)=J(t)J(s)$. □

Let $\{J(t)\}_{t\in\mathbb{R}}$ be a mixed C-cosine family of infinitesimal generator J with $\{L(t)\}_{t\in\mathbb{R}}$ is a C-cosine family of infinitesimal generator J_0 with $\alpha\in\mathbb{C}\backslash\{-1\}$.

Theorem 2.15. *Let $\{J(t)\}_{t\in\mathbb{R}}$ be a mixed C-cosine family of infinitesimal generator J with $\{L(t)\}_{t\in\mathbb{R}}$ be a C-cosine family with $\alpha\in\mathbb{C}\backslash\{-1\}$ such that for all $t,s\in\mathbb{R},\ L(s)J(t)=J(t)L(s)$. If $x\in D(J)$, then for all $t\in\mathbb{R}, J(t)x, L(t)x\in D(J)$, $JJ(t)x=J(t)Jx$ and $JL(t)x=L(t)Jx$.*

Proof. Let $x\in D(J)$ and let $s\in\mathcal{R}^*$ and $t\in\mathcal{R}$. Utilizing Proposition 2.14, $J(t)J(s)=J(s)J(t)$ so

$$2\left(\frac{J(s)J(t)x-CJ(t)x}{s^2}\right)=2J(t)\left(\frac{J(s)x-Cx}{s^2}\right)\to CJ(t)Jx \text{ as } s\to 0.$$

Consequently, $J(t)Jx\in D(J)$ and $JJ(t)x=J(t)Jx$.
Let $x\in D(J)$ and let $s\in\mathcal{R}^*_+$ and $t\in\mathcal{R}$. So

$$2\left(\frac{J(s)L(t)x-CL(t)x}{s^2}\right)=2L(t)\left(\frac{J(s)x-Cx}{s^2}\right)\to CL(t)Jx \text{ as } s\to 0.$$

Consequently, $L(t)x\in D(J)$ and $JL(t)x=L(t)Jx$. □

Put $L_1=(1+D)J-DL_0$, where $D\in\mathcal{L}(\mathcal{F})$ and L_0 is the infinitesimal generator of the $C-$cosine family $\{L(t)\}_{t\in\mathbb{R}}$ and J is the infinitesimal generator of an H-C-cosine family $\{J(t)\}_{t\in\mathbb{R}}$, analogous the proof of Theorem 2.13, we get with the next theorem.

Theorem 2.16. *Let $\{J(t)\}_{t\in\mathbb{R}}$ be a commuting H-C-cosine family on $\mathcal{F}$ with for any $s\in\mathbb{R},\ DL(s)=L(s)D$ and $DJ(s)=J(s)D$. Put for any $t\in\mathbb{R},\ L_1(t)=(I+D)J(t)-DL(t)$, we have*
(i) *$\{L_1(t)\}_{t\in\mathbb{R}}$ is a $C-$cosine family whose infinitesimal generator is an extension of L_1.*
(ii) *If $I+D$ is invertible, then for any $x\in\mathcal{F}$ and $t\in\mathbb{R}$,*

$$J(t)x=(1+D)^{-1}L_1(t)x+D(I+D)^{-1}L(t)x.$$

We finish with an application related to the following second order abstract Cauchy problem in a p-Banach space.
Application. Set for all $t\in\mathbb{R}, J(t)=\sum_{k=0}^{\infty}\frac{t^{2n}}{(2n)!}CJ_0^n$ where $J_0,C\in\mathcal{L}(\mathcal{F})$ such that C is injective and $J_0C=CJ_0$, then $u(t)=J(t)x$ is a solution of the homogeneous second differential equation given by

$$\frac{d^2u(t)}{dt^2}=J_0u(t), u'(0)=0,\ t\in\mathbb{R},$$

and $u(0)=x$ and $x\in\mathcal{F}$.

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