

## PSEUDO QUASI-ORDERED RESIDUATED SYSTEMS, AN INTRODUCTION

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**ABSTRACT.** The concept of quasi-ordered residuated systems was introduced in 2018 by S. Bonzio and I. Chajda as a generalization of both hoop-algebras and commutative residuated lattices ordered by quasi-orders. The substructures of ideals and filters in such algebraic structures were considered by the author. This paper introduces and analyzes the concept of pseudo quasi-ordered residuated systems as a non-commutative generalization of quasi-ordered residuated systems with left and right residuum operations. Also, this paper discusses the concepts of ideals and filters in pseudo quasi-ordered residuated systems.

### INTRODUCTION

Quasi-ordered residuated system is a commutative residuated integral monoids ordered under a quasi-order, introduced by S. Bonzio and I. Chajda in [2]. In the last few years, the theory of quasi-ordered residuated systems was enriched with more results on ideals and filters in them (see [11–13, 15, 16]).

Hoop algebras were presented by B. Bosbach in [4, 5]. Pseudo-hoop algebras were presented as non-commutative generalizations of hoop algebras by Georgescu, Leuştean and Preoteasa in [8], following after the notions of pseudo-MV algebras in [7] and pseudo-BL algebras ([6]). The pseudo KU-algebras and the pseudo UP-algebras were studied in [9, 10] by this author.

In this article, the author deals with the introduction and analysis of the concept of pseudo quasi-ordered residuated systems. The paper is divided into three sections. In the second section we recall some facts about quasi-ordered residuated systems. Section 3 is the main part of this paper. In the subsection 3.1. we introduce the concept of pseudo quasi-ordered residuated systems and we prove some of their fundamental properties. Subsection 3.2 is devoted to the concept of filters in this newly determined class of algebraic structures. Further on, in subsection 3.3 of this paper, concept of ideals in a pseudo quasi-ordered residuated system is studied is defined.

### 1. PRELIMINARIES

In this section, the necessary notions and notations and some of their interrelationships, mostly taken from paper [2, 11, 12, 17], are listed in the order to enable a reader to comfortably follow the presentation in

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Submitted on Jul. 1, 2022.

2020 *Mathematics Subject Classification.* 06A11, 06F25, 08A02.

*Key words and phrases.* quasi-ordered residuated system (QRS), non-commutative QRS, pseudo-QRS, ideals in pseudo QRS, filters in pseudo-QRS.

this report. It should be pointed out here that the notations for logical conjunction, logical implication and others have a literal meaning. The notation  $=:$  in the formula  $A =: B$  serves to indicate that  $A$  in it is the abbreviation for the formula  $B$ .

**1.1. Concept of quasi-ordered residuated systems.** In article [2], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

**Definition 1.1** ([2], Definition 2.1). A *residuated relational system* is a structure  $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, R \rangle$ , where  $\langle A, \cdot, \rightarrow, 1 \rangle$  is an algebra of type  $\langle 2, 2, 0 \rangle$  and  $R$  is a binary relation on  $A$  and satisfying the following properties:

- (1)  $(A, \cdot, 1)$  is a commutative monoid;
- (2)  $(\forall x \in A)((x, 1) \in R)$ ;
- (3)  $(\forall x, y, z \in A)((x \cdot y, z) \in R \iff (x, y \rightarrow z) \in R)$ .

We will refer to the operation  $\cdot$  as (commutative) multiplication, to  $\rightarrow$  as its residuum and to condition (3) as residuation.

Recall that a *quasi-order relation*  $\preceq$  ' on a set  $A$  is a binary relation which is reflexive and transitive.

**Definition 1.2** ([2]). A *quasi-ordered residuated system* is a residuated relational system  $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, \preceq \rangle$ , where  $\preceq$  is a quasi-order relation in the monoid  $(A, \cdot)$

The following proposition shows the basic properties of quasi-ordered residuated systems.

**Proposition 1.3** ([2], Proposition 3.1). *Let  $\mathfrak{A}$  be a quasi-ordered residuated system. Then*

- (4) *The operation  $\cdot$  preserves the pre-order in both positions;*

$$(\forall x, y, z \in A)(x \preceq y \implies (x \cdot z \preceq y \cdot z \wedge z \cdot x \preceq z \cdot y));$$

- (5)  $(\forall x, y, z \in A)(x \preceq y \implies (y \rightarrow z \preceq x \rightarrow z \wedge z \rightarrow x \preceq z \rightarrow y));$
- (6)  $(\forall y, z \in A)(x \cdot (y \rightarrow z) \preceq y \rightarrow x \cdot z);$
- (7)  $(\forall x, y, z \in A)(x \cdot y \rightarrow z \preceq x \rightarrow (y \rightarrow z));$
- (8)  $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preceq x \cdot y \rightarrow z);$
- (9)  $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preceq y \rightarrow (x \rightarrow z));$
- (10)  $(\forall x, y, z \in A)((x \rightarrow y) \cdot (y \rightarrow z) \preceq x \rightarrow z);$
- (11)  $(\forall x, y \in A)((x \cdot y \preceq x) \wedge (x \cdot y \preceq y));$
- (12)  $(\forall x, y, z \in A)(x \rightarrow y \preceq (y \rightarrow z) \rightarrow (x \rightarrow z));$
- (13)  $(\forall x, y, z \in A)(y \rightarrow z \preceq (x \rightarrow y) \rightarrow (x \rightarrow z)).$

It is generally known that a quasi-order relation  $\preceq$  on a set  $A$  generates a equivalence relation  $\equiv_{\preceq} := \preceq \cap \preceq^{-1}$  on  $A$ . Due to properties (4) and (5), this equality relation is compatible with the operations in  $A$ . Thus,  $\equiv_{\preceq}$  is a congruence on  $A$ .

In the light of the previous note, it is easy to see that the following applies:

- (7) and (8) give:

$$(14) (\forall x, y, z \in A)(x \cdot y \rightarrow z \equiv_{\preceq} x \rightarrow (y \rightarrow z)).$$

Due to the universality of formula (9) (or, due to the commutativity of the multiplication from (14)), we have:

$$(15) (\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \equiv_{\preceq} y \rightarrow (x \rightarrow z)).$$

Also, from (11) and (2), it follows

$$(16) (\forall x \in A)(x \rightarrow x \equiv_{\preceq} 1)$$

In the general case,

$$(17) (\forall x, y \in A)(x \preceq y \iff x \rightarrow y \equiv_{\preceq} 1)$$

is valid, which is obtained by referring to (11) and (2).

From the previous analysis it can be concluded that a quasi-ordered residuated system is a generalization of a hoop-algebra (in the sense of [3]) because the following formula

$$(\forall x, y \in A)(x \cdot (x \rightarrow y) \equiv_{\preceq} y \cdot (y \rightarrow x))$$

does not have to be a valid formula in the observed algebraic structure in the general case.

A quasi-ordered residuated system  $\mathfrak{A}$  is said to be a strong quasi-ordered residuated system ([14], Definition 6) if additionally the following

$$(18) (\forall x, y \in A)((x \rightarrow y) \rightarrow y \equiv_{\preceq} (y \rightarrow x) \rightarrow x)$$

is valid. If we recall that a hoop is a Weisberg hoop if condition (18) is added to the axioms that determine the concept of hoops, then we can conclude that a strong quasi-ordered residuated system is a generalization of Weisberg hoops. It is well known that the underlying ordering of a Wajsberg hoop is a lattice ordering, and that the join is term-definable by  $a \sqcup b =: (a \rightarrow b) \rightarrow b$ . Since any hoop satisfies the equation ([4])  $(a \rightarrow b) \rightarrow (b \rightarrow a) = b \rightarrow a$ , any Weisberg hoop satisfies the pre-linearity condition  $(a \rightarrow b) \sqcup (b \rightarrow a) = 1$ . However, a strong quasi-ordered residuated the system, in the general case, does not have to satisfy the pre-linearity condition. Thus, a strong quasi-ordered residuated system is a generalization of Weisberg hoops.

## 1.2. Concept of filters.

**Definition 1.4** ([11], Definition 3.1). For a subset  $F$  of a quasi-ordered residuated system  $\mathfrak{A}$  we say that it is a *filter* of  $\mathfrak{A}$  if it satisfies conditions

$$(F2) (\forall u, v \in A)((u \in F \wedge u \preceq v) \implies v \in F), \text{ and}$$

$$(F3) (\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F).$$

Let it note that the empty subset of  $A$  satisfies the conditions (F2) and (F3). Therefore,  $\emptyset$  is a filter in  $\mathfrak{A}$ . It is shown ([11], Proposition 3.4 and Proposition 3.2), that if a non-empty subset  $F$  of a quasi-ordered system  $\mathfrak{A}$  satisfies the condition (F2), then it also satisfies the following conditions

$$(F0): 1 \in F \text{ and}$$

$$(F1): (\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F)).$$

Also, it can be seen without difficulty that  $((F3) \wedge F \neq \emptyset) \implies (F2)$  is valid. Indeed, if (F3) holds, then the formula  $u \in F \wedge u \preceq v$ , can be transformed into the formula  $u \in F \wedge u \rightarrow v \equiv_{\preceq} 1 \in F$  by (F0) so from here, according to (F3) it can be demonstrate the validity of implications (F2). However, the reverse does not have to be valid.

If  $\mathfrak{F}(A)$  is the family of all filters in a QRS  $\mathfrak{A}$ , then  $\mathfrak{F}(A)$  is a complete lattice ([11], Theorem 3.1).

In the papers [15, 16], prime and irreducible filters in strong quasi-ordered residuated systems and their mutual relations are considered.

**1.3. Concept of ideals.** The article [12] introduces and discusses the concepts of pre-ideals and ideals in a quasi-ordered residuated system. In doing so, the following conditions were analyzed:

$$(J1) (\forall u, v \in A)((u \in J \vee v \in J) \implies u \cdot v \in J)$$

$$(J2) (\forall u, v \in A)((u \preceq v \wedge v \in J) \implies u \in J) \text{ and}$$

$$(J3) (\forall u, v \in A)((u \rightarrow v \notin A \wedge v \in A) \implies u \in J).$$

In addition, the following implications

$$(J2) \implies (J1) \quad ([12], \text{Proposition 4.1})$$

$$(J2) \implies (\forall u, v \in A)(u \preceq v \implies u \rightarrow v \notin J) \quad ([12], \text{Proposition 4.2})$$

$(J \neq \emptyset \wedge (J3)) \implies (J2)$  ([12], Proposition 4.3)

have been shown to be valid formulas. However, the implication  $(J2) \implies (J3)$  do not have to be a valid formula in the general case.

**Definition 1.5.** ([12], Definition 4.1) Let  $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preceq \rangle$  be a quasi-ordered residuated system. For a subset  $J$  of the set  $A$  we say that it is an *pre-ideal* in  $\mathfrak{A}$  if the condition (J2) holds.

**Definition 1.6.** ([12], Definition 4.2) Let  $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preceq \rangle$  be a quasi-ordered residuated system. For a subset  $J$  of the set  $A$  we say that it is an *ideal* in  $\mathfrak{A}$  if  $J = A$  or the condition (J3) is valid.

In addition, of course, every ideal in a quasi-ordered residuated system  $\mathfrak{A}$  is a pre-ideal in  $\mathfrak{A}$  ([12], Proposition 4.2).

## 2. THE MAIN RESULTS

Section 3 is the main part of this paper. In the subsection 3.1. we introduce the concept of pseudo quasi-ordered residuated systems and we prove some of their fundamental properties. Subsection 3.2 is devoted to the concept of filters in this newly determined class of algebraic structures. Subsection 3.3 is devoted to an introduction of the concept of ideals in a pseudo quasi-ordered residuated system.

As can be seen in the material presented below, the notion of filters determined here for pseudo quasi-ordered residuated systems is somewhat different from the corresponding notion of filters in pseudo-hoops analyzed in the article [1]. Also, for determining the concept of ideals in pseudo quasi-ordered residuated systems, a different way was used than it was done for the concept of ideals in pseudo-hoops, as it was done in article [18].

### 2.1. Basic definitions and properties.

**Definition 2.1.** A *pseudo quasi-ordered relational system* is a structure  $\mathfrak{p}\mathfrak{A} = \langle A, \cdot, \rightarrow, \rightsquigarrow, 1, \preceq \rangle$ , where  $\langle A, \cdot, \rightarrow, \rightsquigarrow, 1 \rangle$  is an algebra of type  $\langle 2, 2, 2, 0 \rangle$  and  $\preceq$  is a quasi-order on  $A$  satisfying the following properties:

- (1)  $(A, \cdot, 1)$  is a monoid;
- (2)  $(\forall x \in A)(x \preceq 1)$ ;
- (3L)  $(\forall x, y, z \in A)(x \cdot y \preceq z \iff x \preceq y \rightarrow z)$ ;
- (3R)  $(\forall x, y, z \in A)(x \cdot y \preceq z \iff y \preceq x \rightsquigarrow z)$ .

We will refer to the operation  $\cdot$  as (non-commutative) multiplication, to  $\rightarrow$  as its left residuum and to  $\rightsquigarrow$  as right residuum.

This system of axioms is hereinafter referred to as pQRS.

Let  $\mathfrak{p}\mathfrak{A} = \langle A, \cdot, \rightarrow, \rightsquigarrow, 1 \rangle$  be a pseudo quasi-ordered residuated system. Then the operation  $\cdot$  in  $A/\equiv_{\preceq}$  is commutative if and only if  $\rightarrow \equiv \rightsquigarrow$ . Indeed, for arbitrary elements  $x, y, z \in A$  the following extended equivalence

$$x \cdot y \preceq z \iff x \preceq y \rightarrow z \iff x \preceq y \rightsquigarrow z \iff y \cdot x \preceq z$$

is valid. In particular, for  $z \equiv y \cdot x$  (or for  $z \equiv x \cdot y$ ) we get  $x \cdot y \preceq y \cdot x \preceq x \cdot y$ , i.e. we get  $x \cdot y \equiv_{\preceq} y \cdot x$ . Conversely, for arbitrary elements  $x, y, z \in A$  the following extended equivalence

$$x \preceq y \rightarrow z \iff x \cdot y \preceq z \iff y \cdot x \preceq z \iff x \preceq y \rightsquigarrow z$$

is valid. From here, for  $x \equiv y \rightarrow z$ , or for  $x \equiv y \rightsquigarrow z$ , we get  $x \rightarrow y \preceq x \rightsquigarrow y \preceq x \rightarrow y$ . In this case,  $\mathfrak{p}\mathfrak{A}/\equiv_{\preceq}$  is a quasi-ordered residuated system. Thus, we can conclude that the pseudo quasi-ordered residuated system is a generalization of a quasi-ordered residuated system.

In the following theorem, we collect and reformulate some results proved in [2]. For the sake of completeness, we shall include some proofs of these results.

**Theorem 2.2.** Let  $\mathbf{p}\mathfrak{A} = \langle A, \cdot, \rightarrow, \rightsquigarrow, 1, \preceq \rangle$  be a pseudo quasi-ordered residuated system. Then:

- (a)  $(\forall x, y \in A)((x \preceq y \iff x \rightarrow y \equiv_{\preceq} 1) \wedge (x \preceq y \iff x \rightsquigarrow y \equiv_{\preceq} 1))$
- (b)  $(\forall x \in A)((x \rightarrow x \equiv_{\preceq} 1) \wedge (x \rightsquigarrow x \equiv_{\preceq} 1))$
- (cL)  $(\forall x, y, z \in A)(x \cdot y \rightarrow z \equiv_{\preceq} x \rightarrow (y \rightarrow z))$
- (cR)  $(\forall x, y, z \in A)(x \cdot y \rightsquigarrow z \equiv_{\preceq} y \rightsquigarrow (x \rightsquigarrow z))$
- (d)  $(\forall x, y, z \in A)(x \preceq y \implies ((x \cdot z \preceq y \cdot z) \wedge (z \cdot x \preceq z \cdot y)))$
- (e)  $(\forall x, y, z \in A)(x \preceq y \implies ((z \rightarrow x \preceq z \rightarrow y) \wedge (y \rightarrow z \preceq x \rightarrow z)))$
- (f)  $(\forall x, y, z \in A)(x \preceq y \implies ((z \rightsquigarrow x \preceq z \rightsquigarrow y) \wedge (y \rightsquigarrow z \preceq x \rightsquigarrow z)))$
- (g)  $(\forall x, y \in A)(x \cdot y \preceq x \wedge x \cdot y \preceq y)$
- (h)  $(\forall x, y \in A)(x \preceq y \rightarrow x \wedge x \preceq y \rightsquigarrow x)$
- (i)  $(\forall x \in A)(x \equiv_{\preceq} 1 \rightarrow x \wedge x \equiv_{\preceq} 1 \rightsquigarrow x)$
- (j)  $(\forall x, y, z \in A)((y \rightarrow z) \cdot (x \rightarrow y) \preceq x \rightarrow z)$
- (k)  $(\forall x, y, z \in A)((x \rightsquigarrow y) \cdot (y \rightsquigarrow z) \preceq x \rightsquigarrow z).$

*Proof.* (a) Let  $x, y \in A$  be arbitrary elements such that  $x \preceq y$ . Then  $1 \cdot x \preceq y$  and  $x \cdot 1 \preceq y$  by (1). Thus  $1 \preceq x \rightarrow y$  by (3L) and  $1 \preceq x \rightsquigarrow y$  by (3R). Hence  $1 \preceq x \rightarrow y \preceq 1$  and  $1 \preceq x \rightsquigarrow y \preceq 1$  by (2).

(b) is a direct consequence of (a) since  $\preceq$  is a reflexive relation.

(c) Let  $x, y, z \in A$  be arbitrary elements. Then  $x \cdot y \rightarrow z \preceq x \cdot y \rightarrow z$  by reflexivity of the quasi-order. Thus  $(x \cdot y \rightarrow z) \cdot (x \cdot y) \preceq z$  by (3L). We transform the previous formula into the form  $((x \cdot y \rightarrow z) \cdot x) \cdot y \preceq z$  taking into account that the multiplication is an associative operation. From here we get  $x \cdot y \rightarrow z \preceq x \rightarrow (y \rightarrow z)$  by applying twice (3L). Conversely, they proved similar to the previous one. For arbitrary elements  $x, y, z \in A$  holds  $x \preceq x \rightarrow (y \rightarrow z)$  by transitivity of the quasi-order. Then  $(x \rightarrow (y \rightarrow z)) \cdot x \preceq y \rightarrow z$  and thus  $((x \rightarrow (y \rightarrow z)) \cdot x) \cdot y \preceq z$  by (3L). Hence  $(x \rightarrow (y \rightarrow z)) \cdot (x \cdot y) \preceq z$ . Therefore,  $x \rightarrow (y \rightarrow z) \preceq x \cdot y \rightarrow z$ .

Finally, we have  $x \rightarrow (y \rightarrow z) \equiv_{\preceq} x \cdot y \rightarrow z$ .

Condition (cR) can be proved completely analogous to the previous proof, taking into account the determination of the right residuum.

(d) Let  $x, y, z \in A$  be arbitrary elements such that  $x \preceq y$ . From  $y \cdot z \preceq y \cdot z$ , by reflexivity of the quasi-order, we have  $y \preceq z \rightarrow y \cdot z$  by (3L) and  $x \preceq z \rightarrow y \cdot z$  by transitivity of the quasi-order. Then  $x \cdot z \preceq y \cdot z$  by (3L). On the other hand, we have that  $zy \preceq zy$ , implies  $y \rightsquigarrow z \cdot y$  by (3R). Then  $x \preceq y \rightsquigarrow z \cdot y$  by transitivity of the quasi-order relation in  $A$ . Thus  $z \cdot x \preceq z \cdot y$  by (3R).

(e) Let  $x, y, z \in A$  be arbitrary elements such that  $x \preceq y$ . Since the quasi-order in  $A$  is a reflexive relation, we have  $z \rightarrow x \preceq z \rightarrow x$  and  $y \rightarrow z \preceq y \rightarrow z$ . Then  $(z \rightarrow x) \cdot z \preceq x$  and  $(y \rightarrow z) \cdot y \preceq z$  by (3L).

From  $(z \rightarrow x) \cdot z \preceq x$  and  $x \preceq y$  it follows  $(z \rightarrow x) \cdot z \preceq y$  by transitivity of the quasi-order. Thus  $z \rightarrow x \preceq z \rightarrow y$  by (3L).

From  $x \preceq y$  it follows  $(y \rightarrow z) \cdot x \preceq (y \rightarrow z) \cdot y$  according to (d). Now, from  $(y \rightarrow z) \cdot x \preceq (y \rightarrow z) \cdot y$  and  $(y \rightarrow z) \cdot y \preceq z$ , it follows  $(y \rightarrow z) \cdot x \preceq z$  by transitivity of the quasi-order. Hence  $y \rightarrow z \preceq x \rightarrow z$  by (3L).

(f) Let  $x, y, z \in A$  be arbitrary elements such that  $x \preceq y$ . Since the quasi-order in  $A$  is a reflexive relation, we have  $z \rightsquigarrow x \preceq z \rightsquigarrow x$  and  $y \rightsquigarrow z \preceq y \rightsquigarrow z$ . Then  $z \cdot (z \rightsquigarrow x) \preceq x$  by and  $y \cdot (y \rightsquigarrow z) \preceq z$  by (3R).

From  $z \cdot (z \rightsquigarrow x) \preceq x$  and  $x \preceq y$  it follows  $z \cdot (z \rightsquigarrow x) \preceq y$  by transitivity of the quasi-order. Thus  $z \rightsquigarrow x \preceq z \rightsquigarrow y$  by (3R).

From  $x \preceq y$  it follows  $x \cdot (y \rightsquigarrow z) \preceq y \cdot (y \rightsquigarrow z)$  according to (d). Now, from  $x \cdot (y \rightsquigarrow z) \preceq y \cdot (y \rightsquigarrow z)$  and  $y \cdot (y \rightsquigarrow z) \preceq z$ , it follows  $x \cdot (y \rightsquigarrow z) \preceq z$  by transitivity of the quasi-order. Hence  $y \rightsquigarrow z \preceq x \rightsquigarrow z$  by (3R).

(g) For any  $x, y \in A$ , according to (b), we have  $x \rightsquigarrow x \equiv_{\leq} 1$  and  $y \rightarrow y \equiv_{\leq} 1$ . Now, from  $y \leq 1 \equiv_{\leq} x \rightsquigarrow x$  and  $x \leq 1 \equiv_{\leq} y \rightarrow y$  we get  $x \cdot y \leq x$  and  $x \cdot y \leq y$ , according to (3R), or according to (3L), respectively.

(h) The validity of statements in (h) is a direct consequence of statements in (g) with respect to (3L) and (3R) respectively.

(i) Let  $x \in A$  be an arbitrary element of  $A$ . By reflexivity  $1 \cdot x = x \leq x$  we have  $x \leq 1 \rightarrow x$  by (3L). Similarly,  $1 \rightarrow x \leq x$  is obtained by residuation from  $1 \rightarrow x \leq 1 \rightarrow x$ . So,  $x \equiv_{\leq} 1 \rightarrow x$ .

The second part of this claim can be proven analogously. Due to the reflexivity of the quasi-order on  $A$ , from  $x \cdot 1 = x \leq x$ , it follows by  $x \leq x \rightsquigarrow 1$  according to (3R). On the other hand, the validity of  $1 \rightsquigarrow x \leq 1 \rightsquigarrow x$  implies the validity for  $1 \rightsquigarrow x \leq x$  according to (3R). Hence,  $1 \rightsquigarrow x \equiv_{\leq} x$ .

(j) First, for arbitrary  $x, y, z \in A$ ,  $x \rightarrow y \leq x \rightarrow y$  holds. Then  $(x \rightarrow y) \cdot x \leq y$  by (3L). Multiplying from the left the previous inequality with  $y \rightarrow z$ , we get  $(y \rightarrow z) \cdot (x \rightarrow y) \cdot x \leq (y \rightarrow z) \cdot y \leq z$  by (d). Thus  $(y \rightarrow z) \cdot (x \rightarrow y) \leq x \rightarrow z$  by (3L).

(k) Let  $x, y, z \in A$  be arbitrary elements. Then  $x \rightsquigarrow y \leq x \rightsquigarrow y$  by reflexivity of the quasi-order. Then  $x \cdot (x \rightsquigarrow y) \leq y$  by (3R). Thus  $x \cdot (x \rightsquigarrow y) \cdot (y \rightsquigarrow z) \leq y \cdot (y \rightsquigarrow z) \leq z$  by (d). Hence  $(x \rightsquigarrow y) \cdot (y \rightsquigarrow z) \leq x \rightsquigarrow z$  by (3R).  $\square$

Having regard to (3L) and (3R), it follows directly from statements (j) and (k):

$$(\forall x, y, z \in A)(x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)),$$

$$(\forall x, y, z \in A)(x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)),$$

$$(\forall x, y, z \in A)(x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)),$$

$$(\forall x, y, z \in A)(x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)),$$

**Definition 2.3.** A pseudo quasi-ordered residuated system  $\mathfrak{p}\mathfrak{A}$  is called a *strong pseudo quasi-ordered residuated system* if additionally it satisfies the following conditions:

$$(S1) (\forall x, y \in A)((x \rightarrow y) \rightsquigarrow y \equiv_{\leq} (y \rightarrow x) \rightsquigarrow x), \text{ and}$$

$$(S2) (\forall x, y \in A)((x \rightsquigarrow y) \rightarrow y \equiv_{\leq} (y \rightsquigarrow x) \rightarrow x).$$

As, in the general case, the (strong) pseudo quasi-ordered residuated system does not meet the conditions

$$(\forall a, b \in A)((a \rightarrow b) \cdot b \equiv_{\leq} (b \rightarrow a) \cdot a) \text{ and}$$

$$(\forall a, b \in A)(a \cdot (a \rightsquigarrow b) \equiv_{\leq} b \cdot (b \rightsquigarrow a)),$$

we can conclude that the concept of pseudo quasi-ordered residuated systems is a generalization of the concept of pseudo-hoops and that the concept of strong pseudo quasi-ordered residuated systems is a generalization of the concept of Weisberg pseudo-hoops. (To determine these last algebras, see [8].)

Following the idea presented in [8], we define on a strong pseudo quasi-ordered residuated system  $\mathfrak{p}\mathfrak{A}$  two binary operations that are a join operation: If  $x, y \in A$ , then the joins of  $x$  and  $y$  are

$$x \sqcup_1 y =: (x \rightarrow y) \rightsquigarrow y \equiv_{\leq} (y \rightarrow x) \rightsquigarrow x \text{ and}$$

$$x \sqcup_2 y =: (x \rightsquigarrow y) \rightarrow y \equiv_{\leq} ((y \rightsquigarrow x) \rightarrow x).$$

In what follows, we will need the following Lemma:

**Lemma 2.4.** In a strong pseudo quasi-ordered residuated system  $\mathfrak{p}\mathfrak{A}$  the following holds:

$$(18) (\forall u, v \in A)(u \leq v \implies v \equiv_{\leq} (v \rightarrow u) \rightsquigarrow u) \text{ and}$$

$$(19) (\forall u, v \in A)(u \leq v \implies v \equiv_{\leq} (v \rightsquigarrow u) \rightarrow u).$$

*Proof.* Assume that  $\mathfrak{p}\mathfrak{A}$  is a strong pseudo quasi-ordered system. For given  $u, v \in A$ , let be  $u \leq v$ . First, as  $v \leq (u \rightarrow v) \rightsquigarrow v$  is valid according to (g), from here we immediately get the following  $v \leq (u \rightarrow v) \rightsquigarrow v \leq$

$(v \rightarrow u) \rightsquigarrow u$  with respect to (S1). Secondly, from  $u \preceq v$  it follows  $1 \preceq u \rightarrow v$  by (3L). From here, applying (f), we obtain  $(u \rightarrow v) \rightsquigarrow v \preceq 1 \rightsquigarrow v$ . Thus  $(v \rightarrow u) \rightsquigarrow u \equiv_{\preceq} (u \rightarrow v) \rightsquigarrow v \preceq 1 \rightsquigarrow v$  with respect (S1). Applying the claim (i), from here we get  $(v \rightarrow u) \rightsquigarrow u \preceq v$ . This proves that  $u \preceq v \implies v \equiv_{\preceq} (v \rightarrow u) \rightsquigarrow u$  is valid.

Proof for (19) can be demonstrated by analogy for evidence of the implication (18).  $\square$

As a significant consequences of this lemma we can prove some important properties of strong quasi-ordered residuated systems. In what follows, we need the following lemma:

**Lemma 2.5.** *Let  $\mathbf{p}\mathfrak{A}$  be a strong pseudo quasi-ordered residuated system. Then the following holds:*

$$(20) (\forall u, v \in A)(u \rightarrow v \equiv_{\preceq} ((u \rightarrow v) \rightsquigarrow v) \rightarrow v) \text{ and}$$

$$(21) (\forall u, v \in A)(u \rightsquigarrow v \equiv_{\preceq} ((u \rightsquigarrow v) \rightarrow v) \rightsquigarrow v).$$

*Proof.* Let  $\mathbf{p}\mathfrak{A}$  be a strong pseudo quasi-ordered residuated system and let  $u, v \in A$  be arbitrary elements. As  $v \preceq u \rightarrow v$  holds according to (g), we conclude that the required formula (20) is valid by (19).

Similarly, from  $u \cdot v \preceq v$ , it follows  $v \preceq u \rightsquigarrow v$  according to (g) and (3R). If we apply (18) to this, we get  $u \rightsquigarrow v \equiv_{\preceq} ((u \rightsquigarrow v) \rightarrow v) \rightsquigarrow v$  which is the need to prove.  $\square$

**Theorem 2.6.** *Let  $\mathbf{p}\mathfrak{A}$  be a strong pseudo quasi-ordered residuated system. For any  $u, v \in A$ , the elements  $u \sqcup_1 v$  and  $u \sqcup_2 v$  are the least upper bounds of  $u$  and  $v$  and, according that,  $u \sqcup_1 v = u \sqcup_2 v$  holds.*

*Proof.* It is clear that the following holds  $u \preceq (v \rightarrow u) \rightsquigarrow u = u \sqcup_1 v$  and  $v \preceq (u \rightarrow v) \rightsquigarrow v \equiv_{\preceq} (v \rightarrow u) \rightsquigarrow u = u \sqcup_1 v$ . by (g) and (S1). This shows that  $u \sqcup_1 v$  is an upper bound for  $u$  and  $v$ . If  $z$  is any common upper bound of  $u$  and  $v$ , then by (18),  $z \equiv_{\preceq} (z \rightarrow u) \rightsquigarrow u$  and  $z \equiv_{\preceq} (z \rightarrow v) \rightsquigarrow v$ . On the other hand, from  $v \preceq (z \rightarrow v) \rightsquigarrow v$  it follows  $((z \rightarrow v) \rightsquigarrow v) \rightarrow u \preceq v \rightarrow u$  and  $(v \rightarrow u) \rightsquigarrow u \preceq (((z \rightarrow v) \rightsquigarrow v) \rightarrow u) \rightsquigarrow u \equiv_{\preceq} (z \rightarrow u) \rightsquigarrow u \equiv_{\preceq} z$  according to (e), (f) and (21). Therefore,  $u \sqcup_1 v$  is the least upper bound of  $u$  and  $v$  in the system  $\mathbf{p}\mathfrak{A}$ .

It can be proved that  $u \sqcup_2 v$  is the least upper bound for elements  $u$  and  $v$  by analogy according to previous evidence. As the least upper bound for elements  $u$  and  $v$  is unique, must be  $u \sqcup_1 v = u \sqcup_2 v$ .  $\square$

**Corollary 2.7.** *Let  $\mathbf{p}\mathfrak{A}$  be a strong pseudo quasi-ordered residuated system. Then:*

$$(\forall y, v \in A)(u \sqcup v = v \sqcup u),$$

$$(\forall u, v \in A)(u \preceq u \sqcup v \wedge v \preceq u \sqcup v),$$

$$(\forall u, v \in A)(u \sqcup v \equiv_{\preceq} v \iff u \preceq v),$$

$$(\forall u, v, z \in A)(u \preceq v \implies u \sqcup z \preceq v \sqcup z)$$

*Proof.* We will prove the last formula. Let  $u, v, z \in A$  be such that  $u \preceq v$ . Then  $v \rightarrow z \preceq u \rightarrow z$  by (e). Thus  $(u \rightarrow z) \rightsquigarrow z \preceq (v \rightarrow z) \rightsquigarrow z$  by (f). This means  $u \sqcup z \preceq v \sqcup z$ .  $\square$

**2.2. Filters in  $\mathbf{pQRS}$ .** Filters in pseudo-hoop algebra have been studied in [8] and [1], for example. In this algebraic structure, a filter  $F$  is a non-empty sub-semigroup that satisfies the standard condition for filters. In addition, in [8], Proposition 3.1 it is shown that the condition

$$1 \in F \wedge (\forall a, b \in A)((a \in F \wedge a \rightarrow b \in F) \implies b \in F)$$

is equivalent to the condition

$$1 \in F \wedge (\forall a, b \in A)((a \in F \wedge a \rightsquigarrow b \in F) \implies b \in F).$$

In pseudo quasi-ordered residuated systems, the concept of filters in them is determined somewhat differently.

**Definition 2.8.** Let  $\mathfrak{p}\mathfrak{A} = \langle A, \cdot, \rightarrow, \rightsquigarrow, 1, \preceq \rangle$  be a pseudo quasi-ordered residuated system. A subset  $F$  of  $A$  is a filter of  $\mathfrak{p}\mathfrak{A}$  if it satisfies the following conditions:

- (F2)  $(\forall x, y \in A)((x \in F \wedge x \preceq y) \implies y \in F),$
- (F3L)  $(\forall x, y \in A)((x \in F \wedge x \rightarrow y \in F) \implies y \in F),$
- (F3R)  $(\forall x, y \in A)((x \in F \wedge x \rightsquigarrow y \in F) \implies y \in F).$

It is easy to conclude that the sets  $\emptyset$  and  $A$  are filters in  $\mathfrak{p}\mathfrak{A}$ . These are trivial filters in  $\mathfrak{p}\mathfrak{A}$ . A filter  $F$  of  $\mathfrak{p}\mathfrak{A}$  is proper if  $F \neq A$ . Besides, if  $F$  is a non-empty filter in  $\mathfrak{p}\mathfrak{A}$ , then  $1 \in F$ . Indeed, if  $F \neq \emptyset$ , then there exists an element  $x \in F$ . Since  $x \preceq 1$ , according to (2), we conclude that  $1 \in F$  according to (F2). Thus,  $(F \neq \emptyset \wedge (F2)) \implies 1 \in F$ .

In addition to the above, the following implications are valid  $(F3L) \implies (F2)$  and  $(F3R) \implies (F2)$  assuming  $F \neq \emptyset$ . Assume that (F3L) (i.e. (3FR), respectively) is a valid formula and let elements  $x, y \in A$  be such that  $x \in F \wedge x \preceq y$ . Then  $x \in F \wedge x \rightarrow y \equiv_{\preceq} 1 \in F$  and  $x \in F \wedge x \rightsquigarrow y \equiv_{\preceq} 1 \in F$ . Hence  $y \in F$  according to (3FL) or (3FR) respectively.

Also, for a non-empty filter  $F$  in a pseudo quasi-ordered residuated system  $\mathfrak{p}\mathfrak{A}$ , conditions (F3L) and (F3R) are equivalent. Indeed. Let the non-empty set  $F$  in a pseudo quasi-ordered residuated system  $\mathfrak{p}\mathfrak{A}$ , satisfy the condition (3FR). The assumption  $F \neq \emptyset$ , implies  $1 \in F$ . Let  $u, v \in A$  be arbitrary elements such that  $u \rightarrow v \in F$ . Then  $u \rightarrow v \preceq (v \rightsquigarrow z) \rightarrow (u \rightsquigarrow z)$ . If we take  $z = v$ , we get  $u \rightarrow v \preceq (v \rightsquigarrow v) \rightarrow (u \rightsquigarrow v)$ . This means  $u \rightarrow v \preceq 1 \rightarrow (u \rightsquigarrow v)$  by (b) and  $u \rightarrow v \preceq u \rightsquigarrow v$ , due to (i). Thus  $u \rightsquigarrow v \in F$  by (F2). Therefore,  $v \in F$  by (F3R). This means that the implication  $F \neq \emptyset \wedge (F3R) \implies (F3L)$  is proved. The implication  $F \neq \emptyset \wedge (F3L) \implies (F3R)$  can be proved analogously to the previous proof.

The concepts of implicative filters and comparative filters can be determined in pseudo-quasi-ordered residuated systems by looking at the corresponding concepts in pseudo-hoop algebras as done in [1]:

**Definition 2.9.** Let  $\mathfrak{p}\mathfrak{A}$  be a pseudo quasi-ordered residuated system and  $F$  be a non-empty subset of  $A$ .  $F$  is said to be an *implicative filter* in  $\mathfrak{p}\mathfrak{A}$  if the following holds:

- (F2)  $(\forall x, y \in A)((x \in F \wedge x \preceq y) \implies y \in F),$
- (IF3)  $(\forall x, y, z \in A)((x \in F \wedge x \rightarrow ((y \rightarrow z) \rightsquigarrow y) \in F) \implies y \in F)$  and
- (IF4)  $(\forall x, y, z \in A)((x \in F \wedge x \rightsquigarrow ((y \rightsquigarrow z) \rightarrow y) \in F) \implies y \in F).$

**Theorem 2.10.** Every implicative filter in a pseudo quasi-ordered residuated system  $\mathfrak{p}\mathfrak{A}$  is a filter in  $\mathfrak{p}\mathfrak{A}$ .

*Proof.* Let  $F$  be an implicative filter and  $x, y \in A$  be such that  $x \in D$  and  $x \rightarrow y \in F$ . According to the previous analysis, it is sufficient to prove (3FL), since (3FR) is equivalent to (3FL). We have  $x \rightarrow ((y \rightarrow 1) \rightsquigarrow y) \equiv_{\preceq} x \rightarrow y \in F$  with respect to (a) and (i). Now from this and from  $x \in F$  it follows  $y \in F$  according to (IF3).  $\square$

**Definition 2.11.** Let  $\mathfrak{p}\mathfrak{A}$  be a pseudo quasi-ordered residuated system and  $F$  be a non-empty subset of  $A$ . A subset  $F$  of  $A$  is called a *comparative filter* of  $\mathfrak{p}\mathfrak{A}$  if the following holds:

- (F2)  $(\forall x, y \in A)((x \in F \wedge x \preceq y) \implies y \in F),$
- (CF3)  $(\forall x, y, z \in A)((x \rightsquigarrow y \in F \wedge x \rightarrow (y \rightarrow z) \in F) \implies x \rightarrow z \in F),$
- (CF4)  $(\forall x, y, z \in A)((x \rightarrow y \in F \wedge x \rightsquigarrow (y \rightsquigarrow z) \in F) \implies x \rightsquigarrow z \in F).$

**Theorem 2.12.** Every comparative filter in a pseudo quasi-ordered residuated system  $\mathfrak{p}\mathfrak{A}$  is a filter in  $\mathfrak{p}\mathfrak{A}$ .

*Proof.* Suppose that  $F$  is a comparative. Then due to the equivalence of conditions (3FL) and (3FR), it is enough to prove that if  $x \in F \wedge x \rightsquigarrow y \in F$ , then  $y \in F$ , for any  $x, y \in A$ . First, we have  $1 \rightsquigarrow (x \rightsquigarrow y) \equiv_{\preceq} x \rightsquigarrow y \in F$  and  $1 \rightarrow x \equiv_{\preceq} x \in F$  according to (i). Since  $F$  is a comparative filter in  $\mathfrak{p}\mathfrak{A}$ , we have  $1 \rightsquigarrow y \equiv_{\preceq} y \in F$ . Therefore,  $F$  is a filter.  $\square$



**2.3. Ideals in pQRS.** Ideal theory plays a fundamental role in many algebraic structures, such as lattices, rings and pseudo-MV algebras. Georgescu and Iorgulescu in [7] introduced the notion of ideals in pseudo-MV algebras, which was shown effective in studying structure properties of pseudo-MV algebras. In [18], F. Xie and H. Liu introduced the notions of ideals in pseudo-hoop algebras. As pseudo-hoop algebras may not have lattice structures, not all pseudo-hoop algebras are general residuated lattices. Since pseudo-MV algebras are particular cases of general residuated lattices, the notion of ideals in pseudo-hoop algebras can not be similar to that in pseudo-MV algebras. This will also apply to ideals in pseudo-quasi-ordered residuated systems since this latter structure is a generation of pseudo-hoop algebras. Apart from the above, the procedure applied in article [18] cannot be applied for determining the concept of ideal in pseudo quasi-ordered residuated systems.

**Definition 2.13.** Let  $\mathfrak{p}\mathfrak{A}$  be a pseudo quasi-ordered residuated system and  $J$  be a subset of  $A$ .

- For  $J$  we say that it is an *ideal* in  $\mathfrak{p}\mathfrak{A}$  if the following conditions holds:

$$(J2) (\forall x, y \in A)((x \preceq y \wedge y \in J) \implies x \in J)$$

$$(J3L) (\forall x, y \in A)((x \rightarrow y \notin J \wedge y \in J) \implies x \in J) \text{ and}$$

$$(J3R) (\forall x, y \in A)((x \rightsquigarrow y \notin J \wedge y \in J) \implies x \in J);$$

- For  $J$  we say that it is an *pre-ideal* in  $\mathfrak{p}\mathfrak{A}$  if the condition (J2) is valid.

It is not difficult to conclude:

**Proposition 2.14.** Let  $\mathfrak{p}\mathfrak{A}$  be a pseudo quasi-ordered residuated system and  $J$  be a proper subset of  $A$ . Then (J2) implies the following

$$(J1) (\forall x, y \in A)((x \in J \vee y \in J) \implies x \cdot y \in J).$$

*Proof.* The condition (J1) follows from the condition (J2) according to the (g).  $\square$

The previous proposition proved that any ideal  $J$  in a pseudo quasi-ordered residuated system  $\mathfrak{p}\mathfrak{A}$  is a sub-semigroup in  $A$ .

**Proposition 2.15.** The condition (J2) is equivalent to the condition

$$(J4L) (\forall x, y, z \in A)((x \preceq y \rightarrow z \wedge z \in J) \implies x \cdot y \in J).$$

*Proof.* (J4L)  $\implies$  (J2). Assume that (J4L) is valid and let  $x, y \in A$  be arbitrary elements such that  $x \preceq y$  and  $y \in J$ . Then  $x \preceq 1 \rightarrow y$  and  $y \in J$ . Thus  $x = u \cdot 1 \in J$  by (J4). So, the condition (J2) is proven.

(J2)  $\implies$  (J4L). Suppose (J2) holds and let  $x, y, z \in A$  be arbitrary elements such that  $x \preceq y \rightarrow z$  and  $z \in J$ . Then  $x \cdot y \preceq z$  and  $z \in J$  by (3L). Thus  $x \cdot y \in J$  by (J2). So, the condition (J4L) is proven.  $\square$

Also, the validity of the following proposition can be demonstrated:

**Proposition 2.16.** The condition (J2) is equivalent to the condition

$$(J4R) (\forall x, y, z \in A)((y \preceq x \rightsquigarrow z \wedge z \in J) \implies x \cdot y \in J).$$

As a direct consequence of the previous two propositions, the conclusion follows: (J4L)  $\iff$  (J4R).

Analogous to the article [12], it can be proved:

**Proposition 2.17.** Let  $\mathfrak{p}\mathfrak{A}$  be a pseudo quasi-ordered residuated system and  $J$  be a proper subset of  $A$ .

- The condition (J3R) is equivalent to the condition

$$(J5R) (\forall x, y \in A)((x \notin J \wedge y \in J) \implies x \rightarrow y \in J).$$

- The condition (J3L) is equivalent to the condition

$$(J5L) (\forall x, y \in A)((x \notin J \wedge y \in J) \implies x \rightsquigarrow y \in J)$$

*Proof.* For the sake of illustration, we prove the equivalence  $(J3R) \iff (J5R)$ .

$(J3R) \implies (J5R)$ . Suppose that formula  $(J3R)$  is valid and let  $x, y \in A$  be elements such that  $x \notin J$  and  $y \in J$ . If we assume that  $x \rightsquigarrow y \notin J$  holds, then it would be  $x \in J$  according to  $(J3R)$ . The resulting contradiction refutes the assumption. So, it has to be  $x \rightsquigarrow y \in J$ .

$(J5R) \implies (J3R)$ . Conversely, let  $(J5R)$  be valid and let the elements  $\in A$  be such that  $x \rightsquigarrow y \notin J$  and  $y \in J$ . If we assume that  $x \notin J$  is valid we would have  $x \rightsquigarrow y \in J$  according to  $(J5R)$ . This contradicts the hypothesis. So, it must be  $x \in J$ .  $\square$

The (pre-)ideal  $J$  of a pseudo quasi-ordered residuated system  $\mathfrak{p}\mathfrak{A}$  is said to be a proper ideal if  $J \neq A$ . In that case it is satisfied the following condition

$$(J0) 1 \notin J.$$

### 3. CONCLUSIONS AND THE POSSIBILITY OF ADVANCED RESEARCH

This report presents the results obtained in the research of the concept of pseudo quasi-ordered residuated systems. In this newly determined algebraic structure as a non-commutative generalization of quasi-ordered residuated systems, substructures of filters and ideals in them are considered. In the presented material, a reader can gain an insight into the specificity of this class of algebraic structures in relation to pseudo-hoops, for example. This specificity can be seen in the part that refers to designing ideals.

The material presented in this paper opens the possibility of more advanced research of different types of filters in this algebraic structure as well as their mutual relations and a more precise understanding of the role of ideals in them.

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