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POSITIVE IMPLICATIVE IDEALS IN QI-ALGEBRAS

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ABSTRACT. In this article, we design several QI-algebra constructions for a given QI-algebra, and then besides proving several additional properties of ideals in (right distributive) QI-algebras, we introduce the concept of positive implicative ideals in QI-algebras and prove some of its important properties. Also, it was shown that every ideal in a right distributive QI-algebra is a positive implicative ideal in it.

1. INTRODUCTION

In 1966, Imai and Iséeki ([3]) introduced two classes of algebraic structures named BCK-algebras and BCI-algebras. The concept of BI-algebras was introduced ([2]) in 2017 by A. Borumand Saeid, H. S. Kim and A. Rezaei as a generalization of BCK-algebras. QI-algebras, as a generalization of BI-algebras, were introduced in 2017 in [6] by R. Kumar Bandaru. Then this class of logical algebras was the focus of several authors (see, for example [7,8,10,11]). While in the article [7] the concepts of implicative, normal and fantastic ideals in QI-algebras are discussed, in [8] the properties of ideals and their connection with congruences in (right-distributive) QI-algebras are reviewed. While in the article [10] the so-called the first isomorphism theorem for this class of algebras are proved, in the paper [11], a pseudo-valuation on QI-algebras was created and analyzed.

In this article, as a direct continuation of the research started in [7,8], we first create several constructions of new QI-algebras $([X : A], *, X)$ and $([A : X], *, 0X)$ for a given QI-algebra $(A, \cdot, 0)$ where X is a subset of A. We show how the QI-algebra $(A, \cdot, 0)$ can be extended to the QI-algebra $(A \cup X, *, 0)$, where $A \cap X = \emptyset$. Also, we prove several additional properties of ideals in (right distributive) QI-algebras, and then we introduce and analyze the concept of positive implicative ideals in QI-algebras. A criterion for recognizing positive implicative ideals in QI-algebras was designed. Additionally, it was shown that every ideal in a right distributive QI-algebra is a positive implicative ideal in it.

2. PRELIMINARIES

It should be emphasized here that the formulas in this text are written in a standard way, as is usual in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Longrightarrow$, and so on, are labels for the logical functions of conjunction, disjunction, implication, and so on. Brackets in formulas are used in the standard way, too. All formulas appearing in this paper are closed by some quantifier. If one of the formulas is open, then the variables that appear in it should be seen as free variables.

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In addition to the previous one, the sign =:, in the use of $A =: B$, should be understood in the sense that the mark A is the abbreviation for the formula B.

In this text, to mark recognizable formulas, we will use, as far as possible, their standard abbreviations that appear in a very well-known paper [4].

The notion of BI-algebras comes from the (dual) implication algebra. An algebra $\mathfrak{A} =: (A, \cdot, 0)$ of type $(2, 0)$ is called a BI-algebra ([2], Definition 3.1) if the following holds:

(Re) $(\forall x \in A)(x \cdot x = 0)$,

(Im) $(\forall x, y \in A)(x \cdot (y \cdot x) = x)$.

A BI-algebra $\mathfrak{A}$ is said to be right distributive if the following

(DR) $(\forall x, y, z \in A)((x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z))$

is valid.

The concept of QI-algebras, as a generalization of BI-algebras, was determined in 2017 by R. Kumar Bandaru. An algebra $\mathfrak{A} =: (A, \cdot, 0)$ of type $(2, 0)$ is called a QI-algebra ([6], Definition 3.1) if the following holds:

(Re) $(\forall x \in A)(x \cdot x = 0)$,

(M) $(\forall x \in A)(x \cdot 0 = x)$,

(QI) $(\forall x, y \in A)(x \cdot (y \cdot (x \cdot y)) = x \cdot y)$.

A QI-algebra $\mathfrak{A}$ is said to be right distributive if, additionally, formula (DR) is valid.

Note that every BI-algebra is a QI-algebra but converse need not be true ([6], Example 3.2). Also, every implicative BCK-algebra (in sense of [5]) is a QI-algebra but converse need not be true ([6], Example 3.3).

The terms and symbols used in this paper, which are not determined in it, are taken from the source [4, 6–8].

3. Some constructions of QI-algebras for a given QI-algebra

Let $\mathfrak{A} =: (A, \cdot, 0)$ be a QI-algebra and X be a non-empty subset of A and $a \in A$. Let us define $Xa =: \{x \cdot a : x \in X\}$, $aX =: \{a \cdot x : x \in X\}$, $[X : A] =: \{Xa : x \in A\}$ and $[A : X] =: \{aX : a \in A\}$. Additionally, if we introduce the operation $*$ on the set $[X : A]$, that is, on $[A : X]$, as follows

$$(\forall a, b \in A)(Xa * Xb =: X(a \cdot b)), \text{ and } (\forall a, b \in A)(aX * bX = (a \cdot b)X)),$$

we have:

Theorem 3.1. *Let $\mathfrak{A} = (A, \cdot, 0)$ be a QI-algebra and X be a subset of A. Then the structures $[X : \mathfrak{A}] =: ([X : A], *, X)$ and $[\mathfrak{A} : X] =: ([A : X], *, 0X)$ are QI-algebras. If $\mathfrak{A}$ is a right distributive QI-algebra, then the algebras $[X : \mathfrak{A}]$ and $[\mathfrak{A} : X]$ are also right distributive.*

Proof. By direct verification, we have that it is:

$X0 = \{x \cdot 0 : x \in X\} = X$ and $0X = \{0 \cdot x : x \in X\}$,

$(\forall a \in A)(Xa * X = X(a \cdot 0) = Xa$ and $aX * 0X = (a \cdot 0)X = aX)$,

$(\forall a \in A)(Xa * Xa = X(a \cdot a) = X0 = X$ and $aX * aX = (a \cdot a)X = 0X)$,

$(\forall a, b \in A)(Xa * (Xb * (Xa * Xb)) = ... = X(a \cdot (b \cdot (a \cdot b))) = X(a \cdot b) = Xa * Xb)$.

$(\forall a, b \in A)(aX * (bX * (aX * bX)) = ... = (a \cdot (b \cdot (a \cdot b)))X = (a \cdot b)X = aX * bX)$.

Therefore, we can conclude that the structures $[X : \mathfrak{A}] =: ([X : A], *, X)$ and $[\mathfrak{A} : X] = ([A : X], *, 0X)$: are

QI-algebras. Here, in the QI-algebras constructed in this way, the role of the constant element in the algebra $[X : \mathfrak{A}]$ is played by the set X itself, i.e. the set $0X$ for $[\mathfrak{A} : X]$, respectively.

The second part of this theorem can be proved by direct verification. □

In particular, if $\mathfrak{A}$ is right distributive, then, according to Proposition 3.9(QI5) in [6] and (Re), we have that $[\{0\} : \mathfrak{A}] = (\{\{0\}\}, *, \{0\})$ which can be identified with the trivial QI-algebra $(\{0\}, \cdot, 0)$. Therefore, this construction should be considered only in non-distributive QI-algebras.

On the other hand, in the second case, in accordance with (M) and (Re), we have $[\mathfrak{A} : \{0\}] = ([\{\{a\} : a \in A\}, *, \{0\})$ which can be identified with the algebra $(A, \cdot, 0)$.

To illustrate this construction, let us look at the following example.

Example 3.2. Let $A = \{0, a, b, c\}$ be a set with the operation given with the table

$\cdot$	0	a	b	c
0	0	b	a	0
a	a	0	a	0
b	b	b	0	b
c	c	b	a	0

Then $\mathfrak{A} = (A, \cdot, 0)$ is a QI-algebra ([10], Example 3.2). For example, for a chosen subset $X = \{0, c\}$, we have

$[\{0, c\} : \mathfrak{A}] = (\{\{0\}, \{a\}, \{b\}, \{0, c\}\}, *, \{0, c\})$ and

$[\mathfrak{A} : \{0, c\}] = (\{\{0\}, \{0, a\}, \{b\}, \{0, c\}\}, *, \{0\})$. □

Another way of constructing a QI-algebra for a given QI-algebra is described by the following theorem:

Theorem 3.3. *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a (right distributive) QI-algebra and X be a set such that $X \cap A = \emptyset$. Then $\mathfrak{B} = (B, *, 0)$ is a (right distributive) QI-algebra, where $B = A \cup X$ and the operation $*$ defined as follows:*

$$x * y = \begin{cases} x \cdot y & x \in A \wedge y \in A, \\ 1 & x = y, \\ x & \textit{otherwise}, \end{cases}$$

Proof. It is clear that $\mathfrak{B} =: (B, *, 0)$ satisfies (Re) and (M). To prove (QI) consider the following cases:

(i) If $x = y$, then

$x * (y * (x * y)) = x * (x * (x * x)) = x * (x * 0) = x * x.$

(ii) If $x, y \in A$, then the proof is trivial.

(iii) If $x \in A$ and $y \in X$ or if $x \in X$ and $y \in A$, then

$x * (y * (x * y)) = x * (y * x) = x * y.$

Therefore, $\mathfrak{B}$ is a QI-algebra.

That the formula (RD) is a valid formula in the algebra $\mathfrak{B}$ can be checked in a similar way as it was done previously. Thus, $\mathfrak{B}$ is a right distributive QI-algebra if $\mathfrak{A}$ is a right distributive Qi-algebra. □

In order to illustrate the previous theorem, we give the following:

Example 3.4. Let $\mathfrak{A} =: (A, \cdot, 0)$ be a QI-algebra as in Example 3.2 and $X = \{d\}$. Let $B = \{0, a, b, c, d\} = A \cup \{d\}$ be a set with the operation given with the table

$*$	0	a	b	c	d
0	0	b	a	0	0
a	a	0	a	0	a
b	b	b	0	b	b
c	c	b	a	0	c
d	d	d	d	d	0

Then $\mathfrak{B} = (A \cup \{d\}, *, 0)$ is a QI-algebra according to Theorem 3.3

4. Something more on ideals

In this section we present several new results about ideals in (right distributive) QI-algebras.

The concept of ideal in QI-algebras is determined by the following definition:

Definition 4.1. ([6], Definition 4.1) A subset J of a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is called an ideal of $\mathfrak{A}$ if the following holds:

(J0) $0 \in J$,

(J1) $(\forall x, y \in A)((x \cdot y \in J \wedge y \in J) \implies x \in J)$.

We denote the family of all ideals in the QI-algebra $\mathfrak{A}$ by $\mathfrak{J}(A)$.

Example 4.2. Let $A = \{0, a, b, c\}$ be a set with the operation given with the table

$\cdot$	0	a	b	c
0	0	b	a	0
a	a	0	a	0
b	b	b	0	b
c	c	c	b	0

Then $\mathfrak{A} = (A, \cdot, 0)$ is a QI-algebra ([6], Example 3.2).

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_3 = \{0, c\}$ and $J_5 = \{0, a, c\}$ of the set A are ideals of the QI-algebra $\mathfrak{A}$. The subset $J_2 = \{0, b\}$ is not an ideal in $\mathfrak{A}$ because, for example, for $b \in J_2$ we have $c \cdot b = b \in J_2$ but $c \notin J_2$. The subsets $J_4 = \{0, a, b\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $c \cdot b = b \in J_4$ but $c \notin J_4$ The subset $J_6 = \{0, b, c\}$ also not an ideal in $\mathfrak{A}$ because, for example, for $c \in J_6$ we have $a \cdot c = 0 \in J_6$ but $a \notin J_6$. □

Example 4.3. Let $A = \{0, a, b, c\}$ be a set with the operation given with the table

$\cdot$	0	a	b	c
0	0	b	a	0
a	a	0	a	a
b	b	b	0	b
c	c	b	c	0

Then $\mathfrak{A} = (A, \cdot, 0)$ is a QI-algebra ([10], Example 3.21). This algebra is not right distributive because, for example, we have $(b \cdot c) \cdot a = b \cdot a = b \neq 0 = b \cdot b = (b \cdot a) \cdot (c \cdot a)$.

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_3 = \{0, c\}$, $J_5 = \{0, a, c\}$ and $J_6 = \{0, b, c\}$ of the set A are ideals of the QI-algebra $\mathfrak{A}$. Subset $J_2 = \{0, b\}$ is not an ideal in $\mathfrak{A}$, because, for example, we have $c \cdot b = b \in J_2$ but $c \notin J_2$. Subset $J_4 = \{0, a, b\}$ is also not an ideal in $\mathfrak{A}$, because, for example, we have $c \cdot b = b \in J_4$ but $c \notin J_4$.

Remark 4.4. On the other hand, in accordance with the axiom (QI), the determination of ideals in QI-algebras can be done in the following way: A subset J in a QI-algebra $\mathfrak{A} = (A, \cdot, 0)$ is called an ideal in $\mathfrak{A}$ if, in addition to (J0), it also satisfies the following condition:

(J1a) $(\forall x, y \in A)((x \cdot (y \cdot (x \cdot y)) \in J \wedge y \in J) \implies x \in J)$.

Let $\mathfrak{A} = (A, \cdot, 0)$ be a QI-algebra. The relation $\preccurlyeq$ on a QI-algebra $\mathfrak{A}$ is introduced in the standard way:

$$(\forall x, y \in A)(x \preccurlyeq y \iff x \cdot y = 0).$$

A relation $\preccurlyeq$ is not a partially order, but it is only reflexive.

In addition to the previous one, it is easy to conclude that for an ideal J in a QI-algebra $\mathfrak{A}$ holds:

Proposition 4.1 ([8], Proposition 3.4)**.** *Let J be an ideal in a QI-algebra A. Then:*

(J2) $(\forall x, y \in A)((x \preccurlyeq y \wedge y \in J) \implies x \in J)$.

If $\mathfrak{A}$ is a right distributive QI-algebra, then it additionally holds

(J3) $(\forall x, y \in A)(x \in J \implies x \cdot y \in J)$.

Theorem 4.5 ([8], Theorem 3.3)**.** *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a right distributive QI-algebra and $J \subseteq A$ such that $0 \in J$. Then J is an ideal in $\mathfrak{A}$ if and only if it satisfies the following condition*

(J4) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \wedge y \in J) \implies x \cdot z \in J)$.

The following theorem gives a criterion for recognizing ideals in QI-algebras.

Theorem 4.6. *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a QI-algebra. A subset J of A is an ideal in $\mathfrak{A}$ if and only if it holds*

(J5) $(\forall x, y, z \in A)((y \in J \wedge z \in J \wedge (x \cdot y) \cdot z = 0) \implies x \in J)$.

Proof. Let J be an ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $y, z \in J$ and $(x \cdot y) \cdot z = 0$. Then $x \cdot y \preccurlyeq z \in J$. Thus $x \cdot y \in J$ by (J2). Hence, $x \in J$ by (J1).

Conversely, let (J5) be valid and let $x, y \in A$ be such that $x \cdot y \in J$ and $y \in J$. Then $(x \cdot y) \cdot (x \cdot y) = 0$, $y \in J$ and $x \cdot y \in J$ according to (Re). Thus $x \in J$ by (J5). Let us show that (J0) holds. Putting $x = 0$ in (J5), we get $0 \in J$ with respect (L). Therefore, J is an ideal in $\mathfrak{A}$. □

In what follows, we need the following lemma:

Lemma 4.7 ([6], Proposition 3.9, Proposition 3.10)**.** *If $\mathfrak{A} =: (A, \cdot, 0)$ is a right distributive QI-algebra, then:*

(L) $(\forall x \in A)(0 \cdot x = 0)$,

(1) $(\forall x, y \in A)((x \cdot y) \cdot y = x \cdot y)$.

(2) $(\forall x, y, z \in A)((x \cdot y) \cdot z \preccurlyeq x \cdot (y \cdot z))$

However, if for the element $a \in A$ we introduce the label $L(a) =: \{x \in A : x \preccurlyeq a\}$ and

$$L_0(a) =: \{x \in A : x \preccurlyeq a\} \cup \{0\} = \{x \in A : x \cdot a = 0 \vee x = 0\},$$

we can prove the following proposition.

Proposition 4.2. *Let $a \in A$ be an element of a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$. Then $L_0(a)$ is an ideal in $\mathfrak{A}$ if and only if the following holds*

(J6) $(\forall u, v \in A)((v \preccurlyeq a \wedge (u \cdot v \preccurlyeq a \vee u \cdot v = 0)) \Longrightarrow (u = 0 \vee u \preccurlyeq a))$.

If $\mathfrak{A}$ is a right distributive QI-algebra, then the subset $L(a)$ is an ideal in $\mathfrak{A}$ for every element $a \in A$.

Proof. Let $L_0(a) = \{x \in A : x \preccurlyeq a\} \cup \{0\}$ be an ideal in $\mathfrak{A}$ and let $u, v \in A$ be such that $v \in L_0(a)$ and $u \cdot v \in L_0(a)$. This means ($v \preccurlyeq a$ or $v = 0$) and ($u \cdot v \preccurlyeq a$ or $u \cdot v = 0$). If $v = 0$, then we have $u \cdot v = u \cdot 0 = u = 0$ or $u \cdot v = u \cdot 0 = u \preccurlyeq a$, which gives $u \in L(a)$. Let us assume that $v \preccurlyeq a$ and ($u \cdot v = 0$ or $y \cdot v \preccurlyeq a$). Thus, we have $v \preccurlyeq a \wedge (u \cdot v = 0 \vee u \cdot v \preccurlyeq a) \Longrightarrow (u = 0 \vee u \preccurlyeq a)$ since $L_0(a)$ is an ideal in $\mathfrak{A}$. This proves the validity of (J6).

Conversely, let the subset $L_0(a)$ satisfy the condition (J6). Let us prove, in that case, that $L_0(a)$ is an ideal in $\mathfrak{A}$. For $u, v \in A$, such that $u \cdot v \in L_0(a)$ and $v \in L_0(a)$, we have $(u \cdot v = 0 \vee u \cdot v \preccurlyeq a)$ and $(v = 0 \vee v \preccurlyeq a)$. If $v = 0$, then $(u = u \cdot 0 = 0 \ \vee \ u = u \cdot 0 \preccurlyeq a)$. Thus, $u \in L_0(a)$. Assume that $v \preccurlyeq a$. Then, according to (J6), we have $u = 0 \vee u \preccurlyeq a$, so we again conclude that $u \in L_0(a)$.

Assume that $\mathfrak{A}$ is right distributive. Since $\mathfrak{A}$ is right distributive, we have $0 \cdot a = 0$ for every $a \in A$, according to (L). Therefore, $0 \in L(a)$. Let $u, v \in A$ be such that $v \in L(a)$ and $u \cdot v \in L(a)$. This means $v \preccurlyeq a$ and $u \cdot v \preccurlyeq a$. Then $v \cdot a = 0$ and $(u \cdot v) \cdot a = 0$. Thus $0 = (u \cdot a) \cdot (v \cdot a) = (u \cdot a) \cdot 0 = u \cdot a$ with respect to (RD) and (M). So, $u \preccurlyeq a$ and $u \in L(a)$. □

Example 4.8. Let $A = \{0, a, b, c\}$ be as in Example 4.2. Then $\mathfrak{A} =: (A, \cdot, 0)$ is a QI-algebra. In this case, we have: $L_0(0) = \{0\} \in \mathfrak{J}(A)$, $L_0(a) = \{0, a\} \in \mathfrak{J}(A)$, $L_0(b) = \{0, b\} \notin \mathfrak{J}(A)$ and $L_0(c) = \{0, a, c\} \in \mathfrak{J}(A)$. □

Let $\mathfrak{A} =: (A, \cdot, 0)$ be a QI-algebra, X be a non-empty subset of A and $a \in A$ be an arbitrary element. Let us define $X_a =: \{x \in A : x \cdot a \in X\}$.

Theorem 4.9. *If J is an ideal of a right distributive QI-algebra $\mathfrak{A}$, then for any $a \in A$, the subset J_a is the smallest ideal in $\mathfrak{A}$ containing J and a.*

Proof. It is clear that $0 \in J_a$ holds because $0 \cdot a = 0 \in J$ by (L) and (J0).

Let $x, y \in A$ be such that $x \cdot y \in J_a$ and $y \in J_a$. This means $(x \cdot y) \cdot a \in J$ and $y \cdot a \in J$. Then $(x \cdot a) \cdot (y \cdot a) \in J$ according to (DR). Thus, $x \cdot a \in J$ by (J1). Hence, $x \in J_a$. Therefore, J_a is an ideal in $\mathfrak{A}$.

Since $\mathfrak{A}$ is a right distributive QI-algebra, for all $x \in J$ we have $(x \cdot a) \cdot a \preccurlyeq x \cdot (a \cdot a) = x \cdot 0 = x \in J$ by (1), (Re) and (M). Then $(x \cdot a) \cdot a \in J$ by (J2). Thus, $x \cdot a \in J$ by (2). This means $x \in J_a$. So, $J \subseteq J_a$. Also, it is clear that $a \in J_a$ because $a \cdot a = 0 \in J$.

Suppose that I is any ideal in $\mathfrak{A}$ containing J and a. Is $x \in J_a$, then $x \cdot a \in J \subseteq I$. Thus $x \cdot a \in I$. Hence $x \in I$ by (J1) since $a \in J \subseteq I$. So, $J \subseteq I$. This means that J_a is the least ideal containing J and a. □

5. Positive implicative ideals

In this section, as the central part of this article, we introduce and analyze the concept of positive implicative ideal in this class of logical algebras. Since, according to the Theorem 5.4 proved here, every ideal in a right distributive QI-algebra $\mathfrak{A}$ is a positive implicative ideal in $\mathfrak{A}$, the observation of non-trivial positive implicative ideals is possible only in a non-distributive QI-algebra.

The concept of positive implicative ideals in this class of logical algebras is correlated with the determination of the term positive implicative ideals in BCK-algebras (in terms of [9]) and BCI-algebras (in sense of [1]).

Definition 5.1. A subset J of a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is called a positive implicative ideal of $\mathfrak{A}$ if the following holds:

(J0) $0 \in J$,

(PIJ) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \land y \cdot z \in J) \implies x \cdot z \in J)$.

We denote the family of all such ideals in the QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ by $\mathfrak{J}_{PI}(A)$.

First, we have:

Proposition 5.1. *Any positive implicative ideal in a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is an ideal in $\mathfrak{A}$. Thus, $\mathfrak{J}_{PI}(A) \subseteq \mathfrak{J}(A)$*

Proof. If we put $z = 0$ in (PIJ), with respect to (M), we get (J1). □

Remark 5.2. Similar to Remark 4.4, in accordance with axiom (QI), the determination of positive implicative ideals in QI-algebras can be done in the following way: A subset J in a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is called a positive implicative ideal in $\mathfrak{A}$ if, in addition to (J0), it also satisfies the following condition:

(PIJx) $(\forall x, y, z \in A)(((x \cdot (y \cdot (x \cdot y))) \cdot z \in J \land y \cdot z \in J) \implies x \cdot z \in J)$.

Every ideal in a QI-algebra $\mathfrak{A}$ is not a positive implicative ideal in that algebra, as the examples below show (See, Example 5.5 and 5.8). However, if $\mathfrak{A}$ is a right distributive QI-algebra, then the situation is completely different.

Theorem 5.3. *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a right distributive QI-algebra. Then every ideal in $\mathfrak{A}$ is a positive implicative ideal in $\mathfrak{A}$.*

Proof. Let J be an ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $(x \cdot y) \cdot z \in J$ and $y \cdot z \in J$. Then $(x \cdot z) \cdot (y \cdot z) \in J$ by (DR). Thus $x \cdot z \in J$ according to (J1). □

The result of the previous theorem suggests that we can deal with the concept of positive implicative ideals only in (non-distributive) QI-algebras. Our second proposition about positive implicative ideals in QI-algebras is the following:

Proposition 5.2. *Let J ne an ideal in a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$. If J is a positive implicative ideal in $\mathfrak{A}$, then holds:*

(PIJa) $(\forall x, y \in A)((x \cdot y) \cdot y \in J \implies x \cdot y \in J)$.

Proof. If we put $z = y$ in (PIJ), with the respect to (Re) and (J0), we get (PIJa). □

Corollary 5.1. *If the implication in Proposition 5.2 does not hold for the ideal J in the QI-algebra $\mathfrak{A} = (A, \cdot, 0)$, then J is not a positive implicative ideal in $\mathfrak{A}$.*

Proof. The statement of this Corollary is the contraposition of the previous proposition. □

Proposition 5.3. *Let $\mathfrak{A} = (A, \cdot, 0)$ be a QI-algebra. For any $a \in A$, The set $L(a)$ satisfies the condition (PIJ) if and only if it the following holds*

$$(\forall x, y, z \in A)(((x \cdot y) \cdot z \preccurlyeq a \land y \cdot z \preccurlyeq a) \implies x \cdot z \preccurlyeq a).$$

Proof. It is similar to the proof of Proposition 4.2. □

Theorem 5.4. *The set $\{0\}$ is a positive implicative ideal in a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ if and only if $L(a)$ satisfies the condition (J1) for any $a \in A$.*

Proof. It is obvious that $0 \in \{0\}$ is valid. Assume that $L(a)$ is an ideal in $\mathfrak{A}$ for all $a \in A$. Let $u, v, z \in A$ be such that $(u \cdot v) \cdot z = 0$ and $v \cdot z = 0$. Then $u \cdot v \preccurlyeq z$ and $v \preccurlyeq z$. Thus $u \cdot v \in L(z)$ and $v \in L(z)$. Hence $u \preccurlyeq z$ by (J1). So, $u \cdot z = 0$. Therefore, the set $\{0\}$ is a positive implicative ideal in $\mathfrak{A}$.

Conversely, suppose that $\{0\}$ is a positive implicative ideal in $\mathfrak{A}$. Let $v \in L(a)$ and $u \cdot v \in L(a)$. This means $v \preccurlyeq a$ and $u \cdot v \preccurlyeq a$. Then $v \cdot a = 0 \in \{0\}$ and $(u \cdot v) \cdot a = 0 \in \{0\}$. Thus, $u \in L(a)$. Hence $u \cdot a \in \{0\}$ since $\{0\}$ is a positive implicate ideal in $\mathfrak{A}$. Hence, $u \preccurlyeq a$ and $u \in L(a)$. This proves that $L(a)$ satisfies the condition (J1). □

The following example shows that the ideal $\{0\}$ in a QI-algebra does not have to be a positive implicative ideal in it.

Example 5.5. Let $A = \{0, a, b, c\}$ as in Example 4.2. Then $\mathfrak{A} =: (A, \cdot, 0)$ is a QI-algebra. Since $L_0(b) \notin \mathfrak{J}(A)$ it is not possible to conclude whether $\{0\}$ is a positive implicative ideal in $\mathfrak{A}$. However, since we have $(c \cdot b) \cdot b = b \cdot b = 0 \in \{0\}$ but $c \cdot b = b \notin \{0\}$, we conclude that the ideal $\{0\}$ is not a positive implicative ideal in $\mathfrak{A}$ by Corollary 5.1. □

On the other hand, we have:

Example 5.6. Let $A = \{0, a, b, c\}$ be a set with the operation given with the table

$\cdot$	0	a	b	c
0	0	b	a	0
a	a	0	a	0
b	b	b	0	b
c	c	b	a	0

Then $\mathfrak{A} = (A, \cdot, 0)$ is a QI-algebra ([10], Example 3.2). This QI-algebra is not right distributive because, for example, we have $(a \cdot b) \cdot c = a \cdot c = 0 \neq a = 0 \cdot b = (a \cdot c) \cdot (b \cdot c)$.

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_2 = \{0, b\}$, $J_5 = \{0, a, c\}$ and $J_6 = \{0, b, c\}$ are ideals in $\mathfrak{A}$. Subsets $\{0, c\}$ and $\{0, a, b\}$ are not ideals in $\mathfrak{A}$. For the subset $J_3 = \{0, c\}$ we have $a \cdot c = 0 \in J_3$ but $a \notin J_3$. Also, for the subset $J_4 = \{0, a, b\}$, for example, we have $c \cdot a = b \in J_4$ but $c \notin J_4$.

However, since for the ideal $J_0 = \{0\}$, we have $L_0(0) = \{0\}$, $L_0(a) = \{0, a\}$, $L_0(b) = \{0, b\}$ and $L_0(c) = \{0, a, c\}$ we conclude, in accordance with Theorem 5.4, that the ideal $J_0 = \{0\}$ is a positive implicative ideal in $\mathfrak{A}$. □

In addition to the previous Theorem 5.4, we also have the following stronger claim about positive implicative ideals in QI-algebras:

Theorem 5.7. *Let J be an ideal in a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$. If for all $a \in A$ the subset $J_a \cup \{0\}$ is an ideal in $\mathfrak{A}$, then J is a positive implicative ideal in $\mathfrak{A}$.*

Proof. Let $x, y, z \in a$ be such that $(x \cdot y) \cdot z \in J$ and $y \cdot z \in J$. Then $x \cdot y \in J_z$ and $y \in J_z$. Since J_z is an ideal in $\mathfrak{A}$, then $x \in J_z$ by (J1). So, $x \cdot z \in J$. Therefore, J is a positive implicative ideal in $\mathfrak{A}$. □

Example 5.8. Let $A = \{0, a, b, c\}$ as in Example 4.2. Then $\mathfrak{A} =: (A, \cdot, 0)$ is a QI-algebra. As shown in Example 4.2, the subsets J_0, J_1, J_3 and J_5 are ideals in $\mathfrak{A}$. If we apply Theorem 5.7 to the ideals J_1, J_3 and J_5 we get:

(i) For $K =: \{0, a\}$, we have:
$K_0 = \{x \in A : x \cdot 0 \in J_1 \ \lor \ x = 0\} = \{0, a\} = J_1$,
$K_a = \{x \in A : x \cdot a \in J_1 \ \lor \ x = 0\} = \{0, a\} = J_1$,
$K_b = \{x \in A : x \cdot b \in J_1 \ \lor \ x = 0\} = \{0, a, b\} = J_4$,

$K_c = \{x \in A : x \cdot c \in J_1 \vee x = 0\} = \{0, a, c\} = J_5.$

Since J_4 is not an ideal in $\mathfrak{A}$, we cannot conclude, relying on Theorem 5.7, that the ideal $J_1 = \{0, a\}$ is a positive implicative ideal in $\mathfrak{A}$. However, according to Corollary 5.1, the ideal J_1 is not a positive implicative ideal in $\mathfrak{A}$, because, for example, we have $(c \cdot b) \cdot b = b \cdot b = 0 \in J_1$ but $c \cdot b = b \notin J_1$.

(ii) For $K =: \{0, c\}$, we have:
$K_0 = \{x \in A : x \cdot 0 \in J_3 \vee x = 0\} = \{0, c\} = J_3,$
$K_a = \{x \in A : x \cdot a \in J_3 \vee x = 0\} = \{0, a\} = J_1,$
$K_b = \{x \in A : x \cdot b \in J_3 \vee x = 0\} = \{0, b, c\} = J_6,$
$K_c = \{x \in A : x \cdot c \in J_3 \vee x = 0\} = \{0, a, c\} = J_5.$

Since J_6 is not an ideal in A, we cannot conclude, relying on Theorem 5.7, that the ideal $J_3 = \{0, c\}$ is a positive implicative ideal in $\mathfrak{A}$. On the other hand, the ideal J_3 is not a positive implicative ideal in $\mathfrak{A}$ because, for example, we have $(c \cdot b) \cdot b = b \cdot b = 0 \in J_3$ but $c \cdot b = b \notin J_3$.

(iii) For $K =: \{0, a, c\}$, we have:
$K_0 = \{x \in A : x \cdot 0 \in J_5 \vee x = 0\} = \{0, a, c\} = J_5,$
$K_a = \{x \in A : x \cdot a \in J_5 \vee x = 0\} = \{0, a, c\} = J_5,$
$K_b = \{x \in A : x \cdot b \in J_5 \vee x = 0\} = \{0, a, b, c\} = A,$
$K_c = \{x \in A : x \cdot c \in J_5 \vee x = 0\} = \{0, a, c\} = J_5.$

Since the subsets K_0, K_a, K_b and K_c are ideals in $\mathfrak{A}$, we conclude that, according to Theorem 5.7, the ideal J_5 is a positive implicative ideal in $\mathfrak{A}$. □

Example 5.9. Let $A = \{0, a, b, c\}$ as in Example 4.3. Then $\mathfrak{A} =: (A, \cdot, 0)$ is a QI-algebra.

Since for the ideal $K = J_3 = \{0, c\}$, we have $K_0 = \{0, c\} = J_3$, $K_a = \{0, a\} = J_1$, $K_b = \{0, b, c\} = J_6$ and $K_c = \{0, c\} = J_3$ we can conclude, in accordance with Theorem 5.7, that J_3 is a positive implicative ideal in $\mathfrak{A}$.

Also, since for the ideal $K = \{0, a, c\} = J_5$ we have $K_0 = \{0, a, c\}$, $K_a = \{0, a\}$, $K_b = A$ and $K_c = \{0, a, c\}$, we can conclude, in accordance with Theorem 5.7, that the ideal $J_5 = \{0, a, c\}$ is a positive implicative ideal in $\mathfrak{A}$.

Also, since for the ideal $K = \{0, b, c\} = J_6$ we have $K_0 = K_a = K_b = K_c = \{0, b, c\} = J_6 \in \mathfrak{J}(A)$, we can conclude, in accordance with Theorem 5.7, that the ideal $J_6 = \{0, b, c\}$ is a positive implicative ideal in $\mathfrak{A}$.

On the other hand, for the ideal $J_1 = \{0, a\}$ we have $(J_1)_b = \{0, a, b\} \notin \mathfrak{J}(A)$ so we cannot conclude whether J_1 is a positive implicative ideal in $\mathfrak{A}$. However, since we have $(c \cdot a) \cdot b = b \cdot b = 0 \in J_1$ and $a \cdot b = a \in J_1$ but $c \cdot b = c \notin J_1$, we conclude that J_1 is not a positive implicative ideal in $\mathfrak{A}$. □

Remark 5.10. Let $A = \{0, a, b, c\}$ as in Example 5.6. Then $\mathfrak{A} = (A, \cdot, 0)$ is a non distributive QI-algebra. It can be shown, with little effort, that all ideals in this QI-algebra are positive implicative ideals. Therefore, there are non-distributive QI-algebras in which all ideals are positive implicative ideals.

Let $\mathfrak{A} = (A, \cdot, 0)$ be a QI-algebra. It can be proved that:

Theorem 5.11. *The family* $\mathfrak{J}_{PI}(A)$ *of all positive implicative ideals in a QI-algebra* $\mathfrak{A} =: (A, \cdot, 0)$ *forms a complete lattice.*

Proof. Let $\{J_k\}_{k \in K}$ be a family of positive implicative ideals in $\mathfrak{A}$. It is obvious that $0 \in \cap_{k \in K} J_k$ holds. Let $x, y, z \in A$ be such that $(x \cdot y) \cdot z \in \cap_{k \in K} J_k$ and $y \cdot z \in \cap_{k \in K} J_k$. This means that for every $k \in K$ we have $(x \cdot y) \cdot z \in J_k$ and $y \cdot z \in J_k$. Then $x \cdot z \in J_k$ for each $k \in K$. Thus $x \cdot z \in \cap_{k \in K} J_k$.

Let $\mathcal{X}$ be the family of all positive implicative ideals in $\mathfrak{A}$ that contain $\cup_{k \in K} J_k$. Then $\cap \mathcal{X}$ is a positive implicative ideal in $\mathfrak{A}$ according to the first part of this proof.

If we put $\sqcap_{k\in K} J_k = \cap_{k\in K} J_k$ and $\sqcup_{k\in K} J_k = \cap\mathcal{X}$, then $(\mathfrak{J}_{IP})(A), \sqcap, \sqcup)$ is a complete lattice. □

Therefore, for a given subset X in A, there is a smallest positive implicative ideal in $\mathfrak{A}$ that contains X. Also, for the element $a \in A$ there is a minimal positive implicative ideal in $\mathfrak{A}$, which contains a.

At the end of this section, the following theorem can be proved without difficulty:

Theorem 5.12. *If the ideal J in a QI-algebra $\mathfrak{A} = (A, \cdot, 0)$ satisfies the following condition*

$$(\forall x, y, z \in A)((x \cdot y) \cdot z \in J \implies (x \cdot z) \cdot (y \cdot z) \in J)$$

then J is a positive implicative ideal in $\mathfrak{A}$.

Every positive implicative ideal does not have to satisfy the condition stated in Theorem 5.12. So, for example, the ideal $J_2 = \{0, b\}$ in Example 5.6 is a positive implicative ideal but it does not satisfy the condition we are talking about, because, on example, we have $(a \cdot b) \cdot c = a \cdot c = 0 \in J_2$ but $(a \cdot c) \cdot (b \cdot c) = 0 \cdot b = a \notin J_2$.

It remains an open question whether the property 'being a positive implicative ideal' is hereditary in the sense that if J is a positive implicative ideal in a QI-algebra $\mathfrak{A}$, then every ideal K in $\mathfrak{A}$, which contains the ideal J, is a positive implicative ideal in $\mathfrak{A}$.

6. Conclusion

The main part of this article is the material presented in sections 3, 4, and 5. In the first of them, we show several creations of new QI-algebras for a given QI-algebra. In the second section, we prove some new properties of ideals in (non-distributive) QI-algebras. In the third section we have introduced the notion of positive implicative ideals in QI-algebras, and have investigated their properties. Of course, in some of the following investigations of QI-algebras, it is possible, among other things, not only to recognize some other types of ideals in this class of logical algebras, but also to establish their properties.

For the sake of illustration, we state the following possibility:

Definition 6.1. A subset J of a QI-algebra $\mathfrak{A} = (A, \cdot, 0)$ is a new type ideal in $\mathfrak{A}$ if, in addition to condition (J0), it also satisfies the following condition

(t) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \wedge y \cdot z \in J) \implies x \cdot z \in J)$.

It is easy to conclude that:

Proposition 6.1. *Every new type ideal in a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is an ideal in $\mathfrak{A}$.*

Proof. If we put $z = 0$ in (t), with respect to (M), we get (J1). □

In addition to the previous one, we have:

Proposition 6.2. *If $\mathfrak{A}$ is right distributive, then every ideal in $\mathfrak{A}$ is a new type ideal in $\mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ be a right QI-algebra, J be an ideal in $\mathfrak{A}$ and $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in J$ and $y \cdot z \in J$. Then $x \in J$ by (J1). Thus, $x \cdot z \in J$ by (J3). Hence, J is a new type ideal in $\mathfrak{A}$. □

The claim in Proposition 6.2 can also be demonstrated in the following way:

Proof. According to Proposition 3.10(5) in [6], for arbitrary $x, y, z \in A$ we have $(x \cdot y) \cdot z \preccurlyeq x \cdot (y \cdot z) \in J$, and therefore we conclude $(x \cdot y) \cdot z \in J$ according to (J2). Now from $(x \cdot y) \cdot z \in J$ and $y \cdot z \in J$ it follows $x \cdot z \in J$ since every ideal in the right distributive QI-algebra $\mathfrak{A}$ is a positive implicative ideal in $\mathfrak{A}$ in accordance with Theorem 5.3. □

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